

GALILEO AND THE THEOREM OF PYTHAGORAS

By STANISLAUS QUAN*

A summary of conclusions reached so far on the 'point', 'line', 'side', 'figure', and 'circle'. The aim of this paper.

In my paper, *Galileo and the Problem of Infinity—II* (*Ann Sci.*, 1972, 28, 274-281), certain conclusions were reached on the significance of the geometrical 'point' (i) as the beginning and the end of any straight line segment, (ii) as a single mark, made with a pencil, that comes-to-be two things at the same time, in the division of a straight line, and (iii) in geometrical figures of three or more sides where, again, it has this twofold character, in that while it divides the end of one side from the beginning of the next side, it also brings the sides into unity, and so provides the continuity we find in the perimeter of the 'figure'. In each case, we are involved in indivisibles.¹ Now, what I have said about a 'point' has general application to lines, sides, and planes (surfaces). These belong to the world of things (*res naturae*), and we can say with Aristotle that points, lines, and planes are "all alike either limits or divisions" (*Ar. Metaphysics*: 1002b 10).² As these conclusions—and in particular the significance of the point and line as divisions—have considerable implications, hitherto completely overlooked, not only in Galilean geometrical demonstrations, but in the whole field of Euclidean Geometry and in the development of Mathematics, I now want to summarise and briefly enlarge upon my findings on the 'point', 'line', 'side', 'figure', and 'circle', and so be in a position to make clear the purpose of the present paper.

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¹ For the 'indivisible', see *Ann. Sci.*, 1972, 28, pp. 261, 245, 272 (footnote), and 275. T. L. Heath (*Euclid's Elements*: Dover, vol. i, p. 156) misunderstood the Aristotelian 'indivisible'. Hence, he failed to grasp the significance of the 'point' as a division, and so the grounds for Aristotle's rejection of Plato's 'point' (as the extremity of a line) as unscientific.

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It will help to prevent confusion in the mind of the reader if I state here at the start that, *in the first place*, in what follows I want to deal with points and lines as *limits*, *divisions*, and *indivisibles* in their own right, and then, *in the second place*, and as an altogether different issue, with the geometrical problems connected with the indivisibility of lines when they come to form the 'sides' of geometrical figures, and so with the indivisibility of geometrical figures when they are parts of other figures.

POINTS—as limits, divisions, and indivisibles.

I said that points, lines, and planes (surfaces) belong to the sphere of things (*res naturae*). What makes them obscure to us as *limits* is that their very existence is a consequence of the existence of something else, but they have no more actuality *as limits* than the end, say, of a walk has. They are separable in thought, but not in fact. They are not real to us as substances are—that is, we cannot perceive them, as we perceive the individual things which, in our experience, 'come-to-be and pass-away' in the world of Nature (Ar. *Metaphysics*: 1002a 31 to 1002b 10; 1060b 5–15; 1090b 5. For the Aristotelian 'substance', see *Ann. Sci.*, 1972, 28, 282 ff). It is better, therefore, to approach an explanation from what is more readily perceptible to sense, and we can borrow our illustration from one familiar to Aristotle—a glass, or some such container, filled with water (Ar. *Physics*: 208b 2; 209b 24–30; 211a 30 to 212a 20).

Now, the interior surface of the glass is in contact with the exterior surface of the contained water. Moreover, this contact is neither part of the glass nor part of the water, since we can pour out the water and separate the container and the contained at any time. Where the interior surface of the glass and the exterior surface of the contained water are *in contact* is the *limit*. This limit has come to be two things at the same time. It is the limit of the surface of the glass and the limit of the contiguous surface of the water. As and while this limit is such, it is indivisible. If we pour the water out, then air takes its place and we then have a fresh limit between the container and the contained air, for the "extremities of what contains and of what is contained are coincident" (Ar. *Physics*: IV. 211b 10).³

³ In one way, the 'limit' can be a third thing. This depends on our thinking. We can think of the 'limit' (i) *of the water*, or (ii) *of the container*, or, of neither the one nor the other, but simply (iii) *of where the two meet*. If he had understood Aristotle on 'place' here, Descartes could not have confused 'body', 'extension', 'internal place', and 'space' in the way he did (Descartes *Works*, Cambridge, vol. i, p. 259). But, neither did he understand 'combination' and the Aristotelian 'indivisible' which, like everyone else, he took to be what (because of its smallness) was beyond the power of any creature to divide further (*ibid.*, Principle XX, p. 264). Hence, he resorts to Pope Urban VIII's irrational method of silencing Galileo, and finds some explanation of the paradox of division *ad infinitum* in God's omnipotency (*ibid.*). See Galileo's *Two Chief World Systems* by Stillman Drake, 1962, p. 464 and note, p. 500.

Again, let us take a wooden post 4×4 inches square and some 6 feet tall, one of whose faces is the boundary between property A and property B. This face, as a limit or boundary, cannot have any extension within the wooden post on the side of property A, otherwise the face would not be a limit or boundary; yet, at the same time, this face is also the boundary of property B, for boundaries belong to the bound (Ar. *Physics*: IV. 220a 24). The face is thus two things at the same time, and is indivisible.

Let us apply this to any given straight line segment. When we put pencil to paper and proceed to draw a line, it is the movement of the pencil that brings the beginning (and so one 'limit') of the line into existence, just as the line itself is made continuous by the continuous movement of the pencil, and just as the end of the line is made when the movement comes to a halt. If we rub the line out, then the ends disappear at the same time. We can no more rub away the 'end' of a line, as if it were a separable individual thing, than we can snip off the 'end' of a piece of string. We merely succeed in making fresh ends. Let us now look at the straight line, Fig. 1 below:

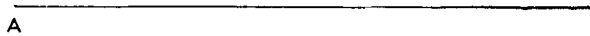


Fig. 1

As the beginning of the line, A is a limit, and it cannot have any extension within the line, otherwise it would not be a limit. But—and this is what is usually forgotten—A is also the limit *where the white surface of the paper, at the junction with the line, is no longer perceptible*. Which of the two we take to be limited depends upon which has our attention at any time. In either case, the limit is a limit of two things at the same time, and as such is indivisible. We imagine that we actually perceive the 'limit' or end, but in truth we infer the end of the white surface from the beginning of the black line, just as we infer the beginning of the black line from the discontinuation of the white surface.

As a *division*, a 'point' presents less difficulty because its twofold character becomes obvious to sense-perception. In the straight line AB, below, let C mark any point of division taken at random.

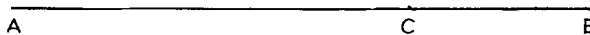


Fig. 2

It is clear that C is the end of AC, and also the beginning of CB. That is why we repeat the use of the C in each instance. Hence, as a division, the point C is two things at the same time, and so indivisible. This same result is also produced by the act of reckoning a division, as well as by the act of dividing the line (Ar. *Physics*: VIII. 263a 24). We can, therefore, say this: As a 'limit' and as a 'division', a point (i) derives its existence from the existence of something else, (ii) it has position—that position being derived from the position of the object to which, as a limit, it belongs, (iii) it serves two functions at the same time, and (iv) it is indivisible.

A LINE—as a limit, a division, and an indivisible.

Just as geometrical 'points' are limits and divisions of geometrical lines, so geometrical lines are limits and divisions of geometrical figures. In the rectangle ABCD below—ignoring the presence of EF, for the moment—the pair of opposite lines AC and BD are the limits of the rectangle in one direction, just as the pair of opposites AB and CD are limits of the same figure in the other direction.

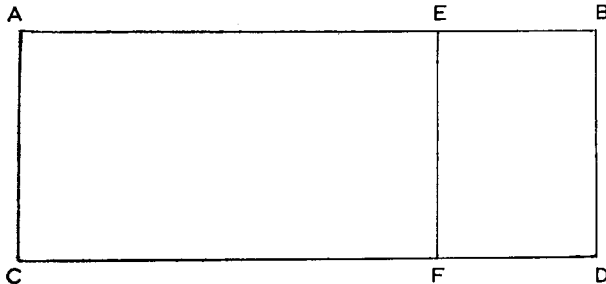


Fig. 3

If we now include the line EF (parallel to AC and BD), then EF divides the rectangle ABCD, and, just as the 'point', *as a division*, was seen to be two things at the same time, so here the line EF comes-to-be two things at the same time, since it is the limit on the one side of the figure AEFC, just as it is also, at the same time, the limit (and a beginning) of the figure EFDB. The line EF, as a division, is a unit and is indivisible (Ar. *Physics*: III. 207b 5; V. 227a 20–30).

This brings me to a consideration of the first importance. Because EF is a division of the rectangular figure ABCD, and because EF is inseparable and indivisibly part of the figures AEFC and EFDB *at the*

same time, then it is quite erroneous for anyone to pretend that the rectangular figures AEFC and EFDB are separate, individual rectangles *in their own right*. They are not. They are PARTS of the original WHOLE rectangle ABCD. If the side EF belongs elsewhere, then so does the figure it helps to form. The figures AEFC and EFDB are not like separable, individual bricks forming a wall. The line EF, *as a division*, creates a kind of combination that is inseparable and indivisible, and so makes the two figures *parts* of a *whole*. For the same reason (Fig. 4 below), it is erroneous for anyone to pretend that a square whose sides are, say, 6 inches, consists of 36 separate, individual squares, each having sides of 1 inch. The 36 *contained* figures are not separable squares at all, but inseparable PARTS of a WHOLE, and—as will become clear when we turn from ‘lines’ to ‘sides’ below later—the sides of these 36 squares are not of 1 inch either. Incidentally, this falsifies the argument in the geometrical passage in the *Meno* 82b-85b (Plato, vol. i, Oxford), and the Platonic theory of reminiscence.

