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256:243, which lies between 20:19 and 19:18. The chromatic semitone is the difference between the diatonic semitone and the tone, or 2187:2048, which lies between 16:15 and 15:14. Thus the diatonic semitone is smaller than the chromatic semitone; in practical terms F# is higher in pitch than Gb. Six tones exceed the octave by the Pythagorean or diatonic comma, with ratio 531441:524288, which lies between 75:74 and 74:73. (The comma exists also as the difference between the diatonic and the chromatic semitones.)

In Table 1 are shown the ratios to the fundamental of all the notes of a chromatic Pythagorean scale. A word of explanation may be necessary regarding cents: The cent is a logarithmic measure of musical intervals, conceived as .01 equal semitone, whence the octave contains twelve hundred cents. If the ratio of the octave is taken as 2 rather than $2^{1/2}$ (in terms of vibration numbers rather than string-lengths), the logarithmic base of the cents system is $2^{1/1200}$. In actual computation, the cents value of an interval is equal to the logarithm of the ratio of the interval, times the constant factor 1200/log 2. The deviation from equal temperament is also shown, based on the deviations of successive semitones; that is, upon the deviations of the differences of successive cents values in the table. Mean deviation (M. D.) and standard deviation (S. D.) are computed in the usual manner.

Table 1. Pythagorean Tuning

Names	C	C#	D	Eb	E	F	F#	G	G#
Ratios	1	2048/2187	8/9	27/32	64/81	3/4	512/729	2/3	4096/6561
Cents	0	114	204	294	408	498	612	702	816

Names	A	Bb	B	C	
Ratios	16/27	9/16	128/243	1/2	M. D. 11.7 cents
Cents	906	996	1110	1200	S. D. 11.8 "

Aristoxenus. To the Pythagoreans the tuning ratios were all-important. Connected as they were with the elaborate mystical doctrine of the harmony of the spheres, these ratios could be divorced entirely from the field of practical music. Aristoxenus (3rd cen., B. C.), pupil of Aristotle, objected to this barren and intellectual toying with musical intervals as if they were but audible ratios. To him the human ear is the sole arbiter of the correctness of pitches. Aristoxenus described scales containing whole, five-quarter, three-quarter, half, three-eighth, third, and quarter tones, and expressed these scales by diagrams in which the whole tone occupied twelve parts, the semitone six, etc. Such a conception of intervals had no counterpart in the mathematics of his day. It is essentially a logarithmic notion of the

THE PERSISTENCE OF THE PYTHAGOREAN TUNING SYSTEM

By J. MURRAY BARBOUR

THE dependence of music upon mathematics is generally known.¹ From the time of the Greeks the tuning of instruments especially has rested upon a firm mathematical basis. The purpose of this article is to show something of the nature of that system of tuning called by the name of Pythagoras, and to trace its later history, modification, and influence—its persistence in the face of radically changed musical practice. The first part of the article contains a brief history of tuning theories, with interpolated expository material. Then comes a section devoted to the ancient doctrine that the comma is the common measure of musical intervals, and to the resurgence of this doctrine in modern forms of multiple division. After a cursory glance at several earlier mathematical accounts of the Pythagorean tuning, a more lengthy discussion is accorded Caramuel and his musical logarithms, with an excursion into other methods of logarithmic representation. The more important modifications of the Pythagorean system are next presented, as well as the names of some strict Pythagoreans of modern times. In conclusion, the present existence of the Pythagorean tuning is considered in relation to the tuning of stringed instruments, introspective concepts of pitches, and theories of harmony.

I. GREEK AND MODERN TUNING THEORIES

Pythagoras. The general philosophical and mathematical theories of Pythagoras (6th cen., B. C.) need not be stated here. The two essential parts of his tuning theory are that two strings of equal diameter and tension, whose lengths are in the ratio of 2:1, differ in pitch by an octave, and that two similar strings whose lengths are in the ratio 3:2 differ by a fifth. The fourth is taken as the difference between the fifth and the octave; that is, the quotient of their ratios, or 4:3. The tone is the difference between the fourth and the fifth, or 9:8. The major third comprises two tones, or 81:64. The diatonic semitone is the difference between the third and the fourth, or

¹For an excellent general treatment of this subject see R. C. Archibald's "Mathematicians and Music," *The American Mathematical Monthly*, vol. 31, 1924, pp. 1-25.

product of ratios, and Aristoxenus is often regarded as anticipating both logarithms and equal temperament.

Ptolemy. Claudius Ptolemy (2nd cen., A. D.), the geographer, attempted to harmonize the principles of Aristoxenus and the disciples of Pythagoras, by the doctrine that proportion and perception must be in agreement if an interval is to be considered musical. Ptolemy advocated the use of superparticular proportions—those in which the antecedent exceeds the consequent by unity. Among the twenty-three principal scales that he has expressed in ratios and string-lengths, seven of his own scales and seven ascribed to other writers are composed entirely of such ratios. Most important of these scales from the modern viewpoint is the diatonic syntonon, which is described below as just intonation.

Boethius. Anicius Manlius Severinus Boethius (6th cen., A. D.), the Roman philosopher and statesman, was the most potent single factor in establishing the continuity of the Pythagorean tuning theory from ancient to modern times.* Although Boethius made clear the distinctions between the three great Greek theorists, he favored the Pythagorean system, and the countless medieval writers to whom Boethius was a sort of musical bible were content to ring the changes upon the doctrines of Pythagoras.

Just Intonation. So long as the music of the medieval chant remained unisonal, the Pythagorean tuning was wholly satisfactory. (Such an orderly system of tuning has always appealed to mathematical logic. Moreover the equal tones and the small semitones appeal to the melodic sense—the singer's logic—and make this tuning method preeminently suitable for unharmonized song.) But the great flaw in this system, harmonically considered, is the sharpness of its major thirds. The Pythagorean third ($64/81$ or 408 cents) is higher than the pure third of the harmonic series ($4/5$ or 386 cents) by the ratio $80/81$ (22 cents), known as the syntonic or Ptolemaic comma—not to be confounded with the Pythagorean comma previously mentioned. During the later medieval period, when singing in thirds and sixths became common, singers unconsciously modified the harsh Pythagorean intervals.

