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L'auteur cherche à décrire quelques-unes des opérations fondamentales de l'arithmétique pythagoricienne, particulièrement celles sur lesquelles s'appuie la théorie pythagoricienne de la musique. La seconde partie de l'étude analyse, selon le témoignage d'Archytas et de Platon, les gammes — autres que celle des Pythagoriciens anciens — fondées sur les notions de moyennes géométrique et « harmonique ».

1

THROUGHOUT WESTERN HISTORY number has from time to time been taken as the rule of art. Although never accepted by all artists of a given period and soon replaced by alternate aesthetic theories, mathematical explanations persistently and recurrently crop up in forms both old and new—sometimes based upon a recently developed mathematical operation, sometimes simply making a new application of an old, familiar one. Vitruvian proportions of architecture, Leonardo's canon of human proportions, Augustine's analysis of poetic rhythm, the lure of the golden rectangle, all illustrate the use of mathematics to clarify the bases of art. Such attempts go back to the Greeks—at least, they tried it first and in some cases most thoroughly—and among the Greeks the earliest and most thorough of all were the pythagoreans.

Pythagoras—so the story goes—invented the theory of music. He, or his disciples, drew attention to the fact that musical intervals could be expressed as numerical ratios and that the more consonant intervals had ratios with very small numbers, like 1:2. Pythagoreans are usually credited with dividing the octave by a fourth and a fifth,

constructing a scale using the whole-tone 8:9, and discovering the "pythagorean comma" or difference between 12 fifths and 7 octaves. There is, however, no easily available account of how these theorems are related to each other and to the mathematics of which they are a part. In this article I will try to describe a few of the basic operations of pythagorean arithmetic and such special operations as have to do with the pythagorean theory of music.

It is not easy to make out the history of pythagorean thought before 400 B.C., nor is it clear, even now, how much Pythagoras himself (who flourished around 532 B.C.) contributed to the arithmetic and musical theorems handed down under his name. There were a group of thinkers called "pythagoreans" active around the end of the fifth century B.C., shortly before Plato. Our information about pythagorean thought comes from or through this group—and not directly but through sources like Plato and Aristotle. Some scholars, Erich Frank in particular,¹ attribute most if not all pythagorean science to these latter-day pythagoreans. Others, however, are willing to grant that Pythagoras himself taught the theorems traditionally associated with his name, while some argue that certain theorems did not originate with Pythagoras but came from the Egyptians, Babylonians, or still more ancient layers of Near-Eastern civilization.² Even if these problems cannot be solved at the present time, we can try to

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Pythagorean Mathematics and Music

I

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put together a summary account of early pythagorean musical theory. Our account will reflect the state of pythagorean mathematics in 400 B.C.; it cannot penetrate earlier than that except by speculative reconstruction. In 400 B.C. the creative phase of pythagorean mathematics was drawing to a close, for thereafter it tended more and more toward academic numerology or mysticism. The same date, 400 B.C., also marks the end of the great period of Greek music, which flourished during the fifth century;³ hence this summary of early pythagorean mathematics gives a picture of the kind of theory that was associated with classic Greek music about which we know so little.

The pythagoreans are reported to have represented numbers with points, each point being a unit.⁴ These points were arranged in some appropriate pattern to give a picture of each number. Here, for example, is the number 4.

Example 1

It is important (Michel observes) to realize that these points give a picture of the very essence of number as conceived by the pythagoreans. These points are not the points of geometry, for they occupy a certain amount of space in the picture: any given number contains only a finite number of points, the number 4 having four points. Yet these points are themselves not geometric areas, but simply units. They are not divisible, but instead are "atoms," indivisible particles. In this respect they reveal a concept of mathematics concerned only with integers—that is, whole numbers—the most important feature of the relationship between pythagorean mathematics and pythagorean theory of music. Furthermore, the essence of a number is revealed by the way in which the point-units are arranged. Without thinking, we call the number 4 a "square," but the pythagorean picture above shows why it should be called a square number. For us, the point-numbers may seem needless effort; for the pythagoreans, they made possible an intuitive grasp of many mathematical truths. Since

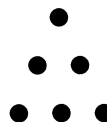
the pythagorean theory of music is dependent upon these truths, we must, for the moment, learn to count with points.