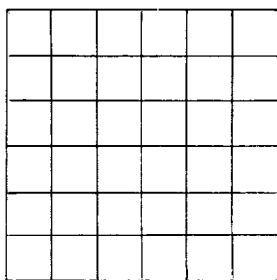


Fig. 4

This confusion of ‘part’ with ‘whole’ makes it possible now to call in question proposition after proposition in what has come down to us as Euclid’s *Elements*. Eudoxus and Archimedes made the same mistake in their mis-conceived attempts to exhaust the circle with their inscribed polygons—whether the polygons are circumscribed makes no difference—and all they have succeeded in demonstrating is that the very process of their thinking puts them among the Democritean Atomists, and nothing else. Later, when our investigations take us further to a consideration of Euclid XII. 2, to the Theory of Number, and lastly to the so-called rigour of the Deductive Method in Mathematics, it will become only too evident that this confusion of ‘part’ and ‘whole’, and ‘whole’ with ‘part’, has run rife throughout the development of mathematics since early Greek times. Let me now turn from the ‘line’ as an *indivisible*

division (in figures) to an altogether different problem—that of the indivisibility of any given straight line segment.

The indivisibility of a straight line segment.

A straight line segment is potentially divisible anywhere, but, if we attempt to divide it, as we do with a pencil to mark the division, we convert a potential divisible into an actual indivisible, and we merely succeed in demonstrating the indivisibility of a straight line (*Ann. Sci.*, 1972, **28**, 279 ff). The reason is this—the segment was made continuous, in the first place, by our uninterrupted and continuous movement from, say, A to B. When we make a fresh movement to mark the division on the line, we interrupt this fresh movement by stopping, to make the mark, and then by re-starting, *from the same mark*. The mark we have made, though numerically one, is theoretically several things at the same time—the end of a movement and of one ‘part’ of the line, and the beginning of a movement and of the next ‘part’ of the line, and so indivisible. This, again, is clear when, for instance, we make a journey from A to B, and decide to stop at C on the way. By the very act of stopping, we divide the movement, and so we divide the time taken, and so the distance to be covered, but, we do not (save potentially) divide the ground under our feet. In the same way, the ‘point’, *as a division*, marks a potential, not the actual cut in the line. This is true even if we *reckon* the movement and the division *in our thoughts*. Hence it is impossible to divide any straight line segment anywhere once, or into halves, let alone *ad infinitum*, because the very dividing mark, though numerically one, is theoretically two things, and so indivisible (Ar. *Physics*: VIII. 262a 21–263b 2; IV. 220a 9).⁴ This is equally true of angles.

If what has been said about the division of a line is true, then no ‘part’ of a straight line segment can be commensurate with another ‘part’, or with the line as a whole, any more than a whole line can be

⁴ There is a unification here similar to that in compounds and combinations found throughout Nature, though it has escaped the notice of geometers. As I have said (*Ann. Sci.*, 1972, **28**, 280), we can ignore the line, *qua line*, and divide the line, *qua matter*; but, even here, division must come to a halt at the constituent ‘parts’ of the compound. ‘Parts’ in this sense find no place whatever in Euclid’s *Elements*. This indivisibility of a straight line segment makes Dedekind’s so-called Postulate groundless and indefensible, and its application (by Killing) to elementary geometry erroneous, though Killing’s position throughout is quite untenable on its own account. See: Heath, *Elements* (Dover), vol. i, pp. 234–240, and Boyer, *Calculus* (Dover), 291. This will receive attention in a subsequent paper. The discussion in Heath (*ibid.*), on the Principle of Continuity, with special reference to Dedekind’s Postulate, makes the *continuous* purely geometrical, and composed of discretess. No account is taken of the *continuous* which has its origin in Nature, in the movement at work in organic and inorganic *combination*—the doctrine on which is central not only to Aristotle’s definitive criticism of the Atomism of Democritus and Leukippus, but also to an understanding of his *Physics*, and so to an understanding of his philosophy as a whole (*Ann. Sci.*, 1972, **28**, 247 ff).

commensurate with any of its 'parts'. The line we make on paper is a *quantum* (Ar. *Metaphysics*: 1020a 15). After all, we drew it, and we perceive it as a stretch. If the straight line segment is a *quantum*, then, the 'point' we make, as a dividing mark, must also be a *quantum*, since it belongs to the whole line. Now, if we attempt to measure the two 'parts' of the straight line separately, with a ruler, we find that we cannot do so, because the very 'point' of division belongs, indivisibly, to both 'parts'. The 'point', as a division, is thus an unknown, unknowable quantity, and hence the 'parts' of the segment, *qua parts*, are incapable of measurement. If 'part' of a line is incapable of separate measurement, then the 'part' cannot be commensurate with the segment as a whole, any more than the whole segment can be commensurate with a 'part', or with any of its 'parts'. The 'point' as a *division*, and so indivisible, must undermine the whole theory of proportion in Euclid's Elements V. and VI.

THE 'LINE' AS A 'SIDE' IN GEOMETRICAL FIGURES.

Now, what has been said about the twofold character of the mark we make with a pencil, in the division of a straight line, becomes still more significant in the formation of geometrical figures. It has been said that, when a geometer draws his inferences, the size of the figure in the diagram is irrelevant (Ar. *Anal. Pr.* I. 49b 35; *Anal. Post.* I. 76b 40). While this is true, it is also true that it is quite impossible for any geometer, or for anyone else, *even in thought*, to make a geometrical figure—be it triangle, square, or polygon of any number of sides—without combining and unifying the sides, if there is to be continuity in the perimeter; and since as a consequence, the end of one side in the figure becomes the beginning of the next side, we are inevitably involved once again in an indivisible, and in an unknown quantity. In the triangle, for instance, the continuity of the three sides is brought about by three junctions, and each side is involved in two—one at each end. If we attempt to measure any one side of the triangle, we find that a part of one side, at either end, belongs to the next side as well. It is thus impossible to measure any one side. How then can one side in a geometrical figure be commensurate with any of the others? However, it is possible for us to ignore the side of the triangle *qua side*—as all people unconsciously do—and then to measure the 'side' *qua line*. We will certainly get a measurement, but it will not be, and cannot be the measurement of the 'side' *qua side*. To convince ourselves of this, all we need do is to take a ruler and attempt to measure each 'side' of the figure round in order, when it will be found that each end comes to be included twice. If the 'sides' of any triangle drawn at random defy measurement, then so do the 'sides' of all geometrical figures. In every instance, we are involved in two things—

with a 'side' that cannot be measured, and with a 'side' that is incommensurable with the other 'sides' of the figure.⁵

A GEOMETRICAL 'SIDE' ALWAYS INVOLVES A SHORTENED 'LINE'.

It will be recalled (*Ann. Sci.*, 1968, **24**, 316; 1972, **28**, 277) when we investigated Galileo's favourite proposition about concentric hexagons, that when the larger makes one complete turn, the smaller stamps out (as it were) each of its six sides separately, and in order. In other words, by the device of concentric hexagons we can reconvert 'sides' back again to 'lines'. When we do, we find that the breaking up of the continuity in the original perimeter changes each 'side' into a corresponding longer 'line'. For this reason, every time 'lines' become the 'sides' of geometrical figures, there is a *loss of length* at each end, and we are involved in an unknown quantitative measurement. We can, in theory, roll our concentric hexagons back again to their original position. If we do, then the separate 'lines' come together again to form 'sides', and so the continuous perimeter of the hexagon. We can do precisely the same with concentric triangles, concentric squares, and so with any geometrical figure of three or more sides. In every instance, each 'side' involves two junctions, two indivisibles, and two unknown quantities.

GEOMETRICAL FIGURES WITHIN FIGURES.

Again, what is true of the external 'sides' of geometrical figures is equally true of the 'sides' of figures inside other figures. Here, the situation is further complicated because (i) the 'sides' of the inside figures are no longer limits only, but *divisions as well*, and so indivisibly *part* of all the other figures, and of the original *whole* figure, and (ii) each 'side' of every inside figure has two junctions, and so two unknown quantities, and so cannot be measured. We were right, therefore, when

⁵ There is a real problem here, but it will have to wait until I come to deal with the Unit and the Theory of Number. We must have a Unit of measure, and this unit must be 'homogeneous with the thing measured' (*Ar. Metaphysics*: 1053a 25). This choice is ours, but, having chosen it, we cannot divide it up, as we mistakenly divide our 'foot' length up into 12 inches, or the inch into tenths, eighths, etc. That is impossible (See *Ann. Sci.*, 1972, **28**, 273 and 276). However, we can use our Unit to measure aggregates, and our measure will be accurate because, as I have already said, the limits of things in contact are coincident. But, any aggregate of such units will be contiguous. The unit, being indivisible, measures similar indivisibles, and these as 'many' cannot as 'parts' combine to form a 'one', i.e. a new unit—a fact recognised by Democritus as being equally true of his indivisible magnitudes (*Ar. Metaphysics*: 1039a 10). Lines as units cannot be continuous or form the perimeter we need to make a 'figure'. The truth is that we *make* a 'line' continuous simply by drawing it, and it is this that creates the problem of division, and that arising from the junction of geometrical 'sides'.

we said (Fig. 4 and text) that a square, whose 'sides' *qua* 'lines' measure 6 inches, does not contain 36 separate and individual squares. It is equally true to say that the 'sides' of this square, *qua* 'sides', are not 6 inches either, anymore than the individual 'sides' of the inside 36 squares are of 1 inch. In fact, we have no right to call these inside squares *squares*. They are figures, and *potential, indivisible parts* of the original square, even though we cannot be sure just what the 'sides' of this square measure.