Finally a variety of tuning in which the most important major thirds had their pure value, $4/5$, gained theoretical recognition. Just intonation, as such a tuning system is called, had its prototype in Ptolemy's diatonic syntonon, or tightly stretched diatonic. The most symmetrical

*The first printed edition of Boethius' text, *De Institutione Musica*, dates from 1492; the standard edition is that of Gottfried Friedlein, 1867.

construction of the chromatic scale in just intonation contains four notes (as F, C, G, D) joined by pure fifths. The remaining eight notes form pure thirds above and below these first four, as in Table 2.

There are two sizes of tone in just intonation with ratios $8/9$ and $9/10$. Unlike the Pythagorean tuning, the diatonic semitone ($15/16$) in just intonation is larger than the chromatic semitone ($24/25$), or $G\flat$ is higher than $F\sharp$. (There are also at least two other sizes of semitone with more complex ratios.)

TABLE 2. JUST INTONATION

Names	C	C $\sharp$	D	E $\flat$	E	F	F $\sharp$	G	G $\sharp$	A
Ratios	1	128/135	8/9	5/6	4/5	3/4	32/45	2/3	16/25	3/5
Cents	0	70	204	316	386	498	590	702	772	884
Names	B $\flat$	B	C							
Ratios	5/9	8/15	1/2							
Cents	1018	1088	1200							
								M. D. 21.3 cents		
								S. D. 23.6 "		

Mean-tone Tuning. In just intonation the diatonic fifth D-A and minor third D-F are a comma too flat, and certain chromatic intervals are also at fault. The simplest way to remedy these grave defects is to diminish the size of each fifth by one-quarter syntonic comma, leaving the major thirds perfect. A system of tuning in which the fifth is taken as $(2/3)$ ($81/80$) $^{3/4}$ or $(1/5)$ $^{3/4}$ —696.5 cents—is called the mean-tone tuning. In Table 3 is shown the mean-tone tuning; in the row of ratios the symbol t is used in place of the ratio $4/5$.

TABLE 3. MEAN-TONE TUNING

Names	C	C $\sharp$	D	E $\flat$	E	F	F $\sharp$	G	G $\sharp$	A	B $\flat$	B	C
Ratios	1	$2^{1/2}t^{3/4}$	$t^{3/4}$	$2^{-1/2}t^{-3/4}$	t	$2^{-1/2}t^{-3/4}$	$t^{3/4}$	$2^{-1/2}t^{3/4}$	t^2	$2^{-1/2}t^{3/4}$	$2^{-1}t^{-3/4}$	$2^{-1/2}t^{3/4}$	2^{-1}
Cents	0	76	193	310	386	503	579	697	773	890	1007	1083	1200
										M. D. 20.0 cents; S. D. 20.2 cents.			

Equal Temperament. The mean-tone tuning effectively remedies the worst defects of just intonation, and is admirable within its own narrow limits. But it too suffers from what might be termed the incommensurability of musical intervals, for three major thirds $(4/5)^3$ constitute an octave that is too flat by the ratio $125/128$ (42 cents), known as the diesis. To overcome this defect, painfully apparent whenever a performer attempted to use tones beyond $E\flat$ or $G\sharp$, the octave was divided into twelve equal semitones. This system of tuning, known as equal temperament, is in general use today. In equal temperament the ratio of each semitone is $2^{1/12}$. Another way of defining equal temperament is: that system of tuning in which the fifth

is tempered by one-twelfth ditonic comma. Table 4 expresses equal temperament in ratios and cents values.

TABLE 4. EQUAL TEMPERAMENT

Names . . .	C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B	C
Ratios . . .	1	$2^{-1/12}$	$2^{-1/6}$	$2^{-1/4}$	$2^{-1/3}$	$2^{-1/2}$	$2^{-2/3}$	$2^{-3/4}$	$2^{-5/6}$	2^{-1}	$2^{-11/12}$	2^{-1}	2^{-1}
Cents . . .	0	100	200	300	400	500	600	700	800	900	1000	1100	1200

M. D. 0; S. D. 0.

Tuning Systems Compared. There is no available evidence that just intonation was seriously considered as a practical tuning system except by a few mathematicians like Kepler and Descartes. Mean-tone tuning and equal temperament, the systems that were in actual conflict over a period of centuries, are the systems that can be fairly compared. It is not always emphasized by theorists that the intervals of equal temperament come closer to the Pythagorean tuning than they do to any other established tuning system. The fifth of equal temperament is only one-twelfth ditonic comma (2 cents) flatter than the Pythagorean fifth; the mean-tone fifth is one-quarter syntonic comma (5.5 cents) flatter. The third of equal temperament is one-third ditonic comma (8 cents) flatter than the Pythagorean third; the mean-tone third is a whole syntonic comma (22 cents) flatter. This same proportion of four to eleven is maintained throughout the scale. Thus it is clear why adherents of the mean-tone system could scoff at the almost Pythagorean thirds of equal temperament. Since the universal adoption of equal temperament, this one-time stigma has become a badge of honor.

Tuning History. The historical progress of these three later types of tuning is full of overlappings and inconsistencies. Generally speaking, both just intonation (as theory) and the mean-tone tuning (as practice, especially for keyboard instruments) were advocated by the end of the first quarter of the sixteenth century, and equal temperament was correctly stated for fretted instruments by the end of the third quarter of that century (probably much older in practice). But equal temperament was not accepted for all kinds of instruments, including organs, until the middle of the nineteenth century.

II. COMMAS AND MULTIPLE DIVISION

Philolaus. The comma has already been implicitly defined as an interval occurring in tuning as the minute difference between two larger intervals, the two specific commas being the ditonic comma

(24 cents) and the syntonic comma (22 cents). The Greeks, seeking a unit of measure for musical intervals, chose the comma. According to Boethius,³ Pythagoras' disciple Philolaus (5th cen., B. C.) held that, since the tone is divisible into minor semitones and a comma, and, since the semitone is divisible into two diaschismata, the tone is then divisible into four diaschismata and a comma. If, now, the diaschisma is taken as two commas exactly, the tone is divided into nine commas. (Of course the arithmetic is only approximate, as $204/9 = 22.7$ cents, which is the value of a mean comma between the ditonic and syntonic.)

