

36 pag
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CONTENTS.		PAGE
A New Fragment of A. T. L., D8. By ANTON E. RAUBITSCHKE,		475
P. Aberdeen 18. By H. C. YOUTIE,		480
REVIEWS:		483
<i>Jaeger's</i> Diokles von Karystos. Die griechische Medizin und die Schule des Aristoteles (LUDWIG EDELSTEIN).— <i>Bignone's</i> Studi sul Pensiero Antico (FRIEDRICH SOLMSEN).— <i>Severyns's</i> Recherches sur la Chrestomathie de Proclo, Première Partie: Le Codex 239 de Photius, Tomes I et II (FREDERICK M. COMBELLACK).— <i>Paratore's</i> Introduzione alle Georgiche (JAMES HUTTON).— <i>Ehrenberg's</i> Alexander and the Greeks (C. A. ROBINSON, JR.).— <i>Glitz, Roussel, and Cohen's</i> Histoire Grecque, IV, Part I: Alexandre et le Démembrement de son Empire (C. A. ROBINSON, JR.).— <i>Becker-Frey's</i> Die Vorgeschichte des philosophischen Terminus 'contingens' (K. v. FRITZ).— <i>Le Blond's</i> Eulogos et l'argument de convenance chez Aristote (WILLIAM C. GREENE).— <i>Mugler's</i> L'Évolution des subordonnées relatives complexes en Grec (JAMES W. POULTNEY).— <i>Westington's</i> Atrocities in Roman Warfare to 133 B. C. (WILLIAM G. FLETCHER).— <i>Memoirs of the American Academy in Rome</i> , XV (AGNES KIRSOPP LAKE).— <i>Müller's</i> Claudians Festgedicht auf das sechste Konsulat des Kaisers Honorius (LESTER K. BORN).— <i>Rolfe's</i> Ammianus Marcellinus, II and III (CHARLES UPSON CLARK).— <i>Walde-Hofmann's</i> Lateinisches etymologisches Wörterbuch, 3. Auflage, Lief. 10. 11 (ROLAND G. KENT).— <i>Sage and Schlesinger's</i> Livy. XII (Books XI-XLII) (NORMAN W. DEWITT).		
BOOKS RECEIVED.		514
INDEX TO VOLUME LXI.		516

AMERICAN JOURNAL OF PHILOLOGY

VOL. LXI, 1

WHOLE No. 241

THE PYTHAGOREANS AND GREEK MATHEMATICS.

Historians are not agreed regarding the relevancy of the history of mathematics to the general history of science and philosophy. While Zeller treated it as negligible, except as mathematical concepts entered expressly into a system, some recent historians have regarded it as far more important, some going so far as to assign to it a leading rôle in the story. A story, of course, requires a hero, and Pythagoras would naturally play that part, were it not for the critical examination of the tradition that began, say, with the publication of Zeller's monumental work. In default of so imposing a figure, historians now tend to fall back upon the "Pythagoreans," as one might tell the story of a nation as that of a reigning dynasty; for the Pythagoreans are conceived as the mathematicians *par excellence* of Greece down to the middle of the fourth century, B. C.

Obviously this view has certain formal advantages, which it is not necessary to emphasize. Moreover, the candid student will gladly acknowledge that the concentration on mathematics that favors and accompanies this point of view has been fruitful in many ways. To be sure it is no uncommon observation that the positive contribution of any discussion is apt to be incidental and nearly or quite independent of the preconceived notion as to the angle from which the subject should be approached. It is important, therefore, to determine with what right and in what measure the critical historian may single out the Pythagoreans as especially worthy of playing the leading rôle, even if one grants the preëminent importance of mathematics.

Though our present concern is with the Pythagoreans as mathematicians, one cannot altogether ignore certain data on

— *The Pythagoreans and Greek mathematics*: AJPh 1940 1-33. | Le rôle des Pythagoriciens dans l'histoire de la mathématique grecque a été fort exagéré. Ce n'est pas en concentrant son étude sur certains individus ou certains groupes que l'on pourra reconstituer le développement de la mathématique grecque, mais en considérant celle-ci dans son ensemble.



The Pythagoreans and Greek Mathematics

W. A. Heidel

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Though our present concern is with the Pythagoreans as mathematicians, one cannot altogether ignore certain data on

which historians rely as evidence of scientific achievements on the part of Pythagoras himself. In this regard contemporary evidence, which we should value highly, is negligible when closely examined. Xenophanes alluded to his belief in the transmigration of souls, significant in reference to his religious views, but of philosophic importance only on certain assumptions that we have no right to make for Pythagoras himself. Much is made of a statement of Diogenes Laërtius¹ that Xenophanes denied that God breathes, it being assumed that he was rejecting the Pythagorean doctrine, attested by Aristotle,² that the cosmos inhales time and empty space from the surrounding infinite. If one adopts this view one does so in spite of several important considerations. For, first of all, the authority of Diogenes is not in itself great, and the passage in which the statement occurs is confused and in part certainly inaccurate, since it asserts that Xenophanes held the doctrine of the four elements. This alone, without other considerations, suffices to show that we have to do with a source on which one may not well rely. However, assuming that Xenophanes really said that God does not breathe, it is not necessary to suppose that he had a philosophical statement to the contrary in mind. He presumably had in mind rather the popular anthropomorphism; for he said that God in no wise resembles man, either in body or in mind—He is all sight, all hearing, all thought.³ That implies that God has neither eyes nor ears. Why should He have lungs? The reference to Pythagoras presupposes that Xenophanes identified God with the cosmos, an assumption that rests on a dubious interpretation of a statement by Aristotle;⁴ but even if one accepts that interpretation as true, there is as good reason to think that he was criticizing Anaximenes as Pythagoras.

Heraclitus also referred to Pythagoras, but in ways that do not warrant one in supposing that he thought of him as in any

¹ IX, 19.

² *Phys.* 213 b 22 ff., *frag.* 201 Rose.

³ The text of Diogenes apparently presupposes this context, for it continues *σύμπαντά τε εἶναι νοῦν καὶ φρόνησιν καὶ αἰδίων*.

⁴ *Metaph.* 986 b 24, *εἰς τὸν ὅλον οὐρανὸν ἀποβλέψας τὸ ἐν εἰναὶ φησι τὸν θεόν*. Aside from the fact that the Laurentian omits *τὸν θεόν*, the word *ἀποβλέψας* raises questions. It is natural to suppose that here, as at 991 a 23, it means "looking at a model," the usual meaning in Plato.

sort concerned with science or mathematics. "Learning of many things," he says,⁵ "teacheth not understanding, else would it have taught Hesiod and Pythagoras, and again Xenophanes and Hecataeus." Another fragment,⁶ of doubtful authenticity, asserts that Pythagoras practised inquiry more than all other men, and constructed for himself a wisdom that was only a knowledge of many things and an imposture. One suggestion of the text, as it has come down to us, is that Pythagoras culled his wisdom from many books. That might indeed be true, but what we otherwise have grounds for believing regarding him would hardly suggest it. The only real clue to the meaning of Heraclitus is the company in which he places the sage; and one will hardly contend that it suggests scientific inquiry in the sense in which it was practised by Pythagoreans in later times. Certainly it is difficult to associate the imposture charged to him with mathematics or mathematical theories.

On the other hand, Epicharmus at a somewhat later date is seriously invoked as a witness to the mathematical interest of Pythagoras or his Order; for a fragment of his refers to odd and even numbers.⁷ Whether the text is genuine or not, it seems to me incredible that one should think of these terms or of the practice of counting with pebbles as originating with Pythagoras. No doubt odd and even numbers, square and oblong figures were almost as old in his day as they are in ours. To make plausible a reference to a particular thinker it does not suffice to point out something that was presumably the common property of many, if not most men; what one has a right to require is something distinctive, for example, in the connection of ideas.

When one comes to Alcmaeon the case is not quite so simple. There is no doubt that he was a physician of Croton, where according to tradition Pythagoras first established his Order. Of the date at which the medical school⁸ of Croton originated we have no definite knowledge, though certainly it existed before the arrival of Pythagoras; neither is it certain whether it had a

⁵ *Frag.* 40 Diels, tr. Burnet.

⁶ 129.

⁷ *Frag.* 2 Diels.

