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- 127 **E. EUGENE HELM** ("The Vibrating String of the Pythagoreans") is associate professor of music at the University of Iowa. A graduate of Southeastern Louisiana College, Louisiana State University and North Texas University (where he received a Ph.D. in musicology). Helm writes: "My interest in resounding strings undoubtedly goes back to my undergraduate days, when I was a misguided physics major. I remember that I was the only student in sophomore physics lab who, during our search for loops and nodes, could make the piece of paper jump off the string of the monochord at the very first pluck. Being a misguided violinist at the time, I knew perfectly well where the loops were."

THE VIBRATING STRING OF THE PYTHAGOREANS

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