Fifty-three Division. *Vanneo.* *Angelo.* *Mersenne.* *Kircher.* If the tone contains nine commas, the octave—which in the Pythagorean tuning is composed of five tones and two minor semitones—contains $5 \cdot 9 + 2 \cdot 4 = 53$ commas. In the early sixteenth century all writers, such as Stefano Vanneo (b. 1493)⁴ and Angelo de Picitono,⁵ who spoke of nine commas in a tone, must have had a mental image of a fifty-three-comma octave. Such a division of the octave was more explicitly stated by two outstanding seventeenth theorists, Marin Mersenne (1588-1648)⁶ and Athanasius Kircher (1602-1680).⁷

Mercator. *Bosanquet.* Many recent historians (Shohé Tanaka, Japanese pupil of Helmholtz, being an exception) have given Nicholas Mercator (1640(?) - 1694)⁸ credit for originating the 53-division. But Mercator's account was wholly theoretical like the accounts of his predecessors. Full credit for constructing an instrument upon which to realize this division in a practical way, belongs to R. H. M. Bosanquet (1841-1912), the eminent English acoustician and astronomer.⁹ In addition to the comparative purity of its fifths, the great advantage that this system has over other forms of multiple division of the octave, lies in the fact that the fifty-three fifths can be tuned practically pure: $(2/3)^{53} 2^{31} \approx 1$, or, more precisely, $53 \cdot 702 - 31 \cdot 1200 = 6$ cents, an insignificant error.

Thirty-one Division. *Marchettus.* *Vicentino.* *Colonna.* Another theory of measuring tones was advocated by Marchettus of Padua (fl. end of 13th cen.), who held that the tone is composed of five

³*Op. cit.*, book 3, chap. 8.

⁴*Recanteum de Musica Aurea*, Rome, 1533.

⁵*Fior Angelico di Musica*, Venice, 1547.

⁶*Harmonicorum Libri XII*, Paris, 1648, *Liber Tertius de Instrumentis Harmonicis*, Prop. XVII, p. 126.

⁷*Musurgia Universalis*, vol. 1, Rome, 1650, p. 135.

⁸See William Holder (1616-1698): *A Treatise of the Natural Ground, and Principles of Harmony*, London, 1694, p. 79.

⁹See esp. his book, *An Elementary Treatise on Musical Intervals and Temperament*, London, 1876.

dieses—diesis being used in a rather general sense. If there are five dieses in a tone and if the diatonic semitone is major as in just intonation, the entire octave will consist of $5 \cdot 5 + 2 \cdot 3 = 31$ dieses. This 31-division was advocated primarily by Nicola Vicentino (b. 1511),²⁰ who constructed instruments with six keyboards for its proper rendition, and who gave a questionable mathematical theory for it, as well as practical tuning rules. Fabio Colonna (1567-1650)²¹ followed Vicentino's lead, with a highly complex numerical theory to guide his choice of notes upon the "Sambuca Lincea."

Rossi. Huyghens. Although Vicentino's 31-division was ridiculed by his contemporaries, two mathematicians of a later generation, Lemme Rossi (d. 1673)²² and Christian Huyghens (1629-1695)²³ accorded it a careful mathematical study based upon logarithms. Huyghens showed that the fifth of this tuning differs but slightly from the fifth of the ordinary mean-tone tuning,²⁴ and that Vicentino was justified in recommending the mean-tone tuning for his archicembalo.

Thirty-four Division. Schneegass. Not unlike Marchettus' theory of dieses was the division of the octave indicated by Cyriac Schneegass (1546-1597).²⁵ Explaining the practical construction of the mean-tone tuning on the monochord, Schneegass enunciated the theory that the diatonic semitone contains $3\frac{1}{4}$ commas and the chromatic semitone $2\frac{1}{4}$. Hence the tone contains $5\frac{1}{2}$ commas, and the octave will contain $5 \cdot 5\frac{1}{2} + 2 \cdot 3\frac{1}{4} = 34$ commas. Apparently Schneegass was the sole advocate of this division of the octave in the history of tuning. Bosanquet, in a special article on the Hindoo tuning,²⁶ showed the excellences of the Hindoo 22-division and also extolled the 34-division, which in his classification is closely related thereto. But he gave no historical account of the 34-division.

Nineteen Division. Aristoxenus. Kornerup. Ariel. Yasser. A recurring type of multiple division for which there is a classical prototype is the 19-division. Aristoxenus' chromatic malakon or soft chromatic used third tones, and an octave made up entirely of third tones would contain nineteen notes, provided the diatonic semitone is major. The nineteen division has eloquent contemporary advocates

²⁰*L' Antica Musica Ridotta alla Moderna Pratica*, Rome, 1555.

²¹*La Sambuca Lincea*, Naples, 1618.

²²*Sistema Musico*, Perugia, 1666, p. 86.

²³"Novus Cyclus Harmonicus," in *Histoire des Ouvrages des Sçavans*, Oct., 1691, reprinted in Huyghens's *Opera Varia*, Leyden, 1724, pp. 747-754.

²⁴ $(1/2)^{13\frac{1}{2}}$ and $(1/5)^4$.

²⁵*Nova & Exquisita Monochordi Dimensio*, Erfurt, 1590.

²⁶"On the Hindoo Division of the Octave," *Royal Society's Proceedings*, 1877, pp. 372-384.

in Ariel," Thorvald Kornerup,²⁷ and Joseph Yasser.²⁸ Stripped of its modern furbelows, the nineteen division is simply a modified mean-tone tuning in which the fifth is tempered by one-third syntonic comma. Mathematically, $(2/3) (81/80)^{13} \approx (1/2)^{13\frac{1}{2}}$ —a close correspondence.

Salinas. It is strange that these theorists and acousticians who have been seeking a historical basis for their 19-division, have neglected to pay homage to the man who should be their patron saint—Francisco de Salinas (1513-1590).²⁹ Of the three types of mean-tone tuning ($1/4$, $2/7$, $1/3$ comma) discussed both by Salinas and by Gioseffo Zarlino (1517-1590)³⁰ Salinas took entire credit for the one-third comma tuning. But he was not overly proud of his creation, believing that the advantage it possessed of ease of execution on a monochord (the cube root was performed by the mesolabium of Archimedes) was offset by the large distortion produced upon the principal consonances—a view from which it is difficult to dissent.