⁸ This term, commonly used, is apt to prove misleading. We must not think of an organized society in the sixth century. Wherever there were physicians who taught their art to their sons or to others whom they approved, there was a "school."

filiation to any other school, though a relation to the Cnidian is not improbable. As a physician Alcmaeon would most naturally derive whatever medical or physiological presuppositions he made from the school to which he belonged, and there is the best of evidence for the belief that in his time as well as later a physician was generally interested in such scientific researches as were being pressed, as the employment of Democedes and Ctesias by the Kings of Persia well illustrates. Alcmaeon dedicated his treatise to three men reported to have been Pythagoreans, but about whose attainments and achievements we know little or nothing. We may assume, therefore, that he was at least on intimate terms with members of the Order, as would be natural in any case since all concerned presumably belonged to the intellectually more conspicuous group of citizens. This association need not of course imply any formal relation to the Pythagorean Order, nor does it afford any grounds for attributing to it any special interest or direction of research considered as a whole. Later tradition, to be sure, regarded Alcmaeon as a Pythagorean, and that view is still generally accepted. So far as one can see, this assumption can be justified only by a statement in Aristotle's *Metaphysics* which, however, proves upon examination to be at least very dubious. After speaking of certain Pythagoreans who set up a table of ten pairs of contraries—limited and unlimited, odd and even, etc.—he proceeds,⁹ "In this way Alcmaeon of Croton seems to have conceived the matter, and either he got the view from them or they got it from him; for he expressed himself similarly to them. For he says most human affairs go in pairs, meaning not definite contraries such as the Pythagoreans speak of, but any chance contraries, e. g. white and black, sweet and bitter, good and bad, great and small. He threw out indefinite suggestions about the other contraries, but the Pythagoreans declared both how many and which their contraries are."

In justice to those who regard Alcmaeon as a Pythagorean it must be added that this version of Aristotle's statement omits a clause asserting that he was <young> in the old age of Pythagoras, which is found in some good manuscripts but wanting in the best. I fully agree with Ross in bracketing it, not only because

⁹ *Metaph.* 986 a 26 ff., tr. Ross.

it is omitted by the Laurentian and is quite ignored by Alexander and is besides otherwise contrary to the usage of Aristotle, but because the text of the MSS that contain the clause is imperfect, since it does not give the inevitable word "young" (νέος). This seems to me strongly to suggest that we have here a marginal note carelessly embodied in the text. One may even conjecture with some confidence the source of the marginal note; for it may well be derived from Porphyry's *Life of Pythagoras* (c. 104), where a long and ill-assorted list is given of the most ancient Pythagoreans who were contemporaries or pupils of Pythagoras, young in his old age (συγχρονίσαντες καὶ μαθητεύσαντες τῷ Πυθαγόρᾳ πρεσβύτη νέοι), including Alcmaeon. If that were true, one readily understands why Alexander could not take account of it.

Now, if we consider the matter more in detail the data regarding Alcmaeon fail to give us much real information. Aristotle, on this view, affords no indication of his date, and Porphyry is as always, except where we can certainly make out his authorities, quite untrustworthy. In this instance he groups as contemporaries and personal pupils of Pythagoras, along with Alcmaeon, such men as Philolaus and his pupil Eurytus as well as Lysis, the teacher of Epaminondas, who must be dated near the turn of the fifth and fourth centuries. Furthermore it is natural to infer that Aristotle had no knowledge whether Alcmaeon was actually a member of the Pythagorean Order; and the way he speaks of the possible relation of their respective doctrines is quite non-committal. To this one must add that from Zeller onward the consensus of scholars has strongly tended to regard the table of ten contrarieties as a relatively late creation of certain Pythagoreans. It is clear, then, that Alcmaeon affords no criterion for determining what is early and late in Pythagoreanism.

What he had in common with certain Pythagoreans of unknown date is, according to Aristotle, a tendency to look upon things as characterized by contraries. Just why Aristotle should have thought it necessary to ask whether such a natural point of view had been borrowed by either from the other remains a profound mystery, because he himself had emphasized the rôle of certain contrarieties in the thought of the Ionians, especially Anaximander. Not to mention the common Greek

practice of setting one state in contrast to its opposite,¹⁰ it was inevitable that Alcmaeon as a physician should concern himself with such phenomena as heat and cold, the opposite effects of summer and winter on his patients. If there was anything distinctive of either Alcmaeon or Pythagoreans in this respect we have no knowledge of it. Burnet, to be sure, in his notes on Plato, *Phaedo* 86b f., would have us believe that the doctrine was Pythagorean, though the special form of it there set forth must have been influenced by Empedocles and the Sicilian School. What one may infer from Aristotle's statement is perhaps only that he was aware that Alcmaeon was sometimes regarded as a Pythagorean or, because of the dedication of his book, in close touch with the Order, but that he was not prepared to commit himself on the question of their relations. That, at least, appears to have been his general attitude toward the Pythagorean tradition as we see it reflected in the works of his maturity.

Our present concern being with the history of Greek mathematics and the rôle played by the Pythagoreans in it, it is clear that so far what we know about Alcmaeon throws no light on the subject. It is true that in the Pythagorean table of contrarieties there are several pairs of mathematical concepts; but none of these is attested for Alcmaeon, and the uncertainty respecting his date and that of the table deprives the question of all possible evidential value. The same is obviously true of the only other datum that may be thought to have a bearing on Pythagorean science. Aëtius states¹¹ that certain *μαθηματικοί*, presumably Pythagoreans, held that the planets moved in a sense contrary to that of the fixed stars, i. e. from west to east, and that Alcmaeon agreed with them. While this would not directly throw light on Pythagorean mathematics, if accepted and interpreted as implying that the theory of the *μαθηματικοί* was at least as old as Alcmaeon, it would confirm one's belief in the scientific interest of Pythagoreans at a date presumably before the middle of the fifth century. The character of the

¹⁰ See Burnet, *Early Greek Philosophy*,* p. 8. Heraclitus only emphasizes what was a common Greek point of view. His merit lies in his attempt to reconcile the oppositions which everyone felt.

¹¹ II, 16, 2-3. The other astronomical views attributed to Alcmaeon are, as Burnet, *E. G. P.**, p. 110, n. 1, truly says, extremely crude.

text of Aëtius, however, is such as hardly to warrant one in accepting the statement as a fact or in so interpreting it, if it were true; for Aëtius is lavish of statements about Pythagoras that sober criticism must reject, and perhaps the earliest dependable evidence regarding the date of Alcmaeon comes from Greek medical writers of the "Hippocratic" and Sicilian schools in the latter half of the fifth century. The earliest of the "planets" (excluding sun and moon) mentioned are the morning and evening stars, the discovery of their identity being attributed to Parmenides or to Pythagoras. Disregarding the latter, it is possible that the former actually referred to their identity. As for the supposed retrograde motion of the planets, including sun and moon, we know that Plato in the *Laws* still thought it worth while to declare that it was false. We are told that Anaxagoras and Democritus held that all the stars moved from east to west. On the other hand Plato in the *Timæus* and the myth of Er in the *Republic* represented the planets as moving from west to east. It may very well be true, therefore, that the notion was originated or at least held by Pythagoreans.

It is certain that Plato and his school owed much to the Pythagoreans, and that Socrates had among his associates men who were somehow affiliated with them. It was, however, a revived Pythagoreanism in both cases, and many questions that cannot be confidently answered arise in connection with it. On the surface it would appear that the associates of Socrates were chiefly concerned with religious and moral problems, while Plato and his school debated mathematical questions with Pythagoreans. This appearance may be deceptive. In any case, as we shall presently see, it is difficult if not impossible for the most part to distinguish between what is Platonic and what is Pythagorean. Above all, we obtain from Plato no certain criteria by which one could differentiate between the fifth and the fourth centuries in Pythagorean thought. Since the revived Pythagoreanism died out at the end of the fourth century one turns expectantly to Aristotle and his pupils for information, the more hopefully because Aristotle and his school diligently studied the earlier history of the several sciences. There were, however, marked differences among them, and unfortunately Aristoxenus of Tarentum, who was most deeply interested in Pythagoreanism,

appears in general to deserve little confidence. It seems probable that he was responsible for a good deal that is reported by later writers.

To begin with Aristotle, there are those who confidently cite a statement made by Apollonius, a writer not earlier than the second century B. C., in his *Historiae Mirabiles*, 6: "After these (*sc.* Epimenides, Aristeas, Hermotimus, Abaris, Pherecydes) came Pythagoras the son of Mnesarchus. At first he busied himself with mathematics, i. e. with numbers,¹² but after a time he did not refrain from the miracle-working of Pherecydes." It will be noted that at best we have here witness to concern about numbers on the part of Pythagoras, entirely credible in itself, but giving no real information, because we are left in the dark regarding the way he was supposed to deal with numbers. So much might probably have been said of any man at the time. The circumstance that makes it worth while to cite the remark is that it is supposed to be derived from Aristotle, who would presumably mean that Pythagoras already began the speculations about numbers mentioned as characteristic of Pythagoreans in his extant treatises. There is, however, no reason whatever to think that the statement derives from Aristotle,¹³ who is expressly cited only as authority for several statements in the sequel. It is interesting, however, to note that from the context it would seem to follow that Pythagoras took up Miracle-working after the fashion of Pherecydes after arriving at Metapontum, whereas his occupation with numbers would thus have begun (and ended?) in Ionia.¹⁴ This would be poor evidence for Pythagoras and his Order as the prime movers in the study of mathematics among the Greeks. We may confidently dismiss this datum as of no significance. Elsewhere Aristotle attributed not a single scientific achievement to Pythagoras. Apollonius is known to have quoted as genuine works admitted to be falsely

¹² *καί* is here, as often, defining.

¹³ Rose includes it in Aristotle, *frag.* 191.