The simplest arrangement of points is in a straight line.

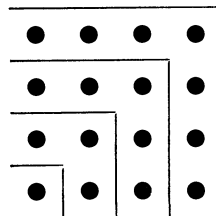
Example 2

All integers, of course, can be represented in this way. But the linear picture, being a trivial one, is for most purposes replaced by some other. Beginning with 3, points can be arranged in a triangle, that is, a plane figure instead of a line. Beginning with 4, as we saw, this plane figure can be a square; with 6, an oblong; with 8 a cube—a solid figure. In general, integers that can be expressed as the product of two factors ($2 \times 3 = 6$) are represented as rectangular numbers, those that are the product of three factors ($2 \times 2 \times 2 = 8$) as solid numbers. But for the purposes of music theory we need consider only plane numbers.

Triangular numbers occur as the sums of series: 3 is the sum of 1 plus 2; 6, the next triangular number, is the sum of 1 plus 2 plus 3.

Example 3

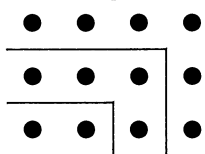
Clearly this series can be extended indefinitely. The generation of numbers in series was one of the most important operations of pythagorean arithmetic. Square numbers, too, were regarded as sums of series; even though squares appear to us to be the products of their sides, the pythagoreans generated square numbers from 1 in the following way.⁵

Example 4

The square number 4 was formed by adding onto 1 the three points enclosed by the right-angled lines. The next square number, 9, was formed by adding on the next 5 points similarly enclosed—and so for all subsequent square numbers. The pythagoreans gave a special name to this group of added points: they called it the *gnomon*, (“index”; compare *gnosis*). When a gnomon of a fixed length L is added to any square of length $L - 1$, it produces the next square. The gnomon for square numbers forms its own series, 3, 5, 7, 9, 11, etc.

The series of square numbers has remained a basic tool of mathematics right up to the present day, while other series studied by the pythagoreans have been discarded. Of these other series perhaps the most important, and the next in order, was formed by adding a rectangular gnomon onto the initial term 2 instead of onto 1.

Example 5



The resulting numbers are, of course, no longer square but oblong, originally called *promēke* or *heteromeke*.⁶ The series, as can be determined from example 5, is (2), 6, 12, 20, 30, 42, 56, 72, 90 . . . which by itself does not seem especially promising. In order to grasp its significance, we must shift our attention to the pictures of the point-numbers themselves, in particular to the relationship of the sides of the rectangles to each other—their ratios. It is clear at once that whereas the sides of any square number are always equal, the sides of oblong numbers (of the series started in example 5) always differ by 1, for the sides are successively 2 and 3, 3 and 4, 4 and 5, and so on. Thus the shape of successive square numbers is always the same, but the shape of numbers in this oblong series is always different.⁷ The oblong shape, however, gets more and more square as the oblong numbers get larger

There is only one series of square numbers, but many series of oblong numbers, for one can start from some number other

than 2; as long as a rectangular gnomon is added, the result will be a series of oblong numbers. Each series will have its own type of ratio that holds for every number in the series; the ratios of the series just described, for example, always show a difference of 1 between their terms—or stated more generally, the larger term of the ratio will exceed the smaller by an aliquot part ($\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc.) of the smaller. But each new series of oblong numbers will be characterized by a different type of ratio, unless the new series is simply a previous series expressed in terms twice as great, three times as great, or some higher multiple. The pythagoreans determined that there were six types of ratios in all.⁸

Example 6

equal	1:1, 2:2, 3:3...
multiple	1:2, 1:3, 1:4...
epimore	2:3, 3:4, 4:5...
multiple-epimore	2:5, 3:7, 4:9..., 2:7..., 3:10
epimere	3:5; 4:7; 5:7, 5:8, 5:9;...
multiple-epimere	3:8; 4:11, 4:15; 5:14,...