III. MATHEMATICAL COMPUTATIONS. TUNING LOGARITHMS.

Boethius. Medieval Theorists. It was possible for men to dilate at length about commas without knowing anything of the finer mathematics of tuning. The writers next to be considered may have lacked musical insight in some cases, but their mathematical ability is unquestioned. Boethius gave a thorough mathematical treatment of the Pythagorean tuning, including tables of string-lengths in least integers for Pythagorean scales in all three genera. These figures were of course available to successive generations of musical theorists who made use of Boethius' text.

Gaffurio. Ornithoparchus. It must be remembered that the orderly progression by fifths was used by the Greeks for the diatonic scale only. With the gradual addition of the five black keys to the organ keyboard, completed during the fifteenth century if not sooner, came the necessity of extending the Pythagorean tuning to these additional notes. Franchino Gaffurio (1451-1522)³¹ gave careful directions for dividing the monochord to include these chromatic notes, and showed a diagram in which their relation to the diatonic notes was indicated by arcs and ratios. A similar set of directions was given by Andreas Ornithoparchus.³²

²⁷*Das Relativitätsprinzip der musikalischen Harmonie*, vol. 1, Leipzig, 1925.

²⁸*Musical Acoustics Based on the Pure Third-System*, Copenhagen and Leipzig, 1922.

²⁹*A Theory of Evolving Tonality*, New York, 1932.

³⁰*De Musica Libri Septem*, Salamanca, 1577.

³¹*Dimostrazioni Harmoniche*, Venice, 1571, *op. cit.*, p. 148.

³²*Theorica Musice*, Naples, 1480; Milan, 1492. Gaffurio's name variously spelled.

³³*Musice Actiue Micrologus*, Leipzig, 1517; Eng. trans. by John Dowland, London, 1609.

Faber Stapulensis. The only contemporary of Gaffurio whose mathematical excursions into Pythagorean tuning theory showed originality was Faber Stapulensis (Jacques le Fèvre, d. 1537).²⁴ Faber felt constrained to repeat all the ancient speculations concerning the sizes of tones, commas, etc., so far as they were valid, and investigated the whole matter independently with a view to stating interval limits with greater accuracy. Faber had a profound influence upon writers on tuning for two centuries, comparable in a measure with the earlier influence of Boethius. Mathematicians especially, unacquainted with the fascinating evolution of musical art, would turn for guidance to the well-worn pages of Faber's text, and would pronounce hoary dicta that only puzzled generations that knew not Pythagoras.

Caramuel's Musical Logarithms. Such a mathematician masquerading as a musical theorist was the versatile Bishop of Vigevano, Juan Caramuel de Lobkowitz (1606-1682). Caramuel has attained a brief and undeserved mention in histories of music for his treatment of solmization.²⁵ But his real contribution to acoustical science has apparently remained buried in his treatise upon logarithms.²⁶ There Caramuel discussed "enharmonic logarithms," and subjoined a table called "musical scale, measuring the string arithmetically and logarithmically." In this table are string-lengths for the Pythagorean diatonic scale and their values in four different kinds of logarithms, including the system of Briggs. In the table below, the string-lengths are given for the lowest of Caramuel's three octaves, and only their musical logarithms are given.

TABLE 5. CARAMUEL'S LOGARITHMS

Chordarum Longitudo	Logarithmi Musicali
C 512,000	1.00000
B 539,391	0.92460
A 606,815	0.75476
G 682,667	0.58492
F 768,000	0.41508
E 809,086	0.33968
D 910,222	0.16984
C 1024,000	0.00000

Later in the same book²⁷ Caramuel compared the "syntonic" ratios

²⁴*Elementa Musicalia*, Paris, 1496.

²⁵*Nova Musica*, Vienna, 1645.

²⁶*Mathesis Nova*, Campana, 1670, tome 2, syntagma 5, article 6, p. 867.

²⁷Syntagma 8, article 12, "De Diabete Musico, seu Enharmonico," pp. 1207-1210.

(just intonation) with what he called the "diatonic"—an error for "ditonic." But his included monochord, like the former, was Pythagorean. His comment upon it is surprising: "Haec est chordae diviso, quam Diatonicam vocant, quod procedat DIA TONOUS. Eâ Antiqui summoperè delectabantur, quos plurimi Recentiores sequuntur." The "plurimi" is astounding. It is almost incredible that the long and bitter polemics over just intonation, mean-tone tuning, and equal temperament should have made no impression on the Bishop of Vigevano, who in all seriousness said that the Pythagorean tuning which had greatly delighted the ancients was being followed by very many of his contemporaries!

However amazed we may be by Caramuel's cloistered conception of musical practice, we must give the musical logarithms their full meed of praise. As in modern ways of representing tonal relations, the logarithms correspond to vibration numbers instead of to string-lengths. If T is any note in Caramuel's table and if L is its string-length, $\text{mus. log. } T = 10 + \text{colog}_2 L$. In general, if F is the length for the fundamental note, $\text{mus. log. } TF = \log_2 F + \text{colog}_2 L$.

Except for the location of the decimal point, an accidental circumstance, Caramuel's logarithms are identical with the "millioctaves" affected by some modern acousticians in place of Ellis' cents. Conversion is simple enough: to obtain the cents from the millioctaves, multiply by 1.2. Since neither measure is obtained directly (unless the investigator is using a table of logarithms to the base 2), it seems preferable to choose cents, for they correspond much more closely to our musical system.

Euler. Seventy years later than Caramuel, Leonard Euler (1707-1783),²⁸ in a famous treatise on acoustics and consonance, discussed the use of logarithms to the base 2 in the study of intervals, and expressed all the intervals of the scale in these logarithms. Quite possibly Euler arrived independently at this method of representation, but it agrees absolutely with the musical logarithms, and the credit for its invention, long ascribed to Euler, should properly be credited to the lively intelligence of Bishop Caramuel.