¹⁴ The supposed connection of Pythagoras with Pherecydes is referred by some to his earlier, by others to his later years. Iamblichus, *Vit. Pyth.*, 184, represents Pythagoras as returning from Italy to attend him in his last illness. This is very improbable. Whether this statement derives from Aristoxenus, who said that he buried Pherecydes in Delos (Diog. Laërt., I, 118), is not certain, though not improbable.

attributed to Aristotle, and in the present case, where no authority is actually cited, it is more likely that he derived the notion from Heraclides of Pontus or a similar source. It is characteristic of a certain kind of the search for sources that this particular statement should be attributed to Aristotle simply because it precedes several others expressly referred to him. There is no doubt that he had written a special treatise about the Pythagorean doctrines,¹⁵ but there remain some difficult questions regarding its scope.

Perhaps the most precise statement regarding the date of the Pythagoreans known to Aristotle occurs in the *Metaphysics*:¹⁶ "Contemporary with these philosophers and before them the Pythagoreans, so-called, devoted themselves to mathematics; they were the first to advance this study, and having been brought up in it they thought its principles were the principles of all things." Of the nature of this study we shall speak presently; for the moment we are concerned chiefly with the temporal relation of the Pythagoreans. It seems clear that the philosophers to whom they were to be compared as to date were Leucippus and Democritus, who had just been mentioned, not the entire series of thinkers previously enumerated. Unfortunately the statement leaves much to be desired, because it is very vague. Assuming that Aristotle meant that the Pythagoreans were contemporaries of both Leucippus and Democritus and in part earlier than either of them, the time indicated might extend from a date before the middle of the fifth century far into the fourth. That is doubtless true as to the later period, and it may have been the case so far as the earlier date is concerned; but it is not certain that Aristotle meant to say just that. It is quite possible that he meant that the first ones were contemporary with Leucippus and earlier than Democritus, for his expression is singularly wanting in precision. What does he imply in saying that they were the first to advance the study of mathematics? Conceivably he might have meant that the study long antedated them and that their merit lay in notably advancing it; but if so, one learns nothing from him that one could not safely infer from a general knowledge of Greek civilization, which had already attained a high degree of advancement. Again, the state-

¹⁵ *Metaph.* 986 a 12.

¹⁶ *Ibid.* 985 b 23 ff.

ment that they had been brought up in the study implies either that it existed as such before they took it up, or that those who advanced it were not the earliest Pythagoreans. At all events careful attention to Aristotle's statement actually assures us that in the latter part of the fifth century Greek mathematics had attained considerable development, which the Pythagoreans were credited with promoting. The precise date would then depend on that of Leucippus, which is unknown.

Another statement similarly vague also occurs in the same work. "Socrates," we are told,¹⁷ "occupied himself with the excellences of character, and in connection with them became the first to raise the problem of universal definitions—for in the realm of physics the problem was only touched on by Democritus, who defined, after a fashion, the hot and the cold;¹⁸ while the Pythagoreans had before then treated of a few things, whose definitions¹⁹ they connected with numbers—e.g. opportunity, justice, or marriage." We may ignore the reference to definitions by predecessors of Socrates, as they were not definitions at all in the logical sense; but it is worth noting that Aristotle at least affords no comfort to those who attribute definitions of geometrical terms to Pythagoreans before the time of Zeno. What interests us here is that Socrates is dated after Democritus and the Pythagoreans are set down before the latter. If this order is accepted as historically true, the occupation of Socrates with universal definitions must be referred to his last years, and the Pythagoreans who proposed the identification of concepts like justice or marriage need not have been earlier than the last third of the fifth century.

In many passages Aristotle either expressly couples Pythagoreans with Plato or Platonists or else makes statements that

¹⁷ *Ibid.* 1078 b 17 ff., tr. Ross.

¹⁸ Aristotle apparently refers to such things as the description of the fire atoms as little spheres.

¹⁹ Cf. *Metaph.* 985 b 29, 987 a 19 ff. It is curious that Aristotle should regard such identifications as definitions. One should expect him somewhere to refer to the definition of mathematical terms, e.g. such as are required in geometry, but he never does so. At best the statement, *Eth. Nic.* 1132 b 21 ff. *ὁρίζοντο γὰρ ἀπλῶς τὸ δίκαιον τὸ ἀντιπεπονθὸς ἄλλῃ* might be compared with attempts of interlocutors in Platonic dialogues. See in general, *De Part. An.* 642 a 24 ff.

may sometimes apply to both and sometimes distinguish between them. Doubtless in his lectures he would make it clear to whom he specifically referred; we cannot always distinguish, and even his ancient commentators were often at a loss. What this means is at once obvious. For Aristotle the Pythagoreans were generally of interest chiefly because of their relation to the teachings of the school of Plato in which he was brought up. The subjects of prime importance to him were those debated in the Academy, while he belonged to it, and between him and the leading Platonists, after he had set up his own school. As has already been said, we are frequently unable to assign the various views to their advocates, just because those who heard his lectures were presumably themselves engaged in the debates and hardly needed to be told whose views were being criticized. Though we cannot hope to go far in solving the historical and personal questions involved in these discussions, a general presumption does seem to be created by this state of the record. Whatever may have gone before (the very point that chiefly concerns us at the moment) it is difficult to accept the view, now tacitly assumed by some scholars, that the debates in the Academy, during Aristotle's connection with it, were chiefly concerned with Pythagorean theories and definitions proposed a century or more earlier. When in search of teachings that might seem to foreshadow his own doctrine Aristotle was often compelled to re-interpret the record, not only in regard to the four kinds of causation, but also in regard to other matters. One might easily point out instances where a rather primitive doctrine is set into a very different light because the point of view had radically changed. Unless one bears this in mind one is likely to misconceive entirely the historical development or at least to build a structure of unverifiable and improbable hypotheses.

So far as concerns the chronology of Pythagorean mathematics it seems to be clear that Aristotle does not warrant us in going beyond the middle of the fifth century. Of course one may conjecturally go much farther back, and many have done so and doubtless will continue to do so. That is a privilege any scholar has, provided he is himself aware, and keeps his readers aware, of the basis of his statements. We have therefore to consider the view of the Pythagorean mathematics as Aristotle

saw it. It is surprising how little positive information he gives us on this head, and some of his definite, and repeated, statements are open to the justifiable suspicion of actual misrepresentation.

His most general statement²⁰ tells us that the Pythagoreans were the first to advance the study of mathematics. He does not say what he means by the term, but it is natural from the context to infer that he was thinking principally of their pre-occupation with *numbers*. They thought numbers, as the first principles of mathematics, were the principles of all things. Because they saw in numbers resemblances to things about them, and because the attributes and ratios of the musical scales could be expressed in numbers, and all other things were modeled after numbers, they thought the elements of numbers were the elements of all things and that the whole heaven was a musical scale and a number. As for the elements of number he elsewhere²¹ tells us that they were the odd and the even, or the limited and the unlimited. The examples he gives of the way in which they noted resemblances in numbers to parts of the heavens and how they identified numbers with certain abstract concepts do not add to our knowledge of mathematics, but rather suggest that their method was fantastic and futile, as were the "definitions" he mentions. If these were the best they could offer, they required little serious attention, and there would be no reason to suspect that their geometry afforded examples of terms properly defined. The same may be said of his report regarding the nature of the One and of the Infinite.²² We are repeatedly

²⁰ *Metaph.* 985 b 22 ff.

²¹ *Ibid.* 986 a 18.

²² Aristotle repeatedly says that the Pythagoreans, like Plato, regarded the One ($\tau\acute{o} \epsilon\nu$) and the Infinite ($\tau\acute{o} \alpha\pi\epsilon\iota\rho\omicron\nu$) as substance and not as attribute, and yet divided the latter (*Phys.* 203, a 4-6; 204 a 20-34; cf. *Metaph.* 987 a 134; 1001 a 3 ff.; 1053 b 9-16). One wonders what basis there was for his statements so far as the Pythagoreans were concerned. Possibly he was transferring Platonic expressions to them. One suspects that he had only the expressions $\tau\acute{o} \epsilon\nu$ and $\tau\acute{o} \alpha\pi\epsilon\iota\rho\omicron\nu$ to go on, and interpreted these as implying the substantial existence of the One and the Infinite apart from any entity of which they might be predicated. This interpretation might well be captious; for, when he professes to cite an actual opinion of theirs, he says (*Metaph.* 1091 a 13 ff., tr. Ross), "There need be no doubt whether the Pythagoreans attribute generation to them (i.e. things) or not; for they obviously say that when the One had been constructed (whether out of planes or of surface or of seed or of elements which they cannot express [rather

told that they knew only one kind of number, the "mathematical" or abstract, but that it also had magnitude, or was spatial. Aristotle is apparently puzzled by this contradiction, and it is difficult to believe that any one consciously held these contradictory views regarding the same thing. One is tempted to think that Aristotle combined two classes of expressions, (1) those relating to mathematics and (2) those used in their cosmology. In their reckonings and in their theory of numbers the numerals they employed were naturally those used by everyone else, whether or not they were expressly characterized as abstract. The difficulty, real or factitious, arose from their use of numbers in cosmology. Here the different ways in which Aristotle represents their views raises the question whether he does them justice, for it is hardly possible to regard as synonymous the statements that the concrete things are made of numbers and that they display resemblances to or are imitations of numbers. There are, of course, ways in which one may explain the different statements without impugning Aristotle's veracity or the fairness of his interpretation; but, when all is said, there remains a reasonable doubt as to the actual views of the Pythagoreans that is not removed by referring to the figurate representation of numbers or to the practice of Eurytus. It is more to the point to note that in any case this practice of Eurytus makes it impossible to distinguish between early and late conceptions of numbers among Pythagoreans. To say, as is sometimes said, that Eurytus "still" used the primitive method merely begs the question. We know that, as the pupil of Philolaus, he was approximately contemporary with Plato.