It is as impossible to divide a figure into halves, or into actual 'parts', with the 'line' *as a division*, as it is impossible to divide a straight line segment into halves, or into actual 'parts', with the 'point' *as a division*. In each case, we convert a potential divisible into an actual indivisible. Hence, Leibniz' definition (see *Elements*, Dover, p. 169)—*A straight line is one which divides a plane into two halves identical in all but position*—must be rejected for a more fundamental reason than that given by T. L. Heath. Again, the formula for the area of a triangle cannot be $\frac{1}{2}$ base $\times$ height because, while it may be possible to measure height independently of the junctions of the 'sides', we cannot halve a base, or any one 'side', since any such division involves three indivisibles, and so three unknown quantities—one at each end of the 'side' or base, and one at the 'point' of division. For the same reason, as will become still more clear below, Euclid's Definition 17 (of a diameter) in the *Elements* (Dover), p. 185, must be false: *A diameter of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.*

Is it not true that we have here in geometrical figures a situation analogous to that organic and inorganic combination which, in his rejection of the Atomism of Democritus and Leucippus, Aristotle found in Nature, and which is central to an understanding of his physical treatises, as it is to an understanding of his philosophy as a whole (*Ann. Sci.*, 1972, 28, 246 ff)? When metal fuses with metal, when flesh is grafted on to flesh, and wood to wood, how is it possible to measure the precise extent of the combination, the unification that has taken place? When three separate 'lines' are joined and converted into 'sides' for the formation of a geometrical figure, how *can* we measure just how much each line loses to the next line to bring about the continuity of all three sides in the triangle?

THE CIRCLE—its indivisibility.

What has been said so far about the twofold character of points and lines as *limits* and *divisions* has direct application to the figure of one side we call a 'circle'. Let me summarise.

In the first place, I have shown (*Ann. Sci.*, 1970, **26**, 133; 1972, **28**, 274) that Galileo, and all geometers alike since, are in error in assuming that there is a one-to-one correspondence between the infinite number of points assumed to exist on the circumference of a circle and on the circumference of any other circle, whatever the size, concentric with it. They are equally in error in assuming that when a circle makes one complete turn it traces out a straight line equal to its circumference. It does not (*Ar. Physics*: VII. 248a 12–248b 5). The straight line traced out is slightly longer, for the reason already given in the case of the hexagon. As a consequence, modern methods of measuring mileages based on this principle (as in contemporary motorcars) are no more accurate than the old trundle wheel used by the early makers of road maps for measuring out distances.

In the second place, it will be recalled (*Ann. Sci.*, 1970, **26**, 131) that it was on the assumption (i) that a circle is a polygon 'having an infinitude of sides', and (ii) that a circle, in making one complete turn, traces out a straight line equal to its circumference, that Galileo was able to provide himself with two straight lines of equal length, and to prove that the one consisted of an 'infinite number of points that completely fill it', while the other (of the same length) consisted of an infinite number of points alternating with an infinite number of 'empty spaces' or vacua. To the contrary I showed (i) that Galileo's circumference cannot consist of an 'infinite number of points' if—as he says they are—these 'points' are at the same time 'sides', because a 'side' always has a 'middle' (i.e. what is between the two extremities), whereas a 'point' has no middle or between. Moreover, (ii) Galileo's first straight line cannot consist of an 'infinite number of points which completely fill it'—as he says—since, if they do, then the points must be *in succession* along the line. If the points are in succession, then they must occupy space and, if they do, then they cannot be points. Again (iii) if Galileo's points are in succession along the line, and if they do occupy space, then they cannot be infinite in number, for, the resulting straight line must then be infinite in length,—a self-contradiction. Again, (iv) in the case of Galileo's second straight line (of the same length as the other), if this consists, as he claims, of an infinite number of points alternating with an infinite number of 'empty spaces' or vacua, then he is again guilty of a self-contradiction. For, as I have already pointed out, just as two boundary posts can have but one space between them, so two points can have only one space between them. Hence, Galileo's infinite number of points along the line must have an infinite number of alternating spaces *minus one* between them,—and that is nonsense. The truth is, as was recognised by Aristotle, that anyone who makes a line consist of 'points' is bound to make Time consist of 'moments', 'instants', or

'nows'. Without being aware of it, he is a Democritean Atomist, and his thinking must be in conflict with the facts of organic and inorganic combination, clearly distinguishable throughout Nature (Ar. *Physics*: VI. 241a 1-5; *De Caelo*: III. 304a 25; and *Ann. Sci.*, 1972, **28**, 247). Just as the 'line' is what is between 'points', so Time is what is between 'moments' or 'nows' (Ar. *Physics*: IV. 220a 5-20; VIII. 263b 24), and the 'points' and 'nows' in each case are limits and indivisibles.

In the third place, I come to two principles which neither Galileo, nor any other geometer since, has taken into his calculations or into his geometrical demonstrations. And first:

A straight line segment and a circumference are not commensurable.

It has already been shown (*Ann. Sci.*, 1972, **28**, 274) that whatever the size of the circle, the straight line which Galileo assumed to be traced out in one complete turn is not equal to the circumference. It is longer. By how much it is longer, we have no means of knowing because the unbroken continuity of a circumference converts any potential dividing mark into an actual indivisible, and so makes the length of the circumference incapable of measurement. Moreover, if, as Galileo assumed, it is possible to bend a straight line into a circle (*Ann. Sci.*, *ibid.*, p. 278) then, if the two ends are to be unified and so made continuous, as the circumference of a circle, the line must suffer a shortening. Hence, just as it is impossible for any circumference whatever to be equal to any straight line, so it is impossible for any straight line segment to be equal to a circumference.

Now, if a circumference cannot equal a straight line, and if a straight line cannot equal a circumference, then it is impossible for either a circumference or a straight line to be *greater than*, or *less than* one another. The circumference of a circle and a straight line are incommensurable (Ar. *Physics*: VII. 248a 10-248b 7). Here, as will become clear in my next paper, lies the clue to the fallacy inherent in the Euclidean method of proof based on the *reductio ad absurdum*, as used for instance in propositions X.I, and XII. 2. It is simply not true to assume that there are only three alternatives, and that one thing must be either *equal to*, or *less than*, or *greater than* another thing. There is a fourth alternative. The things can be incommensurable, and this alters the whole situation. Aristotle grasped the vital distinction, and the reason for it, yet it appears to have escaped the attention of the Pythagoreans, and of Eudoxus, Euclid, and Archimedes—and, we must add, the attention of all geometers and mathematicians alike since. What Aristotle did was to define accurately the 'equal', and how it is opposed to 'the great

and the small' (*Metaphysics*: 1055b 30–1056a 25), and the meaning of 'commensurable' and 'incommensurable' (*Physics*: 248a 10). The proof by *reductio ad absurdum* is effected by what is in fact a bad syllogism (*Ar. Anal. Post.*, 87a).⁶

Again, if it is true that the circumference of a circle, as a whole, cannot be equal to, or greater than, or less than any straight line, then it is equally true for the same reasons that it is impossible for any segment of a circumference to be equal to, or greater than, or less than any straight line segment, just as it is impossible for any straight line to be equal to, or less than, or greater than any circumference. A circumference and a straight line are incommensurable, just as the segment of a circumference and any straight line are incommensurable.

Two corollaries follow: The diameter of a circle cannot be commensurate with its circumference, or the circumference be commensurate with its diameter, anymore than a radius can be commensurate with a diameter, or with its circumference. The two extremities of any diameter are indivisibly 'part' of the circumference, and so involved in what is quantitatively unknowable and incapable of measurement. In the case of the radius, the situation is more complicated. At one extremity, each radius is indivisibly 'part' of the circumference, but, at the centre, each radius is indivisibly 'part' of all the other radii, and so (indirectly) indivisibly 'part' of the circumference, and so equally involved in what is quantitatively unknowable and incapable of measurement. It is clearly impossible, therefore, for there to be any commensurability between a radius and a diameter, between a diameter and a circumference, or between a radius and a circumference. Hence, it is not true to say that "circles are to one another as the squares on their diameters", or that there is any commensurability between a radius and the area of a circle. We are involved here, again, in indivisibles, in unknown and unknowable factors, and such information as we can get about the area of a circle can only be an approximation and conjectural. I now come to my second principle, and it is this:

Since lines, and so the 'sides' of geometrical figures, are specifically different from the units with which Number deals, they are incommensurable with

⁶ The situation is not altered in the least because it is said that one thing, such as a circle, can be seen to be longer than another, such as a line. A flag-pole is seen to be longer than a cigar, an aeroplane is seen to be faster than a bicycle, just as an elephant is seen to be bigger than a mouse. But, these things are specifically different and incommensurable. To be commensurable, there must be no specific difference in things—either in what contains the attribute or in the attribute, and the attribute must be applicable to both things without equivocation.

Number and with algebraical symbols, and hence they cannot be subject to the operations of arithmetic or algebra. (Ar. *Topics*: VI.13.150a 21–25).

Here, the very nature of the development of mathematics makes it necessary to say something about the influence of Descartes. According to E. T. Bell,⁷ after the period of Archimedes, Euclid, and Apollonius, that of Descartes, Fermat, Newton, and Leibniz is the second great age of mathematics (*ibid.*, p. 135), and the half-century from 1637 to 1687 is universally recognised as the fountainhead of modern mathematics because the first date marks the publication of Descartes' *Geometry*, and the second date that of Newton's *Principia* (*ibid.*, p. 131). It was Descartes who reduced geometry to algebra and analysis (*ibid.*, p. 139), and reduced all geometry to a universal method (*ibid.*, p. 149).

The important question is this: What precisely did Descartes do? The answer may well lead us to suggest that the dreams he had on the fatal night of November 10th 1619 (*ibid.*, p. 138), when he received visionary insight into his philosophy and analytic geometry were to be less a Pentecost of reasoning, or of mathematical reasoning, than a disaster, and for a reason that Etienne Gilson never came in sight of.⁸

In our search for certainty and truth, the least deviation from the right track comes to be multiplied later a thousandfold, for the further we proceed, the more we lose sight of our beginnings, and so of the source of our error, and the reason is (Ar. *De Caelo*: 271b 5) that 'a principle is great rather in power than in extent; hence that which was small at the start turns out a giant at the end'. Let us now look at the principles at the very opening and foundation of Descartes' *Geometry* (Dover), pp. 2–6:

'Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction. Just as arithmetic consists of only four or five operations, namely, addition, subtraction, multiplication, division, and the extraction of roots . . . so in geometry, to find required lines it is merely necessary to add or subtract other lines; or else, taking one line which I shall call unity in order to relate it as closely as possible to numbers, and which can in general be chosen arbitrarily, and having given two other lines, to find a fourth line which shall be to one of the given lines as the other is to unity (which is the same as multiplication) . . . And I shall not hesitate to introduce these arithmetical terms into geometry for the sake of clearness'.