Origin of Cents. The origin of Ellis' cents measure may also be of interest, although it bears no particular relation to the Pythagorean system. So far as the idea of 1200 equal parts in the octave is concerned, it was anticipated by Prince Tsai-yü of China,²⁹ who had

²⁸*Tentamen Novae Theoriae Musicae*, Petropolis, 1739, p. 112.

²⁹Cf. my article, "A Sixteenth Century Chinese Approximation for π ," *Amer. Math. Monthly*, vol. 40, 1933, pp. 69-73.

stated that the volume of the fundamental pipe was 1200 grains of millet, or, since the twelve semitones of the octave are contained, as it were, in the fundamental that is their source, "we have one hundred grains for each of these parts."

Geometrical Scale. Lambert. Bellermand. Joseph Yasser²⁰ informs us of a manuscript in the New York Public Library²¹ in which the musical intervals are expressed in a "Geometrical Scale" of twelve inches to the octave, with twelve-place decimals. Apparently the author of this valuable manuscript did not associate inches with semitones, for Yasser²² says that Heinrich Bellermand (1832-1903)²³ "was the first who assumed the tempered semitone as a fundamental measuring unit." However, Johann Heinrich Lambert (1728-1777)²⁴ used the tempered semitone as a measure a century before Bellermand, and extended his calculations to fifteen decimal places.

Common Logarithms. Sauveur. Yasser is inclined to credit the author of the manuscript mentioned above with the "initial introduction of common logarithms in music,"²⁵ since he had expressed all intervals in these as well as in "binary" logarithms. But unless the manuscript can be dated before 1701, it does not have priority in this field, for in that year appeared the first of a notable series of articles on acoustics and temperament by Joseph Sauveur (1653-1716).²⁶ In this article Sauveur explained that he was looking for "a common measure of all the intervals of sounds, able to measure their slightest differences." To obtain that measure, he said: "I divide the octave into 43 equal intervals that I call Merides, and I divide each Meride into 7 small equal intervals that I call Eptamerides."²⁷ Later²⁸ he spoke of "subdividing each Eptameride into ten parts, called Decamerides." But he did not use this more subtle division in practice. Now $43 \cdot 7 = 301$, the first three figures of the mantissa of the common logarithm for 2. As Sauveur himself said, when we have found the logarithm for the ratio of any interval, "the first [3] figures denote the Eptamerides in that interval."²⁹

²⁰*Op. cit.*, p. 21, note.

²¹*The Geometrical Scale in Musick; or Gam-Ut Reduced to Geometrical Proportions*, written before 1705 and revised about 1735.

²²*Op. cit.*, p. 20, note.

²³*Die Grösse der musikalischen Intervalle*, Berlin, 1873.

²⁴"Remarques sur les Tempéraments en Musique," *Mem. de l'Acad. Roy. des Sci. et Bel. Let.*, 1774.

²⁵*Op. cit.*, p. 19, note.

²⁶"Principes d'Acoustique et de Musique," *Mem. de l'Acad. Roy. des Sci.*, 1701, pp. 403-498.

²⁷*Ibid.*, p. 418.

²⁸*Ibid.*, p. 422.

²⁹*Ibid.*, p. 420.

Anglas. Kingsmill. Sauveur's Eptameride has not disappeared entirely from the scene. J. P. L. Anglas (b. 1869)³⁰ used it as a unit of musical measure, but called it "savart"—after the physicist, Félix Savart (1791-1841). It was found useful also by Thomas W. Kingsmill (b. 1837),³¹ in comparing the pitches of the Chinese P'ip'a with those of just intonation, although he is content to call it a "logarithmic point."

Logarithms in Tuning. Mersenne. Nierop. Ever since the time of Aristoxenus, this idea of logarithmic representation has been in men's minds. But it was not until after the invention of logarithms that it became possible for a theorist to subdivide the octave equally with the greatest of ease. Dyrck Rembrantz van Nierop (1610-1682)³² used logarithms to solve a difficult tuning problem as early as 1650, and there is evidence pointing to a somewhat earlier use in a work by Mersenne.³³

But none of these men used logarithms to make direct contact with the reader. For them the logarithms were only a convenient means of obtaining the usual representation of pitches in terms of string-lengths. Today we reverse the process on the principle that the logarithmic method is more universal than divisions of a monochord of any arbitrarily chosen length. The measure in current use is only a slight modification of Caramuel's, now over two hundred and fifty years old. Caramuel and his successors have put us greatly in their debt for a convenient means of approach to the study of tuning and temperament.

IV. MODIFICATIONS OF PYTHAGOREAN TUNING

Ramis. Among writers who adhered to the Pythagorean theory, but who advocated a modification of it in practice, there is no more striking example than Bartolomeus Ramis de Pareja (c. 1440-p. 1491).³⁴ Ramis gave monochord directions for the formation of a chromatic scale containing four pure major thirds.³⁵ Such a radical departure from the Pythagorean system was vigorously opposed by several able theorists of the day, to whom Ramis admitted the theoretical supremacy of the tuning by fifths, explaining that his method was merely

³⁰*Precis d'Acoustique*, Paris, 1910.

³¹"The Music of China," *Jour. Roy. Asi. Soc. N. Ch. Branch*, 1910, p. 56.

³²*Wis-konstige Musyka*, Amsterdam, 1650; 2nd ed., 1659.

³³*Harmonie Universelle*, Paris, 1636/7. Cf. Archibald's opinion (*op. cit.*, p. 17) that Huyghens (1691) first used logarithms in discussing musical intervals.

³⁴*Musica Practica*, Bologna, 1482; new ed. by Johannes Wolf, 1901, as Beiheft der Inter. Musikgesellschaft.