about which they don't know what to say]), immediately the nearest part of the Infinite began to be constrained [rather, reading *εἰσείλετο*, "inhaled"] and limited by the limit." Whatever their *language* might imply, it is fair to assume that they thought of the Infinite, not as a substance, but as an attribute of something that could be inhaled, say breath, or air. Cf. Diels, *Vorsokr.*⁵, I, p. 460, 4 ff. The passage above enclosed in parentheses is explained by *Metaph.* 1080 b 20, *ὅπως δὲ τὸ πρῶτον ἐν συνέστη ἔχον μέγεθος, ἀπορεῖν ἐόικασιν*. In any case, whatever basis he had for the supposition that the One and the Infinite were substances seems to have referred to cosmology, and so had reference to mathematics only on the assumption that *the cosmos was actually constituted of numbers*. In so far as the Pythagoreans meant that to be taken quite literally, his argument would of course hold good; but it throws no light on their mathematical conceptions.

In all this we discover little that goes beyond the concern of the Pythagoreans about numbers. The table of ten contrarities contains several mathematical terms, but they may all relate to numbers rather than to geometry. The same is true of the musical intervals and concords and of the harmony of the spheres. As for Pythagorean geometry proper, in which historians of mathematics are naturally most interested, there is, so far as I can see, no certain reference to it in Aristotle. One naturally thinks of the proposition that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides, which no one, probably, doubts that we owe to the Pythagoreans; but, in the first place, though Aristotle alludes to it,²³ he does not say that it was Pythagorean, and secondly he states the matter as concerned with numbers, using it as an example of demonstration by a *reductio ad absurdum*: the proof that the diagonal is incommensurable results from the fact that odd must be equal to even numbers. He speaks of Pythagoreans as earlier than Plato, but he provides no criterion by which to distinguish later from earlier or to determine the age of a single achievement.

When we come to his pupil Eudemos, who wrote the history of the mathematical sciences, we naturally have great expectations. Unfortunately little remains of his work, and that little has in part to be reconstructed from late sources. In the reconstruction, moreover, due care has not always been taken to avoid unjustifiable assumptions; for it does not follow from the fact that the later tradition dealing with the history of mathematics ultimately depends on Eudemos that it did not suffer additions as well as losses. The situation here seems to be quite parallel to that of the doxographic tradition which ultimately derives from Theophrastus, for it is plain in the latter case that the phase represented by Aëtius was strongly influenced by the school of Posidonius, who followed the Stoic practice of "accommodation" or assimilation of earlier to later doctrines. We know, for example, that Posidonius thought that Parmenides knew the geographical zones.²⁴ When it is reported²⁵ that Py-

²³ *Anal. Pr.* 41 a 26.

²⁴ Strabo, I, 94, repeated by Aëtius, III, 11, 4. Aëtius, II, 12, 1 says that Pythagoras and his followers knew the five celestial zones. This

thagoras and Parmenides regarded the earth as spherical, it is obvious that these views go together. Since every other indication points definitely to the conclusion that the sphericity of the earth was first proposed about the end of the fifth century, one has good reason to suspect that we have in these statements an example of the historical method of Posidonius, of which Galen gives us a good illustration. He says²⁶ that Posidonius ascribed the doctrine of the tripartite soul to Pythagoras, "inferring it from what some of his disciples have written, though no treatise of Pythagoras himself has survived to our time." There being no evidence of Pythagorean writings before the time of Philolaus,²⁷ who lived at most a generation before Plato, the Pythagoreans on whom he relied dated presumably from the fourth century or later, and it is not even necessarily implied that these attributed the doctrine in question to Pythagoras. Probably Posidonius merely found the doctrine stated by some Pythagorean and from its source inferred that Pythagoras himself had held it, because, as he thought, his followers religiously adhered to his views.

It is obvious to any critical student that the reports of Aëtius regarding Pythagoras and the Pythagoreans are so compounded of earlier and later data that they are for historical purposes entirely useless except as they can be checked by reference to others that inspire greater confidence. This does not, of course, mean that Posidonius himself is to be credited with every statement about Pythagoras and Pythagoreans in Aëtius. It is enough for our purposes to know that his method has infected the mass. It is important to bear this in mind in dealing with the history of mathematics, which, as has just been said, derives

looks like another inference to Pythagoras from (late) Pythagoreans, such as we might expect Posidonius to make. Aëtius, II, 24, 9 even attributes the notion of zones to Xenophanes.

²⁶ Diog. Laërt., VIII, 48; IX, 21. From the former passage it seems clear that Theophrastus merely said that Parmenides (first?) used the term *στρογγύλη* as describing the form of the earth. Though it *may* mean spherical, it need not be so interpreted, because it is used in the description of the earth by Diogenes of Apollonia, who thought it a circular disk. Favorinus was the authority of Diog. Laërtius.

²⁶ *De Hippocr. et Platone*, p. 478.

²⁷ Demetrius Magnes, *ap.* Diog. Laërt., VIII, 85.

ultimately from Eudemus, as the doxographic tradition derives from Theophrastus. There is good reason to suspect that the former was subjected to the same influences as the latter. Tannery made out a strong case for the thesis that the summary account of the development of Greek mathematics given by Proclus was directly or indirectly derived from Geminus, and we chance to know that Geminus wrote a commentary on Posidonius. These facts suffice to cast suspicion on the statement of Proclus regarding the mathematical achievements of Pythagoras, which owe their supposed authority to the presumption that they derive from Eudemus. While there are many historians who hold that view, it has been rejected of late by several leading scholars. That it may be questioned is enough for our present purpose. I may add, however, that in my opinion it is certainly not derived from Eudemus. There is, in fact, no satisfactory evidence that Aristotle, Eudemus, or Theophrastus attributed a single scientific achievement to Pythagoras himself: such things they always referred to "Pythagoreans." If other members of Aristotle's school represented Pythagoras as the originator of the interests that marked the scientific pursuits of his Order, it becomes a pertinent question why they did so. We should be especially grateful if we could be quite sure that it was really Dicaearchus, one of the best pupils of Aristotle, who said²⁸ of Pythagoras, "What discourses he held with his associates, no one can affirm, for they observed exceptional silence. Nevertheless what is best attested by all is, first, that he said that the soul is immortal; next, that it migrates into other species of living beings; and in addition, that according to certain periods the things that once were come about again, and nothing is absolutely new; and that all beings that have souls must be considered akin. For it seems that Pythagoras was the first to introduce these beliefs into Greece."

Proclus, then, after saying that Thales first went to Egypt and thence introduced geometry into Greece, himself discovering many propositions and preparing the way for his successors to discover the principles of many others, approaching some solu-

²⁸ Porphyry, *Vit. Pyth.*, 19. Diels (*Vorsokr.*⁵, I, p. 100, 36 ff.) holds, with most scholars, that this statement is part of the text which Porphyry refers to Dicaearchus (c. 18).

tions in a more general (i. e. abstract), others in a more visual manner, and saying that Mamercus, the brother of the poet Stesichorus was mentioned²⁹ as having interested himself in geometry, he proceeds:³⁰ "Following these Pythagoras converted geometrical philosophy³¹ into the form of a liberal education, contemplating its principles deductively and investigating its theorems in an immaterial and rational way. It was he who discovered the theory of proportions (?) and the construction of the (five) regular solids." As I have already said, I cannot believe that this truly represents Eudemus, however much in detail may have been ultimately derived from him. Judging by what we otherwise know of him we may be sure that he did not so speak of Pythagoras; but he may have characterized the method of the *Pythagoreans* in some such terms, which seem to reflect the ideals of Plato as set forth in the *Republic*. Even so, however, the method of (Pythagoras or) the Pythagoreans, as here described, does not differ essentially from that attributed to Thales. We know, to be sure, that Eudemus did credit Thales with the knowledge of certain fundamental propositions of geometry;³² but it is plain that his knowledge of them was inferred from feats with which tradition credited him. The question naturally arises, since we have here no verbatim quotations from Eudemus, whether he attributed these discoveries to Thales unconditionally or merely said that he must have known the propositions *if the traditions were true*. Either view is, of course, possible. In any case, however, the statement of Proclus and the known inferences of Eudemus make it certain that the mathematical tradition did not regard Pythagoras as the *founder* of the science in Greek lands. One must add that the statement that Pythagoras discovered the construction of the cosmic (Platonic) solids is certainly not true. The labored efforts of certain scholars to find some justification for it are based on the indefensible view that we are here dealing with Eudemus rather than with Proclus. If the latter's immediate source was Geminus we

²⁹ This expression suggests Proclus' dependence on general literature rather than on a serious history of mathematics.