The types “equal” and “multiple” are self-explanatory, being the ratios of the sides of square numbers and of the multiples of square numbers. Epimores have also been discussed, being ratios whose lowest terms differ by 1. In multiple-epimore ratios, this difference between the terms is understood to be between the larger term and some multiple of the smaller: 2:5 is understood as $2:(2 \times 2) + 1$, whence the ratio is both multiple and epimore. In epimere ratios, the difference is not an aliquot part of the smaller term, but some more complex part—not “a part” but “parts.” Multiple-epimeres are analogous to multiple-epimores.

If the reader has the patience and inclination, he can verify for himself that all integer ratios fall into one or the other of these six groups. This six-fold classification, no small accomplishment in itself, was a tool for handling ratios, for comparing their size and structure—an especially useful tool when the ratios were large or complex. It is interesting to observe that these six kinds of ratios become more and more diffuse as one goes from equal to multiple-epimere, just as does the integer series itself as it departs from unity towards infinity. In fact, the series of integers seems to have

provided the pythagoreans with a model: not only did it furnish them with their subject-matter but in some sense with their method as well. Again and again one discovers that the truths of pythagorean science are the relationships inherent in the series of integers.

The pythagoreans based their musical theorems directly on their arithmetic. As a result, the kind of problem that could be taken up, as well as the kind of solution that could be found for it, depended upon the nature of pythagorean arithmetic. Since the pythagoreans dealt exclusively with integers (fractions being ratios of integers), they dealt with that aspect of musical sound that could be numbered with integers. This meant that their attention was focused on musical intervals, for these lend themselves readily to numerical expression. But—and this is the crux of the matter—the pythagoreans could deal only with those musical intervals that could be expressed as the ratios of integers. Not all intervals can be so expressed; many intervals, including all those drawn from our modern scale of 12 equal semitones to the octave, are “irrational” quantities having no exact expression in the realm of integers.

Not only did integer arithmetic commit the pythagoreans to the study of particular kinds of musical intervals, but it also injected a certain sense of value into their dealings with these intervals. In working with the integer series, one automatically uses the numbers at the beginning of the series as tools with which to work on the larger numbers. One reduces ratios to lowest terms, one finds “lowest common denominators,” one takes small-number ratios as ideal forms or prototypes of larger ones. Because they consist of fewer elements, the smaller numbers are obviously simpler; thence they easily acquire the virtue of being “better.” The pythagoreans’ reverence for unity is well-known, the resulting cult of number-magic often derided, but such mysticism seems merely to have been a crustation formed on a hard core of arithmetic truth. In any case, the pythagorean theorist of music was led, almost inevitably, to establish a hierarchy of value among musical intervals, favoring those with small-

number ratios. This was not, however, completely a function of his arithmetic approach to music, for these same intervals actually sounded simpler than other intervals. In the case of music, the arithmetic method was solidly confirmed by common-sense, empirical evidence. No wonder that in music the pythagoreans saw a patch of the basic fabric of the universe.

Pythagoreans classified intervals according to the types of ratios associated with them; this principle was applied differently, however, by different generations. Early pythagoreans, it seems, divided intervals into two groups, those whose ratios were multiple or epimore, and those whose ratios were more complex.⁹ Here again, there was empirical support for this grouping: the first group sounded simpler—or, as they said, the notes of these multiple or epimore intervals blended better with each other. But the early pythagoreans included among the consonances only the octave (1:2), twelfth (1:3), double octave (1:4), fifth (2:3), and fourth (3:4), the rest of the epimore series, from 4:5 on, not being considered consonant. For this the early pythagoreans have often been criticized, first by Ptolemy (130 A.D.),¹⁰ then by theorists and historians of later ages for whom the epimores 4:5 and 5:6 have become consonant. But the pythagoreans had good reasons for limiting the number of consonances, reasons no more arbitrary than those of the modern overtone theorist who stops short at the fifth overtone because he, in turn, cannot accept as consonant the interval 6:7. The pythagoreans limited the number of consonances because, as mathematicians, they were deeply involved with the properties of the small-number ratios; as long as these, the first and most obvious types of ratio, offered the mathematician an interesting field of study, there was no reason to move on to other types. One of the most striking facts of pythagorean arithmetic—let me say, of the integer series itself—is the extraordinary wealth of relationships at the beginning of this series. When the numbers are small and the differences between them relatively large, they are related to each other in manifold ways. The first number added to itself produces