³⁵See Table 6; cf. Table 2.

intended to facilitate the division of the monochord for singers. Thus we are confronted with a curious paradox: the man who possibly first constructed a complete chromatic system in just intonation should not be called a tuning revolutionist, but a Pythagorean reformer.⁴⁰

TABLE 6. RAMIS' TUNING

Names . . .	C	C#	D	E♭	E	F	F#	G	A♭	A	B♭	B	C
Ratios . . .	1	128/135	9/10	27/32	4/5	3/4	32/45	2/3	81/128	3/5	9/16	8/15	1/2
Cents . . .	0	92	182	294	386	498	590	702	792	884	996	1088	1200

M. D. 10 cents; S. D. 10.1 cents.

Aron. Stevin. Two other pioneers in tuning science should be grouped with Ramis because of their dependence upon the Pythagorean system. Pietro Aron (c. 1480-1545),⁴¹ who gave the first known account of a practical mean-tone system, said nothing about the ratios of the "sonorous and just" thirds that he used in his tuning. To him the "sistema participato," as the mean-tone tuning was long called, was not a modification of just intonation, but of the Pythagorean tuning. Likewise Simon Stevin (1548-1620),⁴² an advocate of equal temperament and possibly the first European to put the equal ratios into numerical form, apparently knew nothing whatever either of just intonation or of the mean-tone tuning. Both men went directly from the idea of pure fifths to that of tempered fifths.

Grammateus. Henricus Grammateus (Heinrich Schreyber of Erfurt, b. ante 1496) made the most interesting modification of the Pythagorean tuning. As an appendix or second part to his arithmetic (*Ayn new kunstlich Buech*, Nürnberg, 1518) there is a treatise on music with the title *Arithmetica applicirt oder gezogen auff die edel kunst Musica*. For the diatonic notes Grammateus followed the Pythagorean doctrine exactly. But the black keys—the "minor semitones"—were formed by dividing each tone into two equal semitones by the well-known Euclidean method for finding a geometric mean proportional. Grammateus may have owed this construction to Faber Stapulensis (cf. p. 294), who had shown that the halving of a tone, impossible by linear section, could easily be accomplished by geometry. (Faber's method was not applied by him to any tuning system.)

⁴⁰In a section of Montucla's *Histoire des Mathematiques* (vol. 4, new ed., 1802, p. 650) called "history of music" there is a table of string-lengths for a seventeen note tuning, twelve notes of which are identical with those of Ramis. The other five notes are formed by pure fifths above and below the semitonal scale—a useless extension in this case because such enharmonic pairs as D♭ and C# differ by only two cents.

⁴¹*Toscanello in Musica*, Venice, 1523; rev. ed., 1529.

⁴²*Van de Spiegeling der Singconst*, 1595 (?); Ms. edited by Dr. D. Bierens de Haan, Amsterdam, 1884. Suggestion of his theory also in *Wyskonstige Gedachtenissen*, Leyden, 1608, in 1st book of Geography.

Bermudo. Grammateus may have taken his geometrical construction directly from Euclid; but Juan Bermudo (b. 1510 (?)),⁴³ in repeating the construction independently, definitely credited Faber with the idea. Most of Bermudo's tunings were Pythagorean for the entire chromatic compass instead of for the diatonic genus only.

Merits of System. Of all irregular systems founded upon the Pythagorean tuning, this system of Grammateus and Bermudo most closely approaches equal temperament. Yet it was ignored by later writers. So far as we know, the sole reference to it was a disparaging criticism by Johann George Neidhardt (1685 (?)-1739).⁴⁴

TABLE 7. GRAMMATEUS' TUNING⁴¹

Names	C	C#	D	E♭	E	F	F#	G	G#	A
Ratios	1	2·2 ^{1/2} /3	8/9	16·2 ^{1/2} /27	64/81	3/4	2 ^{1/2} /2	2/3	4·2 ^{1/2} /9	16/27
Cents	0	102	204	306	408	498	600	702	804	906
values										
Names	B♭	B	C							
Ratios	32·2 ^{1/2} /81	128/243	1/2							
Cents	1008	1110	1200							
values										
								M. D. 3.3 cents		
								S. D. 4.5 "		

V. LATER MODIFICATIONS

Kirnberger. After the time of Bermudo there is a gap of over two centuries in the sequence of modifiers of the Pythagorean tuning, during the period when equal temperament and mean-tone tuning were vying for supremacy. Then came a group of writers who preferred irregular systems of a strong Pythagorean cast. Foremost among these belated Pythagoreans was Johann Philipp Kirnberger (1721-1783), who had to his credit an excellent approximation for equal temperament,⁴⁵

⁴³*Declaracion de Instrumentos Musicales*, Ossuna, 1555.

⁴⁴*Sectio Canonis Harmonici*, Königsberg, 1724, p. 21.

⁴⁵A procedure somewhat similar to that of Grammateus was followed by Giovanni Maria Artusi (d. 1613) *Seconda Parte dell' Artusi overo delle Imperfectioni della Moderna Musica*, Venice, 1603, who proposed a geometric halving of the tones of the mean-tone tuning to form the chromatic notes. This method (M. D. 5.7 cents; S. D. 7.6 cents) is definitely inferior to the Grammateus method. Likewise inferior (M. D. 6.5 cents; S. D. 7.7 cents) is the formation of the chromatic notes in just intonation by an arithmetical halving of tones, first advocated by Andreas Reinhard (*Monochordum*, Leipzig, 1604). This is the simplest of all feasible ways of dividing the chromatic octave, for the semitones in the lower tetrachord can be found by dividing the space between the extremes of the tetrachord into five equal parts, and the higher tetrachord can, with one slight change, be formed similarly. Over a century later in England, Alexander Malcolm (b. 1687) (*A Treatise of Musick*, Edinburgh, 1721) described a division that was probably intended to be the same as Reinhard's.

⁴⁶*Konstruktion der gleichschwebenden Temperatur*, Berlin, 1760. The principle followed, in this method is to tune seven pure fifths upward, come down four octaves, add a pure major third, and consider the resulting interval the fourth of equal temperament. In numbers: 2⁷ (2/3)⁷ 4/5 app. (1/2)^{6 1/12}.

but who zealously advocated a system containing nine or ten Pythagorean fifths.⁵² Kirnberger's tuning (M. D. 9.0 cents; S. D. 9.7 cents) must have been wholly unsatisfactory for the performance of contemporary music.

Von Wiese. Stanhope. Another advocate of unequal temperament was Baron C. L. G. von Wiese (1732-1800)⁵³ whose system contained ten pure fifths. He divided the Pythagorean comma between the other two fifths, the best arrangement of these two fifths giving M. D. 6.7 cents; S. D. 8.2 cents. An English nobleman, Charles, Earl Stanhope (1753-1816)⁵⁴ divided the comma among three successive fifths, the other nine fifths being Pythagorean.