³⁰ *In Eucl.*, p. 65, 11, ed. Friedlein.

³¹ Eudemus would hardly have used this expression.

³² Diels, *Vorsokr.*⁵, I, p. 79, 8 ff.

may really have to thank Posidonius for the view generally accepted by historians of Greek mathematics.

We are fully justified, then, in disregarding the supposed testimony of Eudemus to the mathematical achievements of Pythagoras; but of course that does not eliminate the Pythagoreans. If we take the statements of Proclus as applying to them, we have essentially the same view as we obtain from Aristotle. But Eudemus, fortunately, compensates us for the loss of spurious data regarding the founder by giving precious information about specific achievements of those who called themselves Pythagoreans. It is not necessary for our present purpose to review and evaluate the precious data of Eudemus as reported by Proclus in his *Commentary on the Elements of Euclid*. That may safely be left to more competent mathematicians. It is only necessary to emphasize the need of guarding against the same temptation to which Eudemus may have succumbed—the temptation to infer too much from what we may safely accept as fully attested. If that precaution is fully observed we arrive at a body of propositions and demonstrations at least as early as Eudemus and presumably earlier. One wishes that one might add that all this body may be certainly referred to the time before Hippocrates of Chios,³³ who is said to have written the first *Elements of Geometry*; for then we should have an approximate *terminus ante quem*. Obviously this cannot be done, because there were Pythagoreans who lived contemporary with and after him, and we know almost nothing about the contents of his treatise.

To sum up the situation, we may say that from Aristotle and Eudemus we learn that from the middle of the fifth century onward there were Pythagoreans busily and fruitfully occupied with mathematics, especially with the theory of numbers and geometry. Between the middle of the fifth century and Aristotle there are a few data of considerable importance; but about the part played by Pythagoreans (excepting the specific achievements mentioned by Eudemus, all without dates or names of

³³ There exists no evidence for his date, which is commonly put about 450 B. C. This seems to me too early; on the other hand Erich Frank, *Plato und die sogen. Pythagoreer*, p. 227, probably goes too far in the opposite direction in saying that he lived scarcely before 400.

individual geometers) we have no satisfactory evidence. It is one of the most singular facts in the history of Greek thought that individual Pythagoreans are rarely mentioned except by later writers whom one has every reason to suspect. Aristotle, indeed, mentions among others Hippiasus, but only to say ³⁴ that he, like Heraclitus, made fire the material cause. Whether he wrote a book or not we do not know, but we incline to doubt it because it is expressly stated that he did not ³⁵ and that Philolaus was the first Pythagorean to do so. Even if he did, however, we should infer from Aristotle's reference that his book did not deal with mathematics. In later times he became the Judas of the Order, who betrayed the master's secrets and was deservedly destroyed by the gods. He obviously cannot afford firm footing for a reconstruction of the development of Greek mathematics. Aristotle's other references to individual Pythagoreans—to Paron, Xuthus, Eurytus—tell us nothing of importance.³⁶ Philolaus is cited in the *Eudemian Ethics*,³⁷ but not for mathematics. The contention of some scholars that Aristotle in the *Metaphysics* ³⁸ refers to a statement of his is more than dubious. While I believe that Erich Frank has tried to prove too much, I fully agree with Burnet in regarding the so-called fragments of Philolaus as spurious, or at least as pseudepigraphic. What we may safely say about his views depends on Plato, Eudemos,³⁹ and Menon, and that throws no light on his mathematics. Even if one accepts the "fragments," however, there is little gain in this respect, unless they are interpreted and combined with texts

³⁴ *Metaph.* 984 a 7. Theophrastus (Diels, *Vorsokr.*⁵, I, p. 109, 6 ff.) repeats this with amplification probably based only on Aristotle's statement that Hippiasus and Heraclitus made fire their ἀρχή.

³⁵ Diog. Laërt., VIII, 84, citing Demetrius Magnes as his authority. Cf. n. 27.

³⁶ Diog. Laërt., VIII, 46 mentions as the last of the Pythagoreans, whom Aristoxenus knew, Xenophilus, Phanto, Echecrates, Diocles, and Polymnastus, pupils of Philolaus and Eurytus. None of these, so far as we know, contributed anything to mathematics. Ecphantus appears as an astronomer who combined opinions of Democritus and Anaxagoras. Perhaps he was only an imaginary person, playing a rôle in a dialogue of Heraclides Ponticus. We know nothing about his mathematics.

³⁷ 1225 a 33.

³⁸ 1080 b 6: cf. Philolaus, *frag.* 8 Diels.

³⁹ That is, provided the *Eudemian Ethics* was written by him (300 B. C.?).

dating from later centuries. What remains for the would-be historian are inferences. Besides Aristotle and Eudemus, Plato and the Platonists contemporary with Aristotle inevitably demand consideration; but here, as has already been stated, the information is in general rather vague and subject to different interpretations. Above all, it affords no definite chronological data and no assured references to individuals. Even about Archytas we know too little to be of much service. In fine, we may be said to have positive knowledge only of a considerable body of mathematics in which Pythagoreans were certainly concerned; and some of these Pythagoreans were earlier than Plato; how much older, we do not know, and one is strongly inclined to infer from the polemical tone of many references, that the questions at issue were, at least for the most part, the subjects of debate in the schools of Plato and of Aristotle and therefore presumably not dating back a century or more.

Though this state of our actual knowledge is rarely, if ever, frankly confessed, it has evidently troubled the more conscientious historians. That is why so much stress is laid on the connection of the Eleatics with the Pythagoreans. Burnet⁴⁰ confesses that the only means of distinguishing between what is earlier and what is later in Pythagoreanism is furnished by the Eleatics. This assumes, what one has tried to prove, that there are in the doctrines and arguments of the Eleatics adequate proofs of the existence of Pythagorean doctrines. If that can be shown, the student does indeed have a sure foundation for his reconstruction of the history of Greek mathematics, though even so it must remain in good part conjectural. Much as this is to be desired, we must not permit our wishes to influence our judgment as to what we may infer from the evidence at our command. On the other hand we may not lightly regard the theses of such eminent scholars as Tannery, Bäumker, and those who have accepted their conclusion. If we cannot agree with them, we may derive a little comfort from the fact that they often confess that their accounts are largely conjectural.

There is much to be said for the view that the Eleatics had relations more or less intimate with Pythagoreans. This need not be disputed, though there remain certain difficulties

⁴⁰ *Greek Philos. Thales to Plato*, pp. 43 f.

that may not be ignored. By way of illustration one may cite Parmenides. One statement⁴¹ has it that he was "converted to peace" not by Xenophanes but by the Pythagorean Ameinias, to whom he erected an heroön after his death. This implies that Parmenides was converted to Pythagoreanism, and one would naturally think of this occurring in his youth. If there is any basis for this statement, the evidence for it, one would suppose, was the monument to Ameinias and the inscription it bore. Alongside this datum, however, we must place another. In the proëm of his work Parmenides himself tells of being conducted by the Sun-maidens from Darkness into Light, even to a goddess who reveals to him the unshakable heart of Truth. If this means anything, it must symbolize another conversion, again presumably in his youth.⁴² This conversion is taken to be a renunciation of Pythagorean dualism. He thus appears to be (rather strangely for a "stabilizer") as volatile as Schelling and an apostate from the faith of the Order. Though this cannot be regarded as in any way conclusive, it inevitably suggests caution regarding a person and a situation about which unfortunately we know far less than we could wish. We need not dwell upon Parmenides, however, because the only aspect of his teaching that even remotely concerns mathematics is that dealing with the One and the Many, which is the theme of Zeno's arguments.

It is really in Zeno that we have to look for reference to Pythagorean mathematics, if it is to be found in the Eleatics. Now it must appear strange, in view of the assurance of many modern scholars, that there is not, so far as I know, a single hint in our sources that the Greeks themselves were aware of the purpose of Zeno to criticize the fundamental doctrines of the Pythagoreans. Of course our historians have a ready answer to this objection to their thesis. Are we not told⁴³ that Zeno wrote *Against the Philosophers* and that he meant to pay off with

⁴¹ Diog. Laërt., IX, 21.

⁴² I should not insist, as Burnet did, on the fact that the goddess addresses him as *κοῦρος*, for that is hardly decisive. But the tone of the poem is so uncompromising that I can think of it only as the work of a young man fond of paradoxes; its crude form also suggests a first attempt. If this view is right it has obvious bearings on the question of Parmenides' relation, for example, to Heraclitus.