the second, a relationship found nowhere else in the integer series. The second added to itself produces its own square, again a unique result. The first and second added together produce the third, while the second and third added together skip over the fourth to produce the fifth. The squares of the third and fourth add up to the square of the fifth—the first integer example of the “Pythagorean Theorem.” The modern observer, exposed to the possibilities of non-Aristotelian logic and non-Euclidian space, may offer the objection that all these striking relationships among the small integers are true only by tautology, claiming that the integer series is by definition that series that produces such relationships. But perhaps integers still seem so “natural” to us that we can glimpse something of the power their relationships had for earlier minds.

Having found the ratios of the most consonant intervals among the small numbers 1:2, 2:3, 3:4, the pythagoreans then saw that the ratios 2:3 and 3:4, when combined, circled back to form another octave, 2:3:4, which in turn formed a double octave with the first ratio, 1:2:4. This is the only case in the integer series where two consecutive ratios (2:3 and 3:4) produce the ratio immediately preceding (1:2)—another example of a unique relationship among the small numbers. This relationship is no less striking in musical sounds than it is in numbers. But then the early pythagoreans had also observed that the numbers 1, 2, 3, and 4 added up to 10, and the coincidence of this with the musical relationship just described proved irresistible. The “tetrad,” or first four numbers, became a cornerstone of pythagorean theory, and with that the number of consonances was fixed.¹¹

Later pythagoreans broadened this classification of intervals by recognizing that epimores, even when made of numbers higher than 3:4, were still as a class simpler than epimeres. This principle was stated much later by Ptolemy, who called epimore intervals “*emmelic*,” that is, suitable for melodic progressions. He used them as much as possible for the construction of scales—in preference to “*ecmelic*” or unmelodic intervals with epimere ratios.¹² But

the same principle (as we will see later) is evident in interval calculations that go back to Archytas in 400 B.C.; already by that time pythagoreans had begun to distinguish between various kinds of non-consonant intervals according to whether their ratios were epimores or epimeres. The intervals in question are major and minor thirds, whole-tones, and fractions of whole-tones. In musical practice these intervals existed in various sizes: the major third, for example, could vary between the ratios 64:80 (4:5) and 64:81, the first size being—to our ears—flatter and sweeter. Early pythagoreans used the size 64:81, for reasons we will discover further on. Later theorists, however, preferred the size 64:80, which is the epimore 4:5, rejecting 64:81 as a rather complex epimere in which the larger “side” exceeded the smaller by $\frac{17}{64}$ ths. Similarly, epimore ratios such as 7:8, 8:9, 9:10, 10:11 were preferred for whole-tones; 15:16, 16:17, 17:18 for semitones; 27:28, 35:36, 38:39 for quarter-tones.

This may seem to us acoustical quibbling. It seemed that way to Aristoxenus, who around 320 B.C. tried to deal with all intervals smaller than a fourth in terms of their function rather than their acoustical size.¹³ Indeed, what practical basis do we have for judging the whole-tone 800:900 (an epimore) to be simpler, more “melodic” than, say, 805:900 (an epimere)? As long as this interval sounds like a whole-tone and behaves like one, is its quality significantly related to the type of ratio it represents? Perhaps the answer is “no.” Still, there are strong arithmetic reasons for arguing the other way. If the earlier terms of the epimore series are consonant—and they indubitably are—why should the series abruptly become dissonant after 4, or 5, or ∞ terms? And if an interval like 4:5 or 9:10 is in some subtle sense “consonant” by virtue of being an epimore, then it must be more “consonant” than the epimeres (and irrational values) that surround it. Clearly the difficulty arises from the fact that as epimores get more complex their consonant property is harder to perceive, and as a result they seem less different from their dissonant neighbors. In any given century musical taste and style draw a hazy bound-

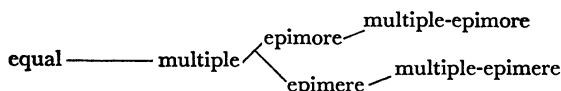
ary between intervals simple enough to be consonant and those complex enough to be dissonant. As time goes by, the consonant group tends to include higher and higher epimores. It may be of interest to note that twentieth-century style is rapidly forcing the acceptance of new consonances, one of which, lying between a major second and a minor third, could easily have the size 6:7—even though orthodox theory has up to now admitted as consonant no interval smaller than 5:6.¹⁴

While the epimore series leads to ever smaller intervals, the multiple series leads in the opposite direction to larger and larger ones. At the beginning of this series stands the octave, whose status has never been questioned. Beyond the octave lie other multiple intervals that also have a claim to consonance, even though they, too, diminish in consonance as they depart from unity. These intervals are the twelfth (1:3), double-octave (1:4), double-octave-and-third (1:5), and so on. While most people would grant the status of consonance to these three, they would probably balk at the double-octave-and-seventh (1:7); but the argument that applies to epimores applies here too. Ptolemy asked why the ratio 1:5 could not be included among the consonances.¹⁵ This objection, a reasonable corollary to the admission of the higher epimores as “emmelic,” attacks the limit

principle as a tool for explaining the phenomenon of musical consonance.

Taken all in all, the six-fold classification of ratios seems to me to give a remarkably true picture of the qualities of rational intervals. It divides these intervals up in a way that is simple and clear, yet manifold enough to reflect sonorous reality. This classification makes it possible to distinguish clearly between consonances of the multiple series (octave, twelfth, etc.) from those of the epimore series (fifth, fourth, etc.). The one series gets larger, the other smaller; both depart from the octave (1:2), since this ratio is the first in the multiple series—and also, in a curious way, the root of the epimore series. The ratio 1:2 is not an epimore, of course; still, its two terms differ by 1, which is a characteristic of epimores in their lowest terms. Occurring immediately before the first epimore 2:3, the ratio 1:2 has something of its form. In the same way, octave and fifth are both perfect consonances, only the octave is more so. Like everything else connected with the integer series, this classification of intervals becomes more diffuse as it moves further from unity; even so, the distinction of epimores and epimeres is not without musical significance. The reader can check this against his own experience, for which purpose it is helpful to arrange the six classes in the following way.

Example 7



the early pythagoreans placed on the group of consonances, while accepting the principle that consonance was a function of multiple or epimore ratios. Then Ptolemy further objected that the octave-and-fourth (the eleventh) should be considered consonant on the grounds that octave-compounds of consonances must also be consonant. Since the ratio of the octave-and-fourth is 3:8 (a multiple-epimere), it was never included in the pythagorean class of consonances. Here Ptolemy was attacking the principle itself, and here again there is much that could be said in defense of this

It is perhaps of interest to the modern reader to observe that the multiple ratios occur—as a group and in order—between a fundamental and each successive overtone; the epimores occur—as a group and in order—between successive pairs of adjacent overtones; the remaining classes occur between non-adjacent pairs of overtones. This is only to say that the overtone series is a sounding image of the integer series. From a pythagorean point of view, such a phenomenon, while interesting in itself, does not make the results any more true or the consonant intervals any more consonant.

For the pythagoreans, intervals are consonant because they sound consonant and because their ratios are simple. It is easy to underestimate the significance of these conclusions, particularly in the light of later developments; we have to remember that the pythagoreans came to their conclusions before there were any later developments. There are, of course, other ways to deal with musical intervals. Aristoxenus in 320 B.C. dealt with intervals at length in a way no less rigorous than that of the pythagoreans, yet without a single reference to arithmetic ratios. But the point is, the pythagoreans came first—and coming first, used a mathematics that counted things simply by means of units. There was no reason to invent a more sophisticated mathematics until this first, most obvious kind had solved all the problems it could and revealed the nature of those it could not solve. In this respect too, the pythagoreans' theory of consonance, dealing as it did with the simplest intervals, was the musical corollary of their arithmetic.

II

Once musical intervals were expressed as arithmetic ratios, they could be combined by using arithmetic operations. Here, however, it seems that it was music that provided the method for arithmetic—at least in the beginning.¹⁶ The notion of compounding ratios was a difficult one to express in Greek mathematics, both in arithmetic and in geometry. Modern musical theory tells us that to compound two intervals we multiply their ratios ($\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$), and to double an interval, or compound it with itself, we raise its ratio to a power, ($(\frac{2}{3})^2 = \frac{4}{9}$). The Greeks did not express powers as exponents but rather as geometrical shapes like squares and cubes.¹⁷ And although the pythagoreans were capable of carrying out a multiplication of ratios—indeed of very extended calculations¹⁸—it is not clear that the early pythagoreans compounded ratios in this way. Even if they did, it seems certain that they would have regarded such multiplication as a purely mechanical operation: it is difficult to see what theoretical significance they could have attached to it.

On the other hand, when expressed in terms of musical intervals, the compounding of ratios is so clear as to be self-evident.¹⁹ A fourth and a fifth “add up” to an octave just by being placed next to one another. Once it was established that their respective ratios were 3:4 and 2:3, the method of compounding these ratios was equally self-evident: they “added up” to the ratio 1:2 by virtue of the series 2:3:4. This operation (at least in the case just described) is presumably as old as Greek mathematics, if not older. It is not hard to imagine it as the model for further combination of ratios.

Upon closer inspection, the intervals of fourth and fifth can be added up so easily only because their ratios happen to be continuous, that is, share a common term. Because of this, no multiplication is necessary; in fact, multiplication is irrelevant, the result being explicit in the number series itself. When the two ratios are *not* continuous, however, their compounding is not self-evident and some further operation is needed. To compound the ratios 3:4 and 8:9, for example, we must change one or the other ratio to make the two continuous. Clearly it is the smaller that must be changed, the larger being incapable of reduction to the terms of the smaller. The smaller ratio is easily expressed as 6:8, whence the continuous series 6:8:9, and the ratio 6:9 or 2:3 as the resulting compound. The Greeks were apparently willing and able to attack such problems by simple trial; out of such trials could have come the purely mechanical operation of multiplying ratios.

The tetrad also contained a model for compounding a ratio with itself, 1:2:4. Stated differently, this is a special case of a continuous series in which every ratio is equal to every other. This kind of series is “in proportion,” *ana logon*,²⁰ and is called an *analogia*. When compounding a fifth (2:3) with itself, we are forced as before to express the ratios in higher terms in order to make them continuous: 4:6:9. But unlike the series for octaves, 1:2:4, this series for fifths comes to an end; in order to compound three fifths, we must raise to still higher terms. This operation, an important one in pythagorean arithmetic, leads to the following super-series.

Example 8

2	4	8	16	32	...
3	6	12	24	48	...
	9	18	36	72	...
		27	54	108	...
			81	162	...
				243	...
					...

Each column represents a continuous series of fifths, in the ratio 2:3.

It is interesting to observe that the gnomon of each column is given by the preceding column. Even more interesting, the top of the second column is the square of the first, the top of the third column the cube of the first, and so on; the same is true of the bottom of each column. Stated more generally, the tops and bottoms of the columns form two continuous proportions, emanating from the terms of the original ratio, and are linked vertically by other continuous proportions in the ratio 2:3.²¹

Once the principle is discerned, any size interval may be compounded with itself by the same method. Thus the arithmetician finds general solutions for the specific problems encountered in combining musical intervals. Confronted with the result, however, the theorist of music is moved to carry out yet another operation that would not occur to an arithmetician. In music the ratio 1:2 has a peculiar auditory force: the notes of the octave are much more alike to the ear than the terms of its ratio are to the eye. The theorist of music, therefore, is inclined to use the octave as a measuring stick for other large intervals, such as the sum of several fifths or fourths. Looking at the series of two fifths (example 8, second column), we can see that a note an octave from 4 would be 8, which note would form with the note two fifths away from 4 the small interval 8:9. Skipping over to the column headed "16," a similar calculation reveals the small interval 64:81 as the difference between four fifths and two octaves; logically enough, 64:81 is double the interval 8:9. In the same way, two fourths (9:12:16) fall short of an octave by 16:18, or 8:9 again.

These small discrepancies came to be very important to the pythagoreans, who found in them a way to handle certain op-

erations that gracelessly refused to come out even. One remarkable feature of the sixfold classification of ratios is that no ratio ever belongs to more than one class. This has the corollary that if any ratio is compounded with itself any number of times, the result will never equal any other such compound of any other ratio. Thus no number of fifths, or of thirds, or of any other interval, ever adds up to any number of octaves.²² Sometimes, to be sure, two different series come within a hair's breadth of coinciding. These extremely small differences have always attracted the attention of theorists. The pythagoreans, who discussed them first, called them "commas," and one comma in particular is still called the "pythagorean comma." It is usually described as the difference between 12 fifths and 7 octaves, but while the pythagoreans could certainly calculate it that way, they probably arrived at the result differently. When Aristoxenus indicated that the octave could be divided into six equal whole-tones, he was taken to task by an anonymous writer²³ who demonstrated what every pythagorean knew, that the sum of six tones each 8:9 missed being an octave by the rather small quantity 524288:531441. Here is the demonstration.

Example 9

8	64	512	4096	32768	262144	($\times 2 =$
9	72	576	4608	36864	294912	
	81	648	5184	41472	331776	
		729	5832	46656	373248	
			6561	52488	419904	
				59049	472392	
					531441	524288)

It is important to notice that pythagorean tunings are not "out-of-tune" by the pythagorean comma—a common misunderstanding. It could never occur to a pythagorean to tamper with the octave, the interval closest to unity. The demonstration was originally carried out, as we just saw, to show that six whole-tones did *not* exactly divide an octave. The pythagoreans drew from this the conclusion that the octave would have to be divided in some other way.

The early pythagoreans did not, then, construct their scale by a "cycle of fifths"; instead, they sought both the principle and

erated in some more mysterious way directly from the womb of unity itself.

(To be concluded in the Spring issue.)

					<i>Example 10</i>			
tone	{	8	64	192		3		
		9	8	72	216			
		9	81	243			256	4
						limma		

This division of the octave, obtained not by a “cycle of fifths” but by projecting the tone 8:9 inside the fourth, is perhaps the oldest one used by the pythagoreans²⁴ and in some ways the most characteristic. It uses the principles inherent in the beginning of the integer series more economically than any other. For if we reflect on the matter, we see that in some sense the fourth itself is a limma, left over from the projection of the fifth back into the octave. The fifth, in its own way, is a limma, left over from the projection of the octave forward into the twelfth (1:3). Only the octave seems to remain aloof from this process, being gen-

³ See I. Henderson, "The History of Greek Music," in *The New Oxford History of Music*, I: *Ancient and Oriental Music*, ed. E. Wellesz (London, 1957), p. 376 ff.; R. P. Winnington-Ingram, "Greek Music (Ancient)" in *Grove's Dictionary of Music and Musicians*, ed. E. Blom (5th edition, London 1954), III, 772.

⁴ Pythagorean arithmetic is expounded at length in three very late writers, Nichomachus of Gerasa

(1st century A.D.), Theon of Smyrna (2nd century A.D.), and Iamblichus (3rd–4th century A.D.). My immediate source for information concerning pythagorean arithmetic is a comprehensive volume by P. H. Michel, *De Pythagore à Euclide* (Paris, 1950), from which I have drawn the materials necessary for this discussion. For point-numbers, see Michel, pp. 295 ff.

⁵ Michel, pp. 304 ff.

⁶ Michel, pp. 311 ff.

⁷ Aristotle comments upon the "same" and the "different" as elements of pythagorean arithmetic in the *Physics*; see Kirk and Raven, *op. cit.*, p. 243.

⁸ Michel, pp. 348 ff. I have omitted the "sub-classes" or inversions, and slightly rearranged those given in order to present a clearer picture.

⁹ This classification is attributed to the "older" pythagoreans by Ptolemy in his *Harmonika*, ed. I. Düring (Göteborg, 1930), trans. Düring in *Ptolemaios und Porphyrios über die Musik* (Göteborg, 1934), p. 29.

¹⁰ Düring, p. 31.

¹¹ See Kirk and Raven, *op. cit.*, pp. 229 ff.

¹² Düring, pp. 32 ff.

¹³ Aristoxenus disposes of the pythagorean approach in one sentence of his *Harmonics*, ed. and trans. H. S. Macran (Oxford, 1902), p. 189.

¹⁴ Cf. the speculations advanced on this subject by D. Kraehenbuehl and C. Schmidt, "On the Development of Musical Systems," *Journal of Music Theory*, VI (1962), 32.

¹⁵ Düring, *op. cit.*, p. 31.

¹⁶ Here I am following an idea put forward by P. Tannery in a very interesting article, "Du rôle de la musique grecque dans le développement de la mathématique pure," *Mémoires scientifiques*, III (Paris, 1915), 68 ff.; the article first appeared in 1902. Tannery felt that Euclid's—that is, Eudoxus'—ap-

proach to the operation of compounding ratios (Euclid's *Elements* VI, proposition 23) suggested an origin in music, on the grounds of certain peculiarities of terminology that made no sense in arithmetic but were self-evident in music. It seems to me, however, that similar peculiarities are apparent in pythagorean arithmetic itself—as preserved in Euclid. One can compare proposition 4 of Book VIII as the arithmetic equivalent of the geometric operation described in VI, 23.

¹⁷ See Tannery, p. 71.

¹⁸ See T. L. Heath, *The Thirteen Books of Euclid's Elements*, trans. and commentary (Cambridge, 1908), II, 119, for an example.

¹⁹ Tannery, p. 72.

²⁰ See Heath, II, 129, 117 ff., 292 ff.

²¹ See Michel, pp. 360 ff.

²² The same principle is expressed differently by J. Yasser, *A Theory of Evolving Tonality* (New York, 1932), pp. 117 ff.

²³ The *Sectio canonis*, formerly attributed to Euclid, ed. C. Jahn, *Musici scriptores graeci* (Leipzig, 1895), p. 113; French trans. Ch. Em. Ruelle, *L'Introduction harmonique de Cléonide; La division du Canon d'Euclide le géomètre; Canons harmoniques de Florence*, in *Collection des Auteurs grecs relatifs à la musique*, III (Paris, 1894), 50 ff.

²⁴ R. P. Winnington-Ingram, "Aristoxenos and the Intervals of Greek Music," *Classical Quarterly*, XXVI (1932), 200, admits that pre-Platonic theorists used the tone 8:9 to construct a scale, but for some reason is reluctant to attribute this construction to the pythagoreans. The fact that Plato describes this procedure when reporting on pythagorean mathematics in the *Timaeus* seems to me—in the absence of contrary indication—sufficient evidence for considering it to be pythagorean