Multiple Division. In the nineteenth century the Pythagorean tuning formed an essential part of several experiments in multiple division. We have noted that the tuning of Bosanquet's 53-system was wholly by pure fifths. So was the tuning of the twenty-four notes of the octave of Helmholtz's harmonium, divided between two manuals.⁵⁵ Shohé Tanaka, Helmholtz's pupil, constructed a Transponir-Harmonium or Enharmonium, the keyboard of which contained twenty or twenty-six notes in just intonation. All the notes of the keyboard could be automatically shifted into any of twelve "keys," and the keynotes of these keys were in Pythagorean tuning.⁵⁶

VI. UNALTERED PYTHAGOREAN SYSTEM

Virdung. Agricola. Papius. Fludd. All of the above modifications and partial utilizations of the Pythagorean system of fifths were seriously presented as the best possible solution of the tuning problem. What further evidence is needed of the vitality of the Pythagorean tuning? A few additional writers urged its use without modification; others noted that it was in common use although they did not personally advocate it—the evidence of the latter group being the more valuable. In the middle of the sixteenth century, Martin Agricola (1486 (?)–1556)⁵⁷ referred to the Pythagorean tuning as in con-

⁵²*Die Kunst des reinen Satzes*, part 2, Berlin, 1774/79.

⁵³*Versuch über die logisch-mathematische Klangeintheilungs- Stimmungs- und Temperatur-Lehre*, Dresden, 1793.

⁵⁴"Principles of the Science of Tuning Instruments with Fixed Tones," *Phil. Mag.*, vol. 25, 1806, pp. 291-312.

⁵⁵The theory of Helmholtz's method was traced to Euler by A. J. Ellis: "On the Condition, Extent, and Realization of a Perfect Musical Scale," *Proc. of Roy. Soc.*, 1864, pp. 93-108.

⁵⁶Shohé Tanaka: "Studien im Gebiete der reinen Stimmung," *Vierteljahrsschrift für Musikwissenschaft*, vi, 1890, pp. 1-90. Tanaka's keyboard was merely an extension of Joachim Steiner's keyboard of twelve semitones to the octave, other features being identical.

⁵⁷*Musica Instrumentalis deutsch*, 4th ed., Wittemberg, 1545; ed. by Robt. Eitner in Band 24, *Publi. ält. prak. u. theo. Musikwerke*, 1896.

temporary practice. Toward the end of the century, Andreas Papius (1542-1581)⁵⁸ unhesitatingly spoke in favor of pure fifths. Early in the seventeenth century Robert Fludd (1574-1637) came into violent conflict with Kepler over matters of theology and tuning, and for metaphysical reasons considered no tuning worthy but the Pythagorean.⁵⁹

Caramuel. Malcolm. In the second half of the seventeenth century came Caramuel. (See p. 294). Then, early in the eighteenth century, Alexander Malcolm (cf. p. 299) related of spinet tuners: "some and even the Generality . . . tune not only their Octaves, but also their 5ths as perfectly Concordant as their Ear can judge, and consequently make their 4ths perfect, which indeed makes a great many Errors in the other Intervals of 3rd and 6th."⁶⁰ Truly a remarkable predilection for the Pythagorean tuning in England, recorded only one year before Bach wrote the first volume of the *Well-Tempered Clavier*!

Roussier. China. With Abbé Pierre Joseph Roussier (1716-1790 (?) the Pythagorean tuning system was in the nature of an obsession.⁶¹ He sought every opportunity to elevate "triple progression" and to scorn other tuning methods. Roussier heartily approved the old Chinese theory of tuning by perfect fifths. As early as 100 B. C. the pipe-lengths for twelve semitones in Pythagorean tuning were set down by Se-ma Ts'ien. King Fāng (3rd (?) cen., A. D.) extended from twelve to sixty the series of notes related by fifths, noting the close correspondence of the fifty-fourth note with the first, made use of in the European 53-division. Later theorists preferred to temper the scale, and Prince Chu Tsai-yü (1536-p. 1610) published a table of equal temperament in 1596.⁶² But in 1713 the Pythagorean system was officially declared the Chinese scale and it remains so in theory until this day, although the practice is far different.

Mersenne. It must be admitted that Roussier's was a solitary voice among European historians and theorists of his generation. Not so among practical musicians. Mersenne,⁶³ who had discussed for in-

⁵⁸*De Consonantiis, seu pro Diatessaron*, Antwerp, 1581.

⁵⁹Fludd's *Veritatis Proscenium*, Frankfurt, 1621, criticized Kepler's *Harmonices Mundi*, Linz, 1619. Kepler's reply was entitled *Apologia pro Opere Harmonices Mundi*, and Fludd made a second attack called *Monochordium Mundi Symphoniacum*, Frankfurt, 1623.

⁶⁰Malcolm, *op. cit.*, p. 307.

⁶¹*Mémoire sur la Musique des Anciens*, Paris, 1770; also in his notes on Père Amiot's *De la Musique des Chinois*, vol. 6 of *Mémoires concernant l'Histoire, les Sciences, les Arts, les Mœurs, les Usages, &c des Chinois*, Paris, 1780. Additional information about Chinese tuning in Maurice Courant's "Chine et Corée," in vol. 1 of Lavignac's *Encyclopédie de la Musique*, Paris, 1913; and in Édouard Chavannes' *Les Mémoires historiques de Se-ma Ts'ien*, vol. 3, Paris, 1893.

⁶²Cf. p. 295.

⁶³*Harmonie Universelle*.

struments a great variety of theoretical tuning systems in just intonation (in addition to the equal temperament that he advocated in practice), thrice mentioned the Pythagorean system as a practical method. He used pure fifths to tune the open strings of the violin family from E $\flat$ to E; the fifteen strings of the lyre from C to A $\sharp$; and the nineteen notes of the cithern—these last consisting of two interlocking series of 8/9 tones.