⁴³ Suidas, s. v. *Ζήνων* (after Hesychius), and Plato, *Parm.* 128 c.

interest those who ridiculed or travestied the view of his master, Parmenides, that All is One? As for the title of Zeno's book, may one confidently assume that it was not given by the Alexandrians but by Zeno himself? The latter supposition is extremely improbable if we date the work *ca.* 465 B. C. With that assumption falls also the presumption that it was specifically directed against the Pythagoreans as the only "philosophers" at the time prominent in Italy. But why should one think especially of "philosophers?" The paradoxical doctrine of Parmenides that All is One and that motion is impossible must have made him the butt of many a ribald remark. Philoponus says,⁴⁴ "Those who introduce plurality are confident of it because of its self-evidence, for there is a horse and a man, etc." Surely Parmenides, as a man, was not a horse? One imagines that Antisthenes the Cynic was not the first to answer the arguments against motion by getting up and walking away.⁴⁵ That is still the way the man of the world answers philosophers and professors. Indeed, if we are to depend on the titles of books attributed to Zeno, why should we not rather think of Empedocles, on whose work he is reported to have written a commentary? Assuming that the Pythagoreans were the profound philosophers we are given to understand, we should hardly think of them as indulging in the kind of ridicule that Plato's statement implies.

It is clear that nothing reported about the purpose of Zeno's book affords the least presumption in favor of the view that it was directed against the Pythagoreans. If there is any evidence pointing to such a conclusion it must therefore be discovered in the arguments themselves. It would be tedious and useless to review the arguments in detail. They are familiar to every student of Greek thought and their subtlety still exercises the students of logic and mathematics. We need only to direct attention to their general assumptions and the form in which the arguments have been handed down to us. Regarding the latter it is important to observe that we have at most three statements⁴⁶ that can be plausibly regarded as preserving the

⁴⁴ *Phys.* 42, 18, ed. Vitelli.

⁴⁵ Diels, *Vorsokr.*⁵, I, p. 251, 20 ff.

⁴⁶ Diels, *ibid.*, p. 255, 14 prints *εἰ μὴ ἔχου μέγεθος τὸ ὄν, οὐδ' ἂν εἶη* as *ipsissima verba* of Zeno. This is in itself improbable and is further

actual words of Zeno, whereas the variant versions of his arguments themselves prove that we have for the most part to deal with paraphrases dating from later times, from which we can at best infer only the drift of the argument. If this is obvious at first glance, it is emphasized also by Aristotle's reference⁴⁷ to those who urge the antinomies of Zeno and by the certainty that at times he is thinking quite as much of Plato as of the Pythagoreans.⁴⁸

It is sometimes urged that Zeno was attacking an hypothesis. But there is really no reason whatever to single out the Pythagoreans as the proponents of the fundamental hypothesis of all his arguments, to wit, that things are a plurality. Whatever specious considerations may be offered in favor of the supposition that Zeno had Pythagoreans in mind are all due to incidental statements in later authors, who are manifestly *interpreting* and not reporting what he said. So far as we know Zeno did not mention the word "number" at all, though he does *imply* a reference to numbers.⁴⁹ In doing so, moreover, he implies no special conception of number but only such as anyone must have who enumerates objects of any sort. Most of the difficulties he raises are connected with the notion of infinity. What we actually know about the Pythagorean notion of infinity, in relation to number, is *nil*. Taking infinity in the strict sense,⁵⁰ Zeno evidently regards a realized or realizable infinite, that is, a numerable infinite, as a *contradictio in adjecto*. Either things are just as many as they are, in which case they are

indicated as not true by comparing with *προδείξας* the passages p. 257, lines 3 and 6 introduced by *προδείξας* and *δεικνύς*. Except when it is expressly stated that a passage is given *verbatim* it is rarely possible to distinguish between a quotation and a paraphrase, which may be quite free and is in fact very often entirely misleading.

⁴⁷ *Phys.* 263 a 5 ff. It is obvious that no inferences can safely be drawn regarding the original intention of an argument from later applications of it.

⁴⁸ E. g. *Metaph.* 1001 a 29-b 13.

⁴⁹ E. g. *frag.* 3, Diels.

⁵⁰ Zeno is the true Parmenidean in doing so; for much of the significance of Parmenides arises just from the insistence on a single and strict sense in the use of the terms *τὸ εἶναι* and *τὸ ὄν*. It was this tendency, especially emphasized by the Eleatics, that created and promoted dialectic by requiring the distinction and definition of terms.

finite in number, or if they are not, they are not numerable at all, and plurality has no meaning. Parmenides had said that what is is One and at least distinctly implied that it is extended. That statement would naturally provoke criticism; for what has extension must have limits that do not coincide and consequently presuppose an interval between them. This criticism is so obvious that it requires no great mathematician to make it. Zeno recognizes the difficulty ⁵¹ and, as Plato says, pays the critic off with interest. We may imagine him retorting, Yes, there is a difficulty here, but is your assumption of plurality any less obnoxious to objection? You insist that the extended is divisible. Well and good: supposing it to be divisible, it must be divisible *ad infinitum*, for so long as it is extended (and parts of the extended must themselves be extended) there is no limit to division. The only alternative is that the parts shall not have extension, in which case they will be nothing, and no multiplication of them can produce an extended body. The horns of the dilemma are equally fatal, and once one takes the conception of infinity seriously the dilemma must be obvious to any man of intelligence. The difficulty thus posed is logical rather than mathematical, and I, for one, cannot see why we need to look beyond Zeno for the author of the alternatives. If one says he must have been attacking someone who held that reality is composed of indivisible entities, why should one think of contemporary Pythagoreans, rather than of Democritus, Plato, and Xenocrates? If the latter could accept, after the dilemma had been stated, the latter horn, why may not Pythagoreans of a

⁵¹ This is certainly implied in the statement Plato represents Zeno as making (*Parm.* 128 d) that the hypothesis that there is a plurality leads to *still more ludicrous consequences* than the Parmenidean hypothesis that the One only exists. Similarly Plato frankly admitted difficulties in his theory of ideal forms. I think it is unwarranted to say, as is often said, that Zeno's arguments against an indivisible unit (*έν*) do not touch the One of Parmenides, but apply only to an atomic unit, supposed to be Pythagorean; for the One of Parmenides, if extended, as is clearly implied in describing it as continuous, is obnoxious to the same objections as the atom. I cannot otherwise understand the statement attributed to Zeno by Eudemos (*Diels, Vorsokr.*⁵, I, p. 251, 25) *εί τις αύτῷ τὸ έν ἀποδοίη τί ποτέ έστιν, έξειν τὰ έντα λέγειν*, for here τὸ έν must refer to the Parmenidean One. The only way of escape would lie in regarding the One as incorporeal; but neither Parmenides nor Zeno took that way.

later age have done the same? The burden of proof rests with those who contend that Pythagoreans, Zeno's predecessors or contemporaries, held that view. *His* argument cannot take the place of such proof.

As has already been said, the supposed reference to the Pythagoreans finds its only documentary support from later writers. There we find such terms as *monad*, *henad*, *point*, etc., but not in Zeno's own statements. He doubtless spoke, as did Parmenides, of the *one* (ἓν); *monad*⁵² and *henad*, which are not known to occur before Plato's time, are obviously abstract terms, suited to a conception of number directly opposed to the view attributed to the Pythagoreans by Aristotle, who insists that their numbers had magnitude. He does, to be sure, say that they were mathematical, but apparently only because they used numbers in ordinary calculations, as one does in every mathematical operation. Moreover he expressly declares that their numbers were not *monadic*. That, it would seem, should dispose of the supposition that they used the term *monad*.⁵³ As for the contention that Zeno was attacking a view that identified the monad with the point, it is clear that there is really no evidence to support it. Simplicius does indeed twice⁵⁴ quote Eudemus on

⁵²Theo Smyrnaeus, p. 20, 19 says Ἀρχύτας δὲ καὶ Φιλόλαος ἀδιαφόρως τὸ ἐν καὶ μονάδα καλοῦσι, καὶ τὴν μονάδα ἓν. One can well believe this statement, for that brings us down to Plato's time; but why should it be made, if Zeno had already used the terms interchangeably?

⁵³*Metaph.* 1080 b 18 τὸν γὰρ ὅλον οὐρανὸν κατασκευάζουσιν ἐξ ἀριθμῶν, πλὴν οὐ μοναδικῶν, ἀλλὰ τὰς μονάδας ὑπολαμβάνουσιν ἔχειν μέγεθος. From any point of view this statement is curious. I take it that Aristotle in the last clause was falling into current terminology, which consequently signifies nothing. (Alexander, *In Metaph.*, p. 746, 1, ed. Hayd. says μοναδικὸν τὸ ἀμερὲς καὶ ἀσώματον ἐνταῦθα δηλοῖ.) I think, however, that he could not have said that the Pythagoreans' numbers were *not monadic* if he had evidence of their using *μονάς* for ἓν. Cf. *De Caelo* 300 a 14 ff. In *Phys.* 227 a 27 ff. he speaks of those who describe point and monad as separate (κεχωρισμέναι), and says that on this view monad and point cannot be identical. If these were Pythagoreans, we have no means of dating them; more probably they were Platonists. Cf. n. 56.

⁵⁴*Phys.*, p. 97, 13 ff.; p. 138, 32. Aristotle, *Metaph.* 1001 b 7 ff. had given a similar interpretation. If Eudemus had found in Zeno anything to justify the identification he would hardly have contented himself with conjecture. Tannery, *Pour l'histoire de la science hellène*, p. 252, says that Eudemus did not know Zeno's arguments except through tradition. This supposition is difficult to credit, since Simplicius still had the

this matter, but when it is stated that Zeno identified the point with 0 this is expressly given as a conjecture (*ὡς ἔοικε*); in other words, it is an *interpretation* Eudemus offers of Zeno's argument. That an interpretation in terms of mathematics was called for would seem to be strong evidence that Zeno's statement was couched in *logical* terms. After the manner of the commentators Simplicius⁵⁵ gives as a fact what Eudemus conscientiously stated as a conjecture.

Much is made of the supposed identification by the Pythagoreans of the unit (monad), point, and atom. That Pythagoreans at some time may have made this identification need not be denied, though the evidence has not been produced. Those who see criticism of Pythagoreans here regard them as maintaining the contradictory positions (a) that space (body) is extended and therefore divisible, and (b) that division may be halted at a given point, leaving discrete ultimate units, which however are equivalent to geometrical points. It is important to observe, however, that Zeno does not say, or imply, that anyone took both these positions, but regards them as *alternative* possibilities under the general hypothesis of a plurality conceived as parts of an extended whole. The conception of non-extended units (points) is itself an alternative under the head of ultimate units. The argument purports to show that each of the conceived possibilities leads to absurd consequences. How difficult it is to determine the special target of these arguments, suppos-

original text at hand. As the historian of mathematics Eudemus would, it seems, certainly have consulted and carefully read the book, had he believed it dealt specifically with the fundamental concepts of mathematics. Tannery there calls attention to another significant fact—that Eudemus, so far as we know, did not mention Zeno in his history of mathematics, but only in his *Physics*. I can account for this only on the supposition that he, like Aristotle, regarded the Eleatic arguments as essentially logical and as concerned with the fundamental concepts of the physical sciences generally rather than with the particular question of number or geometry.

⁵⁵ *Phys.*, p. 99, 7 ff. Burnet, *E. G. P.*³, p. 315, n. 3 quotes part of this statement as if this were actually a quotation from Eudemus. That is hardly fair dealing. Similarly, *ibid.*, p. 314, he says "Plato (*Parm.* 128 c f.) tells us that the premises of Zeno's arguments were the beliefs of the adversaries of Parmenides." The only premise stated is that *things are many*. That premise was certainly not peculiar to Pythagoreans!

ing that there was one, is shown by the fact that they apply perfectly to the Atomists, whom Aristotle (*De Gen. et Corr.* 324 b 25 ff.) represents as trying to meet the Eleatic logic. Similarly *Phys.* 187 a ff. might well be taken as referring to the Atomists; the ancient commentators, however, thought of Plato and Xenocrates, and Ross, *Metaphysics*, I, p. XC, thinks there is an evident reference to Plato's *Sophist*. The equation of the monad and the point having position is attested only by a quite late writer.⁵⁶ Now it is obvious that Zeno did consider, only to reject, the indivisible unit, most pointedly, perhaps, in the "Arrow," where time and space are each conceived as composed of indivisible units. Why one should think this was Pythagorean doctrine I am unable to discover. To make the supposition plausible one must produce evidence that Pythagoreans, and Pythagoreans of Zeno's time, held such a view of time, space, and motion. In the "Stadium" also we have the same elements, only even more sharply defined; for there space, time, and motion are conceived as composed of indivisible units, each precisely corresponding to each. The refinement of the argument is truly wonderful; but what grounds have we for thinking that Pythagoreans expressly held such a view? If Zeno constructed his subtle argument on the sole basis of an extended (spatial) unit, such as Aristotle supposes their numbers to have been, he was presumably capable of conceiving without help from anyone else a *continuum of any sort* as composed of discrete units, which would be a natural way of regarding plurality. Plato makes Zeno say that his arguments were intended to show that the hypothesis of plurality led to even more absurd conclusions than monism *if one adequately followed it out*. I take it, it was Zeno himself ^{56a} who analyzed the assumption of plurality into its ele-

⁵⁶ Proclus, *In Eucl.*, p. 95, 20. If Aristotle, *Phys.* 227 a 27 ff. and *Metaph.* 1002 a 36 ff. represent Pythagorean doctrine, rather than an interpretation of it by Platonists (cf. Plato, *Parm.* 148 d ff.), there is no way of dating the notion that the point and the monad may be identified. Cf. n. 54. In any case we should have to assume, from the usage of Parmenides and Zeno, that the Pythagoreans of the first half of the fifth century spoke of $\tau\acute{o} \epsilon\nu$ and not of the *monad*.

^{56a} When Plato, *Phaedrus* 261 d, called Zeno the Eleatic Palamedes he obviously had in mind the *inventiveness* of Palamedes celebrated in several dramas. Clearly Diogenes Laërtius, IX, 25, or his source, so understood the matter, for it is coupled with the statement of Aristotle

ments, purely as a logical problem,⁵⁷ presenting the alternatives under which it could be made. I would not deny that one or the other of the possibilities he considers had already been stated by others, for it is of course possible: if one holds it to be a fact, one must produce the evidence for thinking so.

Lest the position of this survey be mistaken, it should be clearly stated that we have no satisfactory evidence for the view that Zeno was attacking a particular theory, that is to say, the Pythagorean. There is no pretense, on the other hand, that there is clear evidence that he was not doing so. It suffices for our purpose to point out that the arguments he advanced were, as Plato implies, the result of a thorough canvass of the implications of plurality considered as referred to a world having the property of extension that Parmenides admitted. There is no express reference to number, or if there is, certainly none to a particular conception of number; for, though the arguments are applicable to number, the analysis seems to have been conducted as a dialectical exercise, noting and drawing the necessary conclusions from the alternative forms the primary assumption of plurality may take. If one contends that there were stated hypotheses of schools opposed to Parmenides, it is fair to ask whether one is to assume that every hypothesis of Plato's

that Zeno was the inventor of dialectic, as Empedocles was of rhetoric. The acknowledged originality of Zeno and the fact that no ancient authority suggests that he was criticizing views of the Pythagoreans create a strong presumption that he alone is responsible for both the form and the presuppositions of his arguments.

⁵⁷ It seems clear that Aristotle so regarded the Eleatic method: *De Gen. et Corr.* 325 a 13 ὑπερβάντες τὴν αἴσθησιν καὶ παριδόντες αὐτὴν ὡς τῷ λόγῳ δεόν ἀκολουθεῖν, cf. *De Caelo*, 298 b 20-23. On the other hand, in his Δόξα Parmenides, according to Aristotle, *Metaph.* 986 b 31 ff., set up two causes and principles, ἀναγκαζόμενος ἀκολουθεῖν τοῖς φαινομένοις, καὶ τὸ ἐν μὲν κατὰ τὸν λόγον, πλείω δὲ κατὰ τὴν αἴσθησιν ὑπολαμβάνων εἶναι. The dialectic of Plato's *Parmenides* clearly presupposes the same purely logical approach. Where numbers are mentioned, e. g. 143 a ff., 149 b, this is done incidentally, and in no way suggests that the subject was of special importance. If it be true, as some contend, that the *Parmenides* is partly a criticism of the atomism of Democritus, it is such only by implication, the logical problem of the 'one' being the essential point. If, as it would appear, Theophrastus did not discuss Zeno in his *Φυσικῶν δόξαι*, he also presumably took the position that the arguments were purely logical.

Parmenides is likewise to be so considered. Is it not possible, indeed highly probable, that in that dialogue Plato was imitating the method of Zeno? If so, is it not fairly arguable that the latter also was himself setting up the hypotheses in order to point out their necessary implications?

It is clear that Zeno or others who repeated and applied his arguments might have used them against the theories of Anaxagoras and the Atomists; for they fit their theories as perfectly as the supposed Pythagorean doctrine of numbers. These philosophies did in fact accept the horns of Zeno's dilemma, Anaxagoras adopting the view that matter may be infinitely divisible without therefore being reduced to 0; Leucippus and Democritus, that matter is ultimately constituted of discrete indestructible particles;⁵⁸ and we must assume that both schools applied their principles to mathematics. We need not now inquire how they met the inevitable problems of continuity and infinity. Both these problems continued to exercise the schools of the fourth century; and it is more than likely that Pythagoreans engaged in the debates. Unfortunately we have no satisfactory evidence for them, more especially about the middle of the fifth century.

One readily understands the motives of those who press the claims of Pythagoreanism and seek by all means to reconstruct its doctrines during the obscure period between the times of

⁵⁸ Aristotle pointed out (*De Caelo*, 303 a 8) that the atoms were quasi-numbers, and he represented the theory of the Atomists as an answer to the Eleatic logic (*De Gen. et Corr.* 324 b 25 ff.). It is notable that the dependence of Atomists on the Pythagoreans, which must be evident if the reconstruction of Burnet is sound, was apparently never thought of (Aristotle, *De An.* 404 a 1 ff., 16 ff. really has no significance, even if the text be sound, which, like Diels, I doubt), just as no one hinted at the supposed Pythagorean doctrine as the target of Zeno's devastating arguments. In fact Burnet, *E. G. P.*³, p. vi, insists that "the vital point" of his argument is his contention that Atomism was derived from Eleaticism. That one ignored even the Atomists, again shows how preoccupied one was with the debates in Platonic circles. On the other hand, if one takes Aristotle's view that the Pythagorean numbers had magnitude, one finds it difficult to understand how it could be asserted (cf. Aëtius, I, 3, 9) of Eephantus (who, if an historical person, must have lived in the fourth century), *τὰς Πυθαγορικὰς μονάδας οὗτος πρῶτος ἀπεφώνητο σωματικὰς*, thus making the (numerical) unit virtually an atom.

Parmenides and Plato. We know that there were members of the school who busied themselves with mathematics and contributed much to the advancement of the science, but except for specific discoveries and their necessary implications we actually know little more; above all, we have no chronological data except the fact that a good deal had been achieved before the time of Aristotle and Eudemos. Even respecting the necessary implications of the specific achievements with which Eudemos credits the Pythagoreans we can confidently affirm no more than that they must have been known to them; for it by no means follows that they *discovered* them.

We are thus brought to a point that has been strangely ignored in the reconstruction of Greek mathematics. We have seen that we have no dependable evidence of mathematical achievements of Pythagoras himself, and we know that in later times one inferred his teachings from opinions held by those who were known as Pythagoreans. The best informed Greeks did not regard him or his followers as the creators of Greek mathematics, but thought of them as being active in promoting the science. Eudemos credited Thales with a knowledge of some fundamental geometrical problems: since he did so, as we gather from one instance, by inference from practical achievements traditionally attributed to the sage of Miletus, we may refuse to accept his conclusion; but it is not without significance that the first general historian of mathematics found no difficulty in assuming such knowledge on the part of an Ionian earlier than Pythagoras. We know, moreover, that Anaximander also framed a picture of the cosmos that was essentially geometrical and in principle not unlike that involved in the later Pythagorean theory of the "harmony of the spheres." Indeed, the engineering feat of Eupalinus in constructing the tunnel of Samos implies certain definite geometrical propositions. We thus know that mathematics was cultivated in Ionia before the time of Pythagoras, who left his native Samos about the time the tunnel was built. One readily surmises that Pythagoras had learned some of its rudiments before he went to Italy, whether he and his earliest associates did or did not devote themselves to the study.

There is, in fact, much to be said for the view that mathematics was intensively cultivated by the Ionians from the sixth

century onward. Aside from the geometrical pattern of the cosmos, Anaximander and his successor Hecataeus evidently applied similar principles in the construction of their maps of the earth, and later Ionians applied the same methods in laying out cities. Plato evidently had these schemes in mind, perhaps consciously combining them with cosmological patterns, in describing the capital city of the Atlantians. This aspect of Ionian research should not in the least surprise us when we reflect that almost all the pre-Socratic thinkers, who laid the foundations of Greek science in all fields, were Ionians. But we are not restricted to general considerations and probabilities. Whereas we cannot name a single Pythagorean before Archytas who made a notable contribution to mathematics, we have considerable evidence regarding others whom it will repay one to consider briefly, without attempting to appraise their several merits.

We have referred to Thales, Anaximander, Hecataeus, and Hippodamus of Miletus. Agatharchus of Samos is mentioned⁵⁹ as a scene-painter for Aeschylus in a way to suggest that he was interested in the problem of perspective, which was taken up and advanced by Anaxagoras of Clazomenae and Democritus of Abdera. We have every reason to think that the latter two contributed principles of fundamental importance to the solution of geometrical problems. Oenopides, the astronomer, and Hippiocrates, the author of the earliest known hand-book of geometry, were natives of Chios, and Leodamas of Thasos is mentioned as a contemporary of Plato and Archytas among those who contributed to the improvement of geometry.⁶⁰ One notes with interest that all these were Ionians. One cannot pass over Hippias of Elis, who not only concerned himself with astronomy, but attempted the solution of difficult geometrical problems and touched on the history of mathematics. Though he was presumably not an Ionian (we know nothing of his antecedents), he shared all the interests of the Ionians and in character resembled them rather than the Pythagoreans. Meton was an Athenian, as was Theaetetus. The latter, as a pupil of Theodorus of Cyrene,

⁵⁹ Vitruvius, *Praef.* 7.

⁶⁰ Proclus, *In Eucl.*, prol. II, p. 66, 14, ἐν δὲ τούτῳ τῷ χρόνῳ (i. e. Plato's) καὶ Λεωδάμας ὁ Θάσιος ἦν καὶ Ἀρχύτας ὁ Ταραντίνος καὶ Θεαίτητος ὁ Ἀθηναῖος, παρ' ὧν ἐπηυξήθη τὰ θεωρήματα καὶ προήλθεν εἰς ἐπιστημονικωτέραν σύστασιν.

has of course been thought to belong to the Pythagorean line, though we have no good reason to think of Theodorus as connected with the school. Iamblichus, in his list of Pythagoreans,⁶¹ mentions a Theodorus of Tarentum, but he may quite well have been a different person. In his commentary on Euclid's *Elements* Proclus⁶² gives a list of precursors of Euclid in the composition of geometrical hand-books, each surpassing his predecessor in the number of propositions and the excellence of the demonstrations. Going backward from Euclid he names Hermotimus of Colophon, Theudius of Magnesia, Leon the pupil of Neoclides, and Hippocrates of Chios. Again, where we know anything about the men he names, we are faced with a group of Ionians.

This is certainly a remarkable showing, which it is difficult to understand except on the supposition of a continuous tradition of strong interest in geometry among the Ionians from early times. As against this indisputable evidence it appears reckless to suggest that we owe the entire development of mathematics to the Pythagoreans and to assume that all the necessary implications of the specific achievements credited to Pythagoreans by Eudemus constitute "Pythagorean geometry." Even Iamblichus, who was inclined to claim nearly everything for that school, lends no support to such pretensions, for he says⁶³ that when the mathematical secrets of the Order had been divulged by Hippasus, two men, Theodorus of Cyrene and Hippocrates of Chios did most to advance the science. If we disregard the discredited story of Hippasus and the secret teachings of Pythagoro-

⁶¹ *Vit. Pyth.*, 265.

⁶² P. 67, ed. Friedlein.

⁶³ Diels, *Vorsokr.*⁵, I, p. 108, 10 ff. Aristotle, *Meteor.* 342 b 29 ff., compares and contrasts the views of certain "Italians" (Pythagoreans) and Hippocrates of Chios regarding the comet. The passage decides nothing about the question of their relations. Erich Frank, *Plato und die sogen. Pythagoreer*, p. 233, holds that Aristotle meant to set Hippocrates apart from the Pythagoreans; Loria, *Scienze Esatte*, p. 74, thinks he classed them together. Such instances of partial agreement, with differences in detail, seem to me natural where there is a common interest in a problem. In order to decide whether one thinker depended on the other we should have to know more than we do, especially regarding their chronology. By the end of the fifth century, it seems, many minds were contributing to a more or less common stock of knowledge and opinion.

ras, we have here a confession that neither of these celebrated men belonged to the Order.

If one is to believe that the Ionians above mentioned learned their mathematics from the Pythagoreans one must make some extremely improbable assumptions; for the connection of the individual philosophers and scientists with Pythagoreanism, though often asserted, cannot generally be accepted as based on anything better than the same wishful thinking that inspires some historians of the present day. Iamblichus, to be sure, furnished a long list of "Pythagoreans" assigned to various cities, including Ionian Paros, Cyzicus, and Samos, but Melissus is the only person otherwise known, and he could be connected with Pythagoreanism only through the apostate Parmenides. We do not even know when the Order, originally at home in Italy, was scattered. Zeller thought it could not have been before the middle of the fifth century; and what dependable sources tell us of such representatives as Lysis, Philolaus, and the Pythagoreans who associated with Socrates rather suggests that their interests lay in other directions.

The conclusion to which we are driven by our study is that it is impossible to reconstruct the history of Greek mathematics, as one may to a certain extent tell the story of the development of Greek scientific thought in general, by focusing attention upon individual men or groups. Regarding our knowledge of details and also with respect to the necessary inferences from known facts nothing is changed; but the rôle of the Pythagoreans must appear to have been much exaggerated. If we are to exercise our imagination in order to supplement our knowledge it would seem that we must reckon with the probability of a continuous mathematical tradition in Ionian lands from an early date. Supposing that to be true, the question arises how the achievements of individuals and groups were communicated, so that it became possible from time to time to sum up and integrate the whole. To that question, which arises in other fields of Greek thought also, there is no satisfactory answer.

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