⁷ *The Development of Mathematics* by E. T. Bell: McGraw-Hill, second edition, 1945. To be referred to as: Bell, *Mathematics* (McGraw-Hill).

⁸ *The Unity of Philosophical Experience* by Etienne Gilson: Sheed & Ward, 1938, p. 138.

Having illustrated how lines are to be added to, or subtracted from other lines, Descartes then continues:

‘Often it is not necessary thus to draw the lines on paper, but it is sufficient to designate each by a single letter. Thus, to add the lines BD and GH, I call the one a and the other b , and write $a+b$. Then $a-b$ will indicate that b is subtracted from a ; ab that a is multiplied by b ; a/b that a is divided by b ; aa or a^2 that a is multiplied by itself . . .’

So Descartes here has taken two steps of the gravest consequence.

First, he has taken ‘lines’, and so the ‘sides’ of geometrical figures and converted them into numbers, and he has assumed that because it is possible to say of numbers that ‘5 taken-away from 7 leaves 2’, so it is possible to say that a ‘part’ (the 5) of a line can be taken-away or subtracted from the ‘whole’ line (the 7), and leave a remainder, i.e. the ‘part’ represented by the number 2. Similarly, he has asked us to believe that just as it is possible to say that ‘3 added-to 5 makes 8’, so one ‘line’ can be added-to another ‘line’ with a like result. In other words, Descartes’ ‘lines’ are atomic, and the operation of adding simply makes an aggregate of units.

But, what is true of numbers is impossible for ‘lines’ and ‘sides’ in geometry. If one ‘line’ is to be added-to another, then the ‘lines’ must be either contiguous or continuous. If they are contiguous, then it is clearly impossible for them to form the ‘sides’ of geometrical figures. If the ‘lines’ are unified and made continuous, then they must become ‘sides’, and suffer a shortening at the junction. There is thus in geometry a kind of ‘combination’ analogous to that found throughout Nature. Again, if a mark is to be made on a straight line to suggest the ‘part’ that is to be taken-away or subtracted from the ‘whole’ line, then the mark of division immediately becomes an indivisible, and any such subtraction is impossible. Now, as multiplication and the extraction of roots (even on Descartes’ assumption) is a form of addition and subtraction, then the upshot of this conversion of geometry into arithmetic must necessarily be a source of error in the results obtained. This error cannot be ignored, simply because (as it is) it is small. It is still an error.

Without being aware of what he was doing, in this very act of arithmetising geometry, Descartes was repeating, concealing further, and then helping to perpetuate the very error we have found in Greek geometry, in its confusion of ‘part’ and ‘whole’. When Descartes converted his ‘lines’ and ‘sides’ into numbers, and then proceeded to add, subtract, divide, and multiply as in the operations of arithmetic, then ‘parts’, *qua parts*, were eliminated in the process, and so in his thinking, and came to be treated as ‘wholes’. In his attitude to numbers, Descartes shows

himself to be a Pythagorean; in his identification of 'part' and 'whole' he is a Democritean Atomist.⁹ But what Descartes misguidedly did in his *Geometry* of 1637, Galileo was to do in the *Two New Sciences* of 1638, where the fallacious argument about squares, roots, and cubes—with its further confusion of 'part' and 'whole'—led Galileo to draw an erroneous distinction between finite and infinite classes that later, in the nineteenth century, came to be postulated by the philosopher and theologian Bolzano, and then developed by the medievalist Cantor in his theory of sets of points (*Ann. Sci.*, 1970, **26**, 143 ff; and E. T. Bell: *Mathematics*, pp. 92, 273, 154).

Second: The second step that Descartes took at the beginning of his *Geometry* was to have still more serious consequences, for, having failed to understand the difference between 'lines' and 'sides', and having converted his geometrical 'lines', and so 'sides', into numbers, he then took the fatal step of converting his arithmetic into algebra. The first two mistakes came to be more thoroughly concealed in his third.

When this conversion of geometry into algebra, and of numbers into algebraical symbols, came to be developed in all directions, and then, in the form of applied mathematics, extended to the fields of production and technology, the ramifications of error became incalculable. As I have said before, the fact that the error involved at the start is small (as it is) is no reason for any refusal to take it into account. It is still an error. Can we be surprised that, in spite of all the extraordinary developments in mathematics, scientists still find it absolutely essential in all important productions from motorcars to aeroplanes to build 'prototypes' which are then subjected to the final tests of trial and error and experience?

Now, in drawing the attention of geometers and mathematicians to these things, I am not suggesting in any way how the problems connected with them ought to be met. That is the job of the mathematician. What I am saying is that the best way of solving any problem is to know the grounds giving rise to it in the first place. For centuries, Aristotle has

⁹ In his philosophy, Descartes betrays no knowledge at all of the Aristotelian indivisible, or of the continuous, or of the infinite, or of the importance of the doctrine of organic and inorganic combination to an understanding of the *Physics*. But, there is an error of more fundamental importance in all Descartes' thinking, as a mathematician. In falling into this error, he is in distinguished company for, like all mathematical philosophers since, and like all philosophers in the western philosophic tradition, Descartes failed to make the radical distinction between the 'one' *the unit of measure*, and the 'one' that is the starting-point of all number. Now, this was precisely the mistake made by the Neo-Platonist Porphyry, and prompted him to ask the fatal question which started Boethius, and then the medievals on the fruitless and misguided search for the meaning of the 'universal'. (See Etienne Gilson, *opus cit.* Ch. 1, and *Selections from Medieval Philosophers* by Richard McKeon: New York, vol. i, pp. 67 and 91 (Boethius).) Aristotle's solution of this problem will be dealt with in a subsequent paper on the Unit and the Theory of Number.

been criticised for what Platonist philosophers and mathematicians have too readily assumed to be his shortcomings as a mathematician. Yet, he is the source of the facts which have now been brought to light. The time has come when, in the light of Aristotle's teaching, geometers and mathematicians must take a fresh look *first*, at what has come down to us as the *Elements* of Euclid, and *second*, at the validity of the Cartesian conversion of geometry into arithmetic and then into algebra. Just as arithmetic was, erroneously, to transform the geometrical continuous into undifferentiated units (numbers), so the symbols of algebra were to complete and conceal the error by transforming undifferentiated units into undifferentiated generalisations, and so into the undifferentiated relations of symbolic logic.

This brings me to the purpose of this paper. It so happens that in one of his key geometrical demonstrations in defence of his atomic theory, and of his solution of the problem of division *ad infinitum*, Galileo conducts his proof in two stages by means of two important propositions in the *Elements*. The one proposition (Euclid I. 47)—the famous theorem of Pythagoras—is now recognised as one of the bases of all metrics. The other (Euclid XII. 2)—that circles are to one another as the square on their diameters—is equally important because of its connection with Eudoxus, Archimedes, the theory of exhaustion, and the development of the Calculus. Now, in my original refutation of Galileo's demonstration (*Ann. Sci.*, 1970, **26**, 136), there was no need for me to go beyond Galileo's preliminary explanation. Here, I propose to take his geometrical proof, refute its use of Euclid I. 47 and then, independently, raise the question whether this theorem of Pythagoras can continue to be regarded as valid. In my next paper, I then propose to take a fresh look at Euclid XII. 2 and at what has come to be regarded as the Axiom of Archimedes. In this way, Galileo affords us a convenient point of departure for what, at other hands, must lead to a full investigation into the validity of the whole of the *Elements*. Let me now turn to Galileo's geometrical demonstration and to the Theorem of Pythagoras.

Galileo's geometrical demonstration in the Two New Sciences, p. 27, and the *Theorem of Pythagoras*.

Of this famous theorem, E. T. Bell, *Mathematics*, says:

'Regarding the Pythagorean theorem itself, whoever first guessed it, we recall it is the cornerstone of Euclidean metric geometry and one of the bases of all metrics. It too, like similar triangles, threads all mathematical history, not only in geometry, but also in algebra, the theory of numbers, and mathematical physics' (p. 41).¹⁰

¹⁰ Cp. Descartes: *Geometry* (Dover), p. 10, footnote where in a letter to the Princess Elizabeth of Palatine, Descartes says that in the solution of a geometrical problem, he reduced the question to such terms as made it depend on these two theorems.

Again:

‘The Pythagorean theorem that $x^2 + y^2 = z^2$, where x, y, z are sides of a right triangle, is the basis of metric geometry in Euclidean space’ (p. 69).

Earlier, when explaining what in mathematics is strictly regarded as ‘proof’, E. T. Bell points out that the proof in the *Elements*

‘is vitiated by tacit assumptions that Euclid ignored in laying down the postulates from which he undertook to deduce the theorems in his geometry In Euclid’s day, and for centuries thereafter, the attempted proof of the Pythagorean proposition satisfied all the current requirements of logical and mathematical rigor. A sound proof today does not differ greatly in outward appearance from Euclid’s; but if we inspect the postulates required to validate the proof, we notice several which Euclid overlooked’ (p. 9).

I refer to this at the start because I am not concerned with the ‘proof’ of the Pythagorean proposition, either with that given by Euclid or by anyone else, but rather with the *very validity of the proposition* in the first place, and then with the significance of the place this theorem occupies in mathematics, as one of ‘the bases for all metrics’. I want to suggest that mathematicians should now take a fresh look at this theorem because it is possible to show that its use as a measure is bound to include a margin of error—the greater the distances involved, the greater the inaccuracy, and so the greater the error.

It will be recalled (*T.N.S.*, p. 27, and *Ann. Sci.*, 1970, **26**, 136 ff) that Galileo, through his mouthpiece Salviati, set out to prove that a single point is equal to a line, and to counter any feeling of ‘wonder’ occasioned by this apparent contradiction of sense-perception, he promises to go still further and to demonstrate a ‘miracle’, namely, how a solid cylinder with a bowl-like interior can be made to disappear into the circumference of a circle, while at the same time a solid cone, on the same base, can be equally made to disappear into a ‘point’. He then reaches this conclusion:

‘Hence in conformity with the preceeding we may say that all circumferences of circles, however different, are equal to each other, and are each equal to a single point’ (*T.N.S.*, p.29).

This ‘miracle’ is demonstrated in two stages: *first*, Galileo gives a descriptive explanation how two equal solids disappear into two equals—the one into a circumference, and the other into a ‘point’; and *in the second place*, he proves this geometrically by Euclid I. 47 and XII. 2. Here I want to begin with a summary clarification of Galileo’s descriptive explanation, before refuting the geometrical demonstration based on the theorem itself. The figures below should be self-explanatory.

First: Galileo takes the following geometrical figure—a rectangle made up of two squares having the common side CF. With C as centre, the half-circle AFB is drawn.

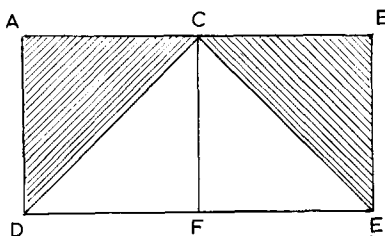


Fig. 5

Second: He then eliminates the shaded portions and so gives himself the figure below

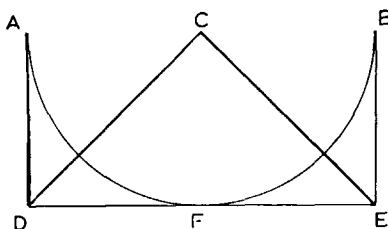


Fig. 6

Third: He then assumes that the whole revolves round the common axis CF with this result:

- (i) the triangle DCE becomes a *solid cone*.
- (ii) the bowl-like shape ADFEB becomes a *solid cylindrical bowl*.
- (iii) since the base line is common in the original figures, then, the circular base of the solid cylindrical bowl is now equal to the circular base of the solid cone. This gives him two solids and two surfaces.

We then get the following:

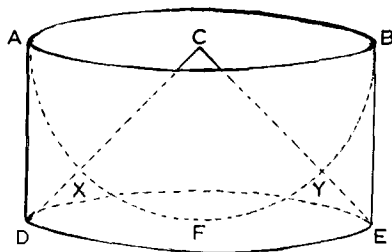


Fig. 7

Fourth: Galileo then—in brief—removes planes of his common base upwards (i.e. the circular surface represented by the ellipse DE, having the centre F), until he is left with (i) the circumference of the circle represented by the ellipse AB (having the centre C), and (ii) the point C (i.e. the ultimate apex-point of the solid cone).

He can then argue that his two original solids have vanished into two equals i.e. the circumference = the centre point C. Before moving on to Galileo's geometrical proof involving the use of Euclid I. 47 and XII. 2, I must make clear how he has already deceived the unsuspecting reader.

First, at the very start Galileo assumes the equality between his solid cylinder and solid cone, of the same height and on the same base; but, there is nothing in Euclid to support this. What we find in the *Elements* XII. 10 is this: 'Any cone is a third of the cylinder which has the same base with it and equal height' (*Elements*: Cambridge, 1926, vol. iii, p. 400. See also the claim of Archimedes in the Historical Note, p. 365). What, therefore, is the use of Galileo's elaborate explanation, or of his so-called 'proof' that follows?

Second, here the reader should refer to Fig. 9 below, and to Fig. 10. Now, it is true that the cylinder and the cone begin by sharing the same base, but, immediately the first plane is removed, this is no longer true. The base of the cone shrinks in diameter concomitantly as the sides DC and EC of the cone become shorter and approach the apex C.

Third, moreover, as this happens, then—since the base of the cone is no longer part of the base of the cylinder—as far as the base of the cylinder is concerned, a hollow must appear round the centre, and this hollow must increase in diameter from the centre F, upwards, along the interior curved surface of the 'bowl', as far as X and Y (Fig. 7).

Fourth, worse still, when, after the removal of successive planes, we reach the points X and Y—where the sides of the cone no longer have any contact with the interior bowl-like surface of the cylinder—a space is bound to occur between the base of the cone and the corresponding plane of the cylinder, and since AXC and CYB is *all space*, then this circular ribbon of space is bound to increase in size until we reach the circumference of the cylinder, and the point C of the cone. If the reader will refer to Figs. 9 and 10 (below) the situation will become more clear. Whatever way we look at this demonstration, it is seen to be riddled with inaccuracies and deception. Why has this sort of thing escaped detection for so long? Let us now see how Galileo attempts

to prove his case by means of Euclid I. 47 and XII. 2. Sagredo is made to introduce the geometrical proof thus:

‘ But for our complete satisfaction pray give us this geometrical proof that there is always equality between these solids and between their bases; for it cannot, I think, fail to be very ingenious, seeing how subtle is the philosophical argument based on the result ’ (*T.N.S.*, p. 29).

We must have Galileo’s own figure before us here.

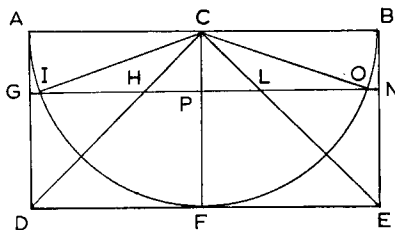


Fig. 8

Here is Galileo’s proof:

‘ *SALV.* The demonstration is both short and easy. Referring to the preceding figure, since $\angle IPC$ is a right angle the square of the radius IC is equal to the sum of the squares on the two sides IP , PC ; but the radius IC is equal to AC and also to GP , while CP is equal to PH . Hence the square of the line GP is equal to the sum of the squares of IP and PH , or multiplying through by 4, we have the square of the diameter GN equal to the sum of the squares on IO and HL . And, since the areas of circles are to each other as the squares of their diameters, it follows that the area of the circle whose diameter is GN is equal to the sum of the areas of circles having diameters IO and HL , so that if we remove the common area of the circle having IO for diameter the remaining area of the circle GN will be equal to the area of the circle whose diameter is HL . So much for the first part ’ (*T.N.S.*, p. 29).

There is so much confused thinking and bad geometry in the fifteen lines of this proof, that any attempt on my part to track the thought-sequence would merely create further confusion in the mind of the reader. The refutation is the important thing here, and this must be done quickly and clearly. What I want to do, therefore, is this: (i) I want to make clear just what Galileo is trying to do. My diagram below should make the situation clear. (ii) I want to summarise briefly the two stages in the proof, and so reach the conclusion, and (iii) I then want to pin-point the basic fallacy in the reasoning.

At the very start, it should be borne in mind that while Galileo directs the reader’s attention to the diagram (i.e. to a plane geometrical figure with its lines, squares, right-angled triangles, diameters etc.),

Galileo's own thinking and conclusion relates to solids—a solid cylinder and a solid cone, assumed to occupy the same base.¹¹

First, what precisely is Galileo trying to do?

Here, a diagram will help to make the situation clearer.

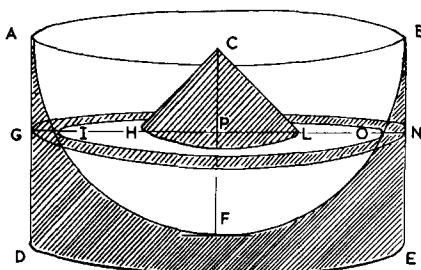


Fig. 9

In the above figure, I have shaded the three areas that play an important part in the proof: (i) the solid cone, (ii) the solid bowl-like interior of the cylinder, and (iii) the plane of solid ring, represented by the width GI, on the left, and the width ON on the right. At the start, the cylinder and the cone do occupy the same base.

What Galileo wants to demonstrate is that this equality continues—as the planes are removed—all the way up to the circumference represented by the ellipse AB with centre C.

He takes a plane straight through—represented by the straight line GHIPLON, and we then have the following:

- (i) the circular base of the cone, with diameter HL.
- (ii) round this circular base of the small cone, there is a circular ring of empty space, represented by the width IH, on the left, and by the width LO, on the right.
- (iii) outside this circular empty space is a second circular ring, but, this is solid, since it is part of the interior bowl-like surface of the cylinder.

Galileo now sets out to prove that the area of the base of the small cone (of diameter HL) is equal to the area of the solid circular ring (having width GI, on the left, and ON on the right).

From this, he thinks he is able to infer that, since the base of the cylinder and that of the cone are the same at the start, and since the plane GHIPLON is *any one* of the planes removed from the base upwards,

¹¹ Other instances like this of Galileo's mental confusion have already been referred to. See *Ann. Sci.*, 1968, **24**, 322 and 328 ff; 1970, **26**, 140 ff; and 1972, **28**, 257 ff.

then, if the area of the base of the small cone can be proved to be equal to the area of the plane of the solid circular ring of width GI, then this must be true likewise of all other planes until we reach the circumference (of the cylinder) = point C (the apex). So much for what Galileo is trying to prove.

Second, here I want to summarise the two stages in the geometrical demonstration, and so reach Galileo's conclusion as quickly as possible, unencumbered by other details of less consequence at the moment. The demonstration falls into two parts.

In part 1, by means of Euclid I. 47, Galileo finds (see the left-hand side of Fig. 8)

The square on GP = the square on IP + the square on HP

Then, since this is equally true of lines PN, PL, and PO, he can then say:

The square on GN = the square on IO + the square on HL

In part 2, Galileo now moves to the second stage, and so to Euclid XII. 2—that circles are to one another as the squares on the diameters. He can then argue thus:

The circle on diameter GN = the circle on diameter IO + circle on diameter HL.

Next, on the axiom that if equals are taken from equals the remainders are equal, he now removes the circle on diameter IO from each side of the equation. He is then left with

circle GN *minus* circle IO = circle HL *minus* circle IO

But, the circle of diameter HL = the base of the small solid cone; While the solid circular ring of width GI represents what is left of the base of the cylinder (after the planes have been removed to this level). Galileo has, therefore, *as he thinks*, proved what he set out to prove. Where then is the fallacy in the situation?

Third, the fallacy will become obvious if we look once more at Galileo's last equation. Here it is:

Circle on diameter GN = the Circle on diameter IO + Circle on
diameter HL

Now, it is possible to take a PART from a WHOLE, and leave a Part. Hence, it is possible to take circle on diameter GN (the WHOLE), and remove the circle on IO (the PART), for, *on the left side of the equation*, we are then left with the plane of solid circular ring of width GI. But, it is manifestly impossible for anyone to take a WHOLE, and then expect to leave a PART. This is precisely what Galileo has done here, *on the*

right side of his equation. The figure below will make this abundantly clear.

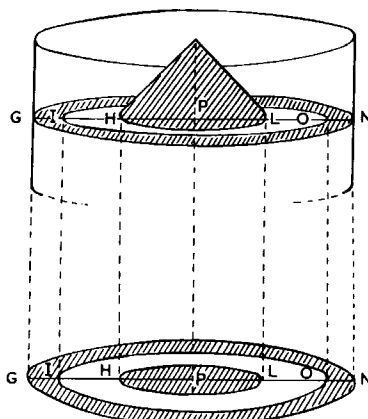


Fig. 10

The ground plan above makes clear the three elements in the equation, namely: The circle of diameter HL = the base of the small cone.

Outside this lies,

A circular ribbon of space (unshaded) of width IH (or LO).

Outside this again, lies,

the circular solid ring of width GI (or of ON).

Now, let us concentrate *on the right-hand side* of Galileo's equation: Circle on diameter GN = circle on diameter IO + circle on HL. It is now clear that the circle on HL is already PART of circle on IO; Hence when Galileo removes the circle of diameter IO from the right-hand side of the equation, he *ipso facto* removes the circle of diameter HL as well. When he takes the WHOLE (circle IO), he also takes the PART (circle HL = the base of the small cone), and he is left with nothing. His equation then is:

The circular solid ring of width GI = 0

This makes nonsense of the geometrical demonstration, and the use of Euclid I. 47 and XII. 2.

This identification of 'part' and 'whole' repeatedly occurs in the *Two New Sciences* (*Ann. Sci.*, 1970, **26**, 133 ff; 1972, **28**, 265 and 283). In his theory about the difference between the finite and the infinite, Galileo could see nothing odd when he found a one-to-one correspondence between a whole class and a sub-class of the whole. Instead of questioning the mis-reasoning that led to such illogicalities as could equate 'part'

and 'whole', he preferred to trust his mathematical conclusions and what he took to be the rigour of his geometrical demonstrations, just as Bolzano and Cantor—following in his footsteps—were to do later. Need there be any cause for surprise to discover that, in the twentieth century, mathematicians—who likewise preferred to question their logic, rather than the validity of their own mathematics—found it necessary to modify the indirect method (*reductio ad absurdum*), to counter objections to current reasoning about infinite classes.¹² Later, when I come to deal with the theory of Number, it will be found that this same fallacy—this confusion of 'part' with 'whole'—has bedevilled the development of mathematics since early Greek times, and, until this is recognised and put right, mathematicians must expect to have their mathematics strewn with paradoxes and with such self-contradiction as 'A is equal to B, and A is not equal to B' (*ibid.*, p. 185).

So far, I have refuted the geometrical demonstration involving Galileo's use of Euclid I. 47 and XII. 2. But, what immediately follows Galileo's demonstration? We find that he leaves off any further demonstration, and so does not prove 'that there is always equality between these solids and between their bases' (*T.N.S.*, p. 29). Instead, the reader is invited to read *De Centro Gravitatis Solidorum* by 'the Archimedes of our age, Luca Valerio, who made use of it for a different object' (*T.N.S.*, p. 30). So Galileo has not only failed to prove what the demonstration was supposed to prove, he has clearly shown now that what he did do is a *non-sequitur*.

In refuting Galileo's position here, I concentrated on the major fallacy, namely the inference leading to the conclusion, because I wanted to avoid details likely to create confusion. I must, however, draw attention now to other errors in the argument, and in the use of Euclid I. 47.

In his preliminary descriptive explanation (*T.N.S.*, p. 28) Galileo assumes that the figures AIG and BON in his diagram (Fig. 8) are triangles. They are not. AI and BO are not straight line segments but rather segments of the semi-circle AFB.¹³ But, the root source of Galileo's errors here is his unexpressed assumption that Euclid I. 47, and what is said about geometrical figures, is concomitantly applicable to and true of what the figures in the diagram represent, namely, a

¹² See E. T. Bell: *Mathematics*, pp. 57, 83, 154, 272–281. Galileo's misreasoning on the infinite has been acclaimed like a message from a new John the Baptist heralding the dawn of a new branch of mathematics. For the refutation of Galileo's position here, see *Ann. Sci.*, 1970, **26**, 143 ff.

¹³ See T. L. Heath: *Elements* (Dover), vol. ii, pp. 39–42, for the 16th century controversies on the angles between circumferences of circles touching one another, internally or externally, and the angles made by 'the contact of a straight line with a circle'. Since Galileo held the same opinion as Vieta here, it is surprising that the point in question is passed over in silence.

solid cylinder and a solid cone, on the same base and of the same height. To appreciate the precise nature of the mistake here, the reader should refer to Galileo's figure (Fig. 8), and to my figures 9 and 10. The following then becomes clear:

- (i) Because Galileo is arguing geometrically, and because his use of the theorem of Pythagoras is made to refer directly to his own diagram, he takes GIHPLON as if it is a continuous straight line from the external part of the cylinder (G) to the opposite side (at N).

In actual fact, this is not so. Reference to my figure on p. 23, and to the ground plan with it, shows that this line has two empty spaces on it. Hence, GN, properly represented, become this:



Fig. 11

No valid demonstration about solids and the areas of solids can be based on an argument about lines. This is to pass from plane to solid geometry, and to suppose that *what is measured* makes no difference to a measurement. We are here involved in incommensurables.

- (ii) This equally applies to the right-angled triangle ICP, where IC represents a length across empty space, and PC is the assumed height of the solid cone. And, as a consequence, this must be equally true of the right-angled triangles OPC and LPC.
- (iii) Again: Does it make no difference to the demonstration that the line IC (the hypotenuse of the right-angled triangle IPC) is said to be equal to the line AC (since they are radii of the semi-circle), and yet, at the same time, equal to GP, when the line GP really consists of GI (measure of solid) + IH (measure of empty space) + HP (measure of half the diameter of the solid cone)? In any case, HP cannot be half the diameter because any division of a line can be only potential, never actual. A line, *qua line* or *qua side*, cannot be divided at all.
- (iv) Again: Galileo argues as if HCL is a triangle. It is not. It is a solid cone with a rounded superficies, and this cannot be ignored in the measurements.

But, there is one consideration that falsifies the whole demonstration because it involves every figure and every line Galileo argues about. The end of every side of every geometrical figure involves indivisibles which, in every measurement, must be included twice. It is impossible, therefore, to measure the 'side' of any geometrical figure. In Galileo's

figure, the points I, H, P, L, and O in the assumed straight line GN are all indivisibles, and since each point involves a duplication, then it is impossible for these points to mark any actual division of the line in question.

It must be obvious now that this fact has never been taken into consideration in geometry. I can find no hint, even, of any reference to anything like it in Euclid's *Elements*, yet, it is something of the utmost importance. When it is realised that an original linear measure, such as 6 cm may, in the first place, be replaced by letters, such as AB, and then replaced by a single letter A, the very possibility of error beggars description. As I have already pointed out in the case of Descartes, what we have here is one error being concealed by another.

Let us now turn from Galileo's geometrical demonstration involving the use of the Pythagorean theorem to the theorem itself. How valid is it? How much longer can it continue to be used in any metric system?

The analysis of Euclid's proof of the Theorem of Pythagoras.

As this analysis involves much detail, the reader should have the following three facts clearly in mind at the start, before accompanying me in the tracking process:

1. Euclid takes no account of 'lines' as indivisible *divisions*. Hence, 'parts' come to be confused with, and treated as 'wholes'.

2. As a consequence, Euclid's original right-angled triangle comes to be eliminated (as a separate triangle, *in its own right*), and he is left with the ILLUSION OF ITS SHAPE, at the centre, created by the positioning of the three squares. In Fig. 12 (below), it will be seen that, if the three squares are linked, we then have the SHAPE and so the (illusory) existence of the right-angled triangle created at the centre of the figure, by the base lines of the three squares.

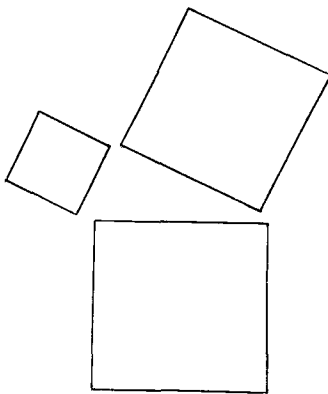


Fig. 12

3. The 'proof' will then be seen to hinge on a mental shift, *first* from a consideration of the original right-angled triangle (as a separate triangle in its own right), *second*, to the three squares (as separate squares, in their own right), and finally, *third*, back again to the triangle for the statement of the conclusion of the proof.

The whole so-called 'proof' thus shows itself to be a rather beautiful paradigm of geometrical self-deception. Let us now track the proof.

First: Euclid gives himself a right-angled triangle, such as ABC below, and so a separate, individual 'whole', *in its own right*.

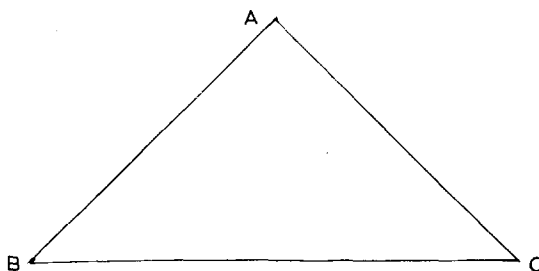


Fig. 13

Second: On each side of this triangle, Euclid now puts a square. The result is that (without being aware of it) he has now made a completely new composite 'figure', namely, DBFGAHKCE, i.e. a new 'whole', consisting of four 'parts'. See Fig. 14 below and Heath: *Elements* (Dover), vol. i, p. 349.

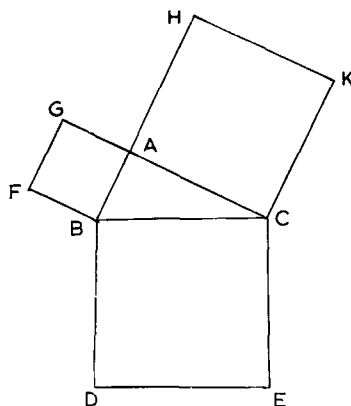


Fig. 14

Note: These four ‘parts’ of Euclid’s new composite ‘whole’ are *figures*. They do not constitute, separately, a triangle and three squares, in their own right, because AB, AC, and BC are not ‘sides’ but *divisions*, and so they indivisibly belong to *all four figures at the same time*. In other words, we have an exact parallel here between the ‘point’ and the ‘line’ as divisions (see p. 6 above). The ‘lines’ AB, AC, and BC divide *potentially* but, actually, they indivisibly join all four ‘figures’ together.

Third: The important step now is to examine (i) the statement of Euclid’s aim, and then (ii) the conclusion of his proof, because these give us the clue to the error in his thinking. Here they are:

- (i) ‘I say that the square on BC is equal to the squares on BA and AC’, and
- (ii) ‘Therefore the square on the side BC is equal to the squares on the sides BA, AC’ (*ibid.*, p. 348–349).

In other words, all through the ‘proof’, Euclid is assuming that BA, AC, and BC are ‘sides’—in the first place of his triangle, and then of his squares; but, in truth, they are no longer ‘sides’, but rather *divisions* of his figures, and so indivisibles. In the same way, Euclid takes BFGA, AHKC, and BCED as separate, individual squares, in their own right. They are not. The bases BA, AC, and BC are *divisions*, and so indivisibles. It is because of this mistake that Euclid can conclude—the emphasis is mine—‘the square *on the side BC* is equal to the squares *on the sides BA, and AC*’.

The consequences of this mistake are very important.

Fourth: So, what WAS, at the start, the hypotenuse of the original right-angled triangle ABC; what THEN—without Euclid’s being aware of it—came to be converted into the *division* BC, has now (by a similar erroneous mental process) been FURTHER CONVERTED into a separate base BC, of the biggest square. As a consequence, the hypotenuse has been removed from the original right-angled triangle altogether. In the same way, what WERE the remaining two ‘sides’ of the original triangle, what WERE unconsciously converted into indivisible *divisions*, have now been converted into the bases AB and AC of the other two squares.

Fifth: As a consequence of this failure to distinguish ‘parts’ from ‘wholes’, and ‘lines’ from ‘sides’ and *divisions*, Euclid has eliminated the right-angled figure, *qua right-angled figure*, at the centre altogether. He is then left with the illusion of its

SHAPE—and this is what deceived him—created by the positioning of the three squares.

Sixth: Euclid now directs his thinking to what he assumes to be three separate squares, in their own right, namely, BCDE, BFGA, and AHKC. His aim now is to prove that square BCDE is equal to the sum of the two squares BFGA and AHKC.

Now, it does not make the slightest difference to our analysis whether Euclid's so-called 'proof' is valid or not. As a matter of fact, his 'proof' is even worse than it appears, for, to effect his proof (see Heath *ibid.*, p. 349, and Fig. 13 above), he joins F to C, B to K, A to E, and A to another point L on the side DE. In other words, he has now made a new composite 'figure' (a 'whole') consisting of 18 'parts'! What is important to us here is that his thinking leads him to establish that the square BCDE is equal to the sum of the other two squares BFGA and AHKC. That is why, near the end of his 'proof', he returns to his starting-point and confirms his inference, by saying—the emphasis is mine—'The square BDEC is described on BC, and the squares GB, HC on BA, AC.' (*ibid.*, p. 350). This brings us to Euclid's last move and so to his conclusion.

Seventh: By this very act of saying—'The square BDEC is described on BC, and the squares GB, HC on BA, AC'—Euclid has made the mental leap that reconverts AB, AC, and BC back again into the 'sides' of his original right-angled triangle, so that it is then possible to make the inference that is his conclusion—'Therefore the square on the side BC is equal to the squares on the sides BA, AC' (*ibid.*, p. 350).

So, in order to make his argument about squares apply to his right-angled triangle, Euclid repeats—this time, in reverse—the mistake he made at the start. But, Euclid cannot theorise about triangles or about squares until he has triangles and squares. Here, he has neither. What he does have is a 'figure', composed of indivisible figures. The so-called 'proof' of the Theorem of Pythagoras is thus seen to be the result of mental confusion caused by a failure to distinguish, *first*, 'parts' from 'wholes', and *second*, 'lines' from 'sides', and 'lines' *qua* lines from 'lines' *qua* divisions. The real blame lies at the door of those who have preferred to heed the effusive outpourings of commentators, like the Neo-Platonist Proclus, when they would have been better employed in trying to understand Aristotle.

Euclid's proposition applies to right-angled triangles in general. However, it has been assumed (*ibid.*, p. 352–361) that while there are

certain combinations of sides in a right-angled triangle that are incommensurable, there are other combinations of sides that are commensurable. I now propose to show that the commensurability of these sides is an appearance only, and that the sides, *qua sides*, of all right-angled triangles, without exception, are incommensurable. Let us take (*ibid.*, p. 355) 'the first *rational* right-angled triangle discovered', namely that of 3, 4, and 5 units though, as will become clear, it is impossible to have sides even of these units.

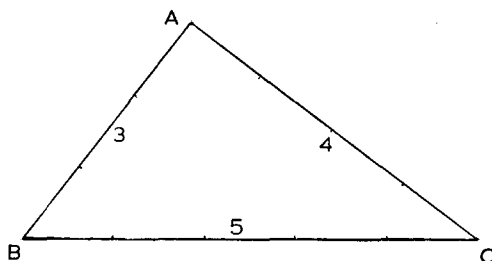


Fig. 15

To all appearances, in geometry, we can say of this right-angled triangle that $AB^2 + AC^2 = BC^2$. If we need any further supporting evidence, then we have it from two sources. In arithmetic, by numbers, we can say $3^2 + 4^2 = 5^2$, and so $9 + 16 = 25$. On the other hand, in geometry it has been assumed that we can divide the whole square on the hypotenuse into 25 small squares, with sides of a unit, just as we can divide the other two squares into 9 small squares and 16 small squares, again with sides of the same unit. It can then be argued that the 25 small squares (in the one) = the 25 small squares (in the other two).

However, when we take a ruler and attempt to measure each side of the right-angled triangle round in order, we find that we cannot avoid including the end of each side twice in the measurement. This also happens whether we attempt to measure the sides of the three larger squares, or the sides of any of the small squares. Hence, the sides of the right-angled triangle cannot be of 3, 4 and 5 units, as we imagined at the start (see p. 8 above). They must be smaller. By how much smaller, we have no means of knowing. The sides are thus incommensurable. This applies equally to the sides of all the squares. Hence, if the 25 squares, in the square on the hypotenuse, *appear* to be equal to the sum of the small squares in the other two larger squares, then, this cannot follow from the sides, *qua sides*, but only from the sides, *qua numbered*, according to the assumed stated lengths of 3, 4, and 5 units. Again,

squares on the sides of any right-angled triangle whatever become indivisible 'parts' of a new composite 'whole', and are incommensurable. Sides of all geometrical figures involve indivisibles, so do figures that are 'parts'.

Now, in what has been said so far about the right triangle, we have made the tacit and all too common assumption that it is possible to draw a right triangle of sides 3, 4 and 5 units; but, this is impossible. The assumption is false. Because we are dealing with a unit of measurement, then, *in the first place*, the units must suffer a shortening if they are to become continuous lengths out of the 3, 4, and 5 units; and, *in the second place*, the continuous lengths we *have* made must now suffer a further shortening when they come to be unified at the ends to form the continuity necessary in the perimeter of the figure. The sides of the right triangle do not consist of multiples of Democritean atomic units, or of aggregates of units, but rather (by our very act of drawing the line) of units of length *in combination*, and that is a very different thing.

Hence, if the sides of the right triangle we have constructed *seem* to be of 3, 4, and 5 units, and hence if, as a consequence, these sides *seem* commensurate, then this is an appearance only because this appearance has its origin not in the sides *qua* sides, or, in the sides *qua* lines, or even in the sides *qua* lines of 3, 4, and 5 units (which they no longer are), but solely in the fact that the 'sides' or 'lines' are *numbered*. We falsely imagine that when we draw a square on the side of the right triangle, we also square the units (falsely imagined to be) in the 'side', but, what we are squaring is a number, and having squared the number we then go on to apply this number to the 'side'. That is why we falsely assume (see Fig. 4 and text) that a square on a line of 6 inches contains 36 squares, each having sides of 1 inch.

Moreover, we have already said (see p. 11 above) that the unbroken continuity of the circumference of a circle converts any potential dividing mark we make with a pencil into an actual indivisible, and so renders the length of the circumference incapable *either* of being divided at all, or, of being measured, because any attempt to measure it must begin and end on the same indivisible mark, which thus comes to be included twice. The circumference of a circle is just as indivisible as is a straight line segment, and the reason is the same. Now, the important and distinctive characteristic of a circumference is that it is continuous *to our perception*, for we do not perceive any break in its continuity. When however, we come to a right-angled triangle, or to any triangle whatever, the perimeter of the figure is still continuous to our perception, but, the continuity has been given three different directions in the 'sides', so that the 'sides' become perceptible *as* 'sides'. Nevertheless, the perimeter is still continuous, and this continuity is now doubly clear

to us because we can prove (in support of sense-perception) that the very ends, where the 'sides' change direction, are junctions, units, combinations, and indivisibles. The end of any one side belongs to the next side, and so cannot be divided. As a consequence, all three sides form a unity—a one, an indivisible, and since we cannot even measure one 'side' we cannot use it to measure another 'side'.

In every instance, the transformation of a 'line' into the 'side' of a geometrical figure alters the length of the 'line', and all our calculations become involved in an indivisible. We are then faced with a linear magnitude that is not subject to measurement. This is true not only of all right-angled triangles of the pattern under consideration (i.e. of 3, 4, and 5), but also of the isosceles right-angled triangle, and of all right-angled triangles whatever, without exception. There is no such thing as a *rational* right-angled triangle (see Heath, *Elements* (Dover), vol. i, p. 355). Hence, whenever the right-angled triangle is used for measuring purposes, and we measure a 'side', then, without being aware of it, we are confusing 'sides' and 'lines', and there is a margin of error in our estimation of length. Small though this error is, it is still an error, and it cannot be ignored.

The extraordinary thing is that if the discovery of the incommensurability of the diagonal of a square with the side was, as Tannery suggests (*ibid.*, vol. ii, p. 112), a veritable logical scandal in geometry among the Pythagoreans, there is not any hint whatever of anything disconcerting like this in Aristotle. Just as he is emphatic that the circumference of a circle and a straight line are not commensurable (Ar. *Physics*: VII. 4, 248a–248b 7), so is he equally emphatic about the incommensurability of the diagonal of a square with the side. What he does tell us is that the latter had ceased to be a wonder, and was no longer an occasion for surprise:

'For all men begin, as we said, by wondering that things are as they are, as they do about self-moving marionettes, or about the solstices or the incommensurability of the diagonal of a square with the side; for it seems wonderful to all who have not yet seen the reason, that there is a thing which cannot be measured even by the smallest unit. But we must end in the contrary and, according to the proverb, the better state, as is the case in these instances too when men learn the cause; for there is nothing which would surprise a geometer so much as if the diagonal turned out to be commensurable'. (Ar. *Metaphysics*, 983a 11–20).

In other words, Aristotle—and the geometers of his day—were quite familiar with the grounds for the incommensurability of the diagonal of a square with the side, and, as far as Aristotle is concerned, the reasons for this incommensurability have been made quite clear, *first*, in the consequences of his accurate definition of the point, line, and plane

(surface) as either limits or *divisions* (Ar. *Metaphysics*, 1002b 5–10), and *second*, in all he has to say about the two fold character of the point in the division of a line:

‘ So in the straight line in question any one of the points lying between the two extremes is potentially a middle point: but it is not actually so unless that which is in motion divides the line by coming to a stand at that point and beginning its motion again: thus the middle point becomes both a starting-point and a goal, the starting-point of the latter part and the finishing-point of the first part of the motion

In the act of dividing the continuous distance into two halves one point is treated as two, since we make it a starting-point and a finishing point: and this same result is also produced by the act of reckoning halves as well as by the act of dividing into halves In the case of reckoning the halves, it is clear that this result follows: for then one point must be reckoned as two: it will be the finishing-point of the one half and the starting-point of the other, if we reckon not the one continuous whole but the two halves ’ (Ar. *Physics*: VIII. 262a 19263b–15).

Why Aristotle’s teaching here never came to be understood and then followed up in the western philosophic tradition is a mystery, but is it possible to put a finger on one source of this failure to grasp the significance of ‘ points ’ and ‘ lines ’ as *divisions*, and then the significance of the difference between a geometrical ‘ line ’ and the ‘ sides ’ of a geometrical figure? I suggest that we can, and it is not merely in Euclid I. 47, but in the very nature of what has come down to us as Euclid’s *Elements*, and particularly in the shortcomings of what, after all, is of fundamental importance—its definitions. The case of the point, line, and plane are instances that immediately come to the mind. There is no adequate attempt made to define, and so clarify the various meanings of ‘ part ’ and ‘ whole ’, and so no understanding of the ‘ unit ’, and still less of the ‘ indivisible ’, and hence, among other things, Euclid’s method of finding the ratio, or relative magnitude, of two commensurable magnitudes, cannot stand (see Heath: *Elements* (Dover), vol. ii, p. 118). But, in the light of Aristotle’s own teaching, let us look, for example, at the following definitions in Book X of the *Elements*:

DEFINITIONS

1. Those magnitudes are said to be COMMENSURABLE which are measured by the same measure, and those INCOMMENSURABLE which cannot have any common measure.

2. Straight lines are COMMENSURABLE IN SQUARE when the squares on them are measured by the same area, and INCOMMENSURABLE IN SQUARE when the squares on them cannot possibly have any area as a common measure. (*Ibid.*, vol. iii, p. 10.)

Definitions of this nature are not merely faulty, they have no right to the name of 'definition' at all.

A gallon of petrol, a gallon of Burgundy, and a gallon of milk are all measured by the same measure (i.e. the liquid measure of 1 gal), but we cannot say that petrol, Burgundy and milk are commensurable. Again, a pound of nails, a pound of beef steak, and a pound of apples are all measured by the same measure, but there is nothing commensurable about nails, beef steak, and apples. Again, a man six feet tall, a road six feet wide, and a hole in the ground six feet deep are all measured by the same measure, but there is nothing commensurable in a man, a road, and a hole in the ground.

Clearly, what is wanted here is the meaning of 'measure' and a proper definition of the 'commensurable' and the 'incommensurable', and we find both in Aristotle:

'The measure is always homogeneous with the thing measured: the measure of spatial magnitude is a spatial magnitude, and in particular that of length is length, that of breadth is breadth, that of articulate sound an articulate sound, that of weight a weight, that of units a unit' (*Metaphysics*: 1053a 25).

'Must we then say that, if two things are to be commensurable in respect of any attribute, not only must the attribute in question be applicable to both without equivocation, but there must also be no specific differences either in the attribute itself or in that which contains the attribute—that these, I mean, must not be divisible in the way in which colour is divisible into kinds? Thus in this respect one thing will not be commensurable with another, i.e. we cannot say that one is more coloured than the other where only colour in general and not any particular colour is meant; but they are commensurable in respect of whiteness' (*Physics*, VII. 2491 2–10).

If we now look at any given straight line segment, it is clear that it cannot be commensurate with the 'side' of a geometrical figure because there is a specific difference between a 'line' and a 'side'. A straight line segment is limited at its extremities, but it can always be extended. We can always lengthen a line. A 'side', on the other hand, belongs to a specific figure. Every 'side' is limited at its ends and cannot be lengthened; yet, unlike the straight line segment, every 'side' is involved in the continuity of the perimeter forming the figure—be it triangle, square, or hexagon.

Again, a straight line segment cannot be commensurate with the circumference of a circle, or with any 'part' of a circumference, because the circumference of a circle is indivisible and has no 'part', and a straight line segment is equally indivisible, has no 'part' (save potentially) and is specifically different from a circumference. For the same reason, neither the diameter of a circle nor a radius can be commensurate with a

circumference. For the same reason, a 'part' of a line cannot be commensurate with the 'whole' line, or with the 'part' of any other line. For the same reason, 'parts' cannot be 'wholes', anymore than a fraction can be a number. For the same reason, a diagonal cannot be commensurate with the side of a square, and the square on the hypotenuse cannot be equal to the sum of the squares on the other two sides—for, lines are lines, and sides are sides, and circles are circles, and squares are squares, and figures are figures, but, not all figures are squares or triangles. However, a figure can be 'like' another figure, a square can be 'like' another square, a circle 'like' another circle, an isosceles triangle 'like' any other isosceles triangle, just as a right-angled triangle can be 'like' any other right-angled triangle, but, *not identical*:

' Things are like if, not being absolutely the same, not without difference in respect of their concrete substance, they are the same in form; e.g. the larger square is like the smaller, and unequal, straight lines are like; they are like but not absolutely the same' (Ar. *Metaphysics*, 1054a 30).

It is clear to me that a new and revolutionary position was created in Greek geometry *first*, by Aristotle's original and correct definition of points, lines, and surfaces as 'all alike either limits or divisions', and *second*, by his definitive clarification of the meaning of the commensurable and incommensurable, and that, as a consequence, he was fully aware of the limitations of geometry as a science; but, we have failed to understand him, just as we have failed to understand what he meant by the 'indivisible', the 'continuum', the 'continuous', the 'infinite', and the 'unit', just as we have failed to understand his definitive criticism of the theory of the Greek Atomists, and so the centrality of his doctrine on organic and inorganic 'combination', not only to the right understanding of the *Physics*, but to his philosophy as a whole. Euclid's *Elements*, proclaimed as the greatest mathematical text-book of all time, may well—in the final analysis—now come to be regarded as the archetype, a paradigm of monumental proportions of the fallacies inherent in the so-called mathematical rigour of Deductive Reasoning. No matter how hallowed by time or by human reputation our postulates, our assumptions may be, if they are false, then how valid can the conclusions be that are derived from them?