Violin. The lyre and the cithern have vanished from our orchestras, but we might expect to find Mersenne's view of violin intonation supported by later writers. It is true that Jean Philippe Rameau (1683-1764)⁸⁰ stated that violinists use tempered intervals in tuning their strings, thus in his opinion approaching equal temperament. Doubtless many violinists today do slightly flatten the fifths of their open strings, especially when playing with rigidly tempered instruments like the piano. But many contemporary violinists are taught to tune their strings by pure fifths, and it is a commonplace of violin technique to stop sharp notes higher than the enharmonic flat notes. A century ago, C. F. Fabricius⁸¹ said that equal temperament was in use for keyboard instruments, just intonation was approached by wind instruments, but "the string instruments on the other hand know nothing whatever of temperament, and the violin player stops B $\flat$, e. g., on a different string from A $\sharp$, and very properly stops the latter somewhat higher than the former." The oft-cited experiments of Cornu and Mercadier⁸² furnished scientific proof of the preference of string players for a Pythagorean intonation in melodic passages. A similar opinion was held by William Pole⁸³ and by Anglas.⁸⁴

Introspection. But not only violinists are inclined to the Pythagorean tuning. The present writer has had practical experience solely with piano and organ, both tempered instruments. He listens with respect, if not with conviction, to the belief that unaccompanied singers use just intonation. But in his own mental conception of pitches—evidently fixed before he had heard of any tuning system whatever—F $\sharp$ is always higher than G $\flat$. A query discreetly put within the circle of one's friends would probably disclose not a few individuals with the determined conviction that the sharped note is higher than the enharmonic flatted note, despite anything the books say.

⁸⁰*Génération Harmonique*, Paris, 1737, pp. 90, 91.

⁸¹"Ueber die Töne und Tonarten unserer Musik," *Allg. Mus. Zeit.* 35, 1832, col. 152.

⁸²Published in *Comptes Rendus* of the French Academy, 1869 and 1871. See Ellis' account in Appendix XX, Section G of his translation of Helmholtz's *Sensations of Tone*, 5th Edition, London, 1930.

⁸³*Philosophy of Music*, 4th ed., 1895, p. 138 ff.

⁸⁴*Op. cit.*, p. 206 ff.

Theory of Harmony. Although just intonation is generally believed to give the theoretical foundation for musical harmony, its claim does not remain unchallenged. If music were a static art like painting, the perfection of the single chord would be paramount. But music is above all things dynamic: harmonically considered, it creates a feeling of stress and then relieves it. The fundamental progression of roots in forming a cadence is by fifths, as E, A, D, G, C. Some musical theorists, forced to choose between just intonation and the Pythagorean tuning as a basis for harmony, believe that the relation of successive chords is the more important element. Thus Percy Goetschius in his harmony text,⁸⁵ says "there is only one scale that is *natural*," and proceeds to build this scale by perfect fifths, or as he calls them later, "harmonic degrees."⁸⁶

Conclusion. This article should not be construed as a plea for a return to the Pythagorean tuning, since equal temperament is far better adapted to harmonic music. It is a plea for a more sympathetic treatment of the Pythagorean system as a tuning theory. The researches of Helmholtz, who was not a trained musician, are largely responsible for renewed interest in just intonation during the nineteenth century, and many recent textbooks of physics contain tables of just intonation and equal temperament only. What these physicists do not realize is that just intonation, as they show it, is the worst of all proposed systems upon which to perform modern music. The beautiful correspondence of a few of its intervals with notes of the harmonic series is more than counterbalanced by the harshness of the remaining intervals.

Nor does the practice of just intonation rest upon an historical basis so far as can be learned from writers of the sixteenth, seventeenth and eighteenth centuries. True, within the past fifty or hundred years, mechanically complicated instruments have been invented upon which a high degree of consonance is possible. But none of these has attained lasting popularity. Edwin H. Pierce,⁸⁷ who was concerned with a fairly successful attempt to commercialize such an instrument, testifies that he and his colleagues found the just intervals insipid, preferring the more incisive thirds of equal temperament.

A further weakness of just intonation lies in the multiplicity of its intervals—two sizes of tone, four sizes of semitone, etc. The principle

⁸⁵*The Material Used in Musical Composition*, 14th ed., New York, 1913, p. 5.

⁸⁶It must be understood that no theory of harmony formulated by a modern theorist for a scale of semitones is intended to combat the long established practice of equal temperament.

⁸⁷"A Colossal Experiment in Just Intonation," *Musical Quarterly*, vol. 10, July, 1924.

of economy in art, logical in conception and aesthetically satisfactory, favors one size of tone only and at most two sizes of semitone. Almost twenty-five hundred years ago Pythagoras, with mathematical exactness, proposed a diatonic tuning system of equal tones and equal semitones. We have modified it somewhat to include in the bond of equality the elements of our chromatic and enharmonic systems. Our equal temperament, with no less of scientific accuracy, revives the ancient principle of equality in the modern world.

THE NABLA

In the *Annals of Mathematics* for 1896 (X, 127-155) a Japanese scholar, Dr. Shunkichi Kimura, published a paper on the *nabla* of quaternions. The word was not unknown to English and American scholars at that time, but it seems never to have been generally accepted in the mathematical vocabulary, at least in this country. Perhaps the reason is that quaternions never commanded the respect or attention of the mathematical world at large.

In an article by Professor A. S. Hathaway, published in 1891 (*Bulletin of the N. Y. Math. Soc.*, I, 66-74), the following passage is quoted from an address given by Lord Kelvin (then Sir William Thompson):

"I took the liberty of asking Professor Ball two days ago whether he had a name for this symbol*; and he mentioned to me *nabla*, a humorous suggestion of Maxwell's. It is the name of an Egyptian harp which was of that shape. I do not know that it is a bad name for it."

The word is found in various languages, as seen in the Latin *nabla*, *nablum*, or *nablium*, a synonym for *psalterium*, a psalter or harp. It appears in Hebrew as NBL, or *nebel*; in Greek as *nablion*, in Spanish as *nabla* ("an ancient musical instrument") and doubtless in numerous other tongues and forms. It would be interesting to search out other similar cases of names in the higher range of mathematics.

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*This was a quaternion symbol, a right triangle with the vertex of an acute angle pointing downward and an exponent 2 at the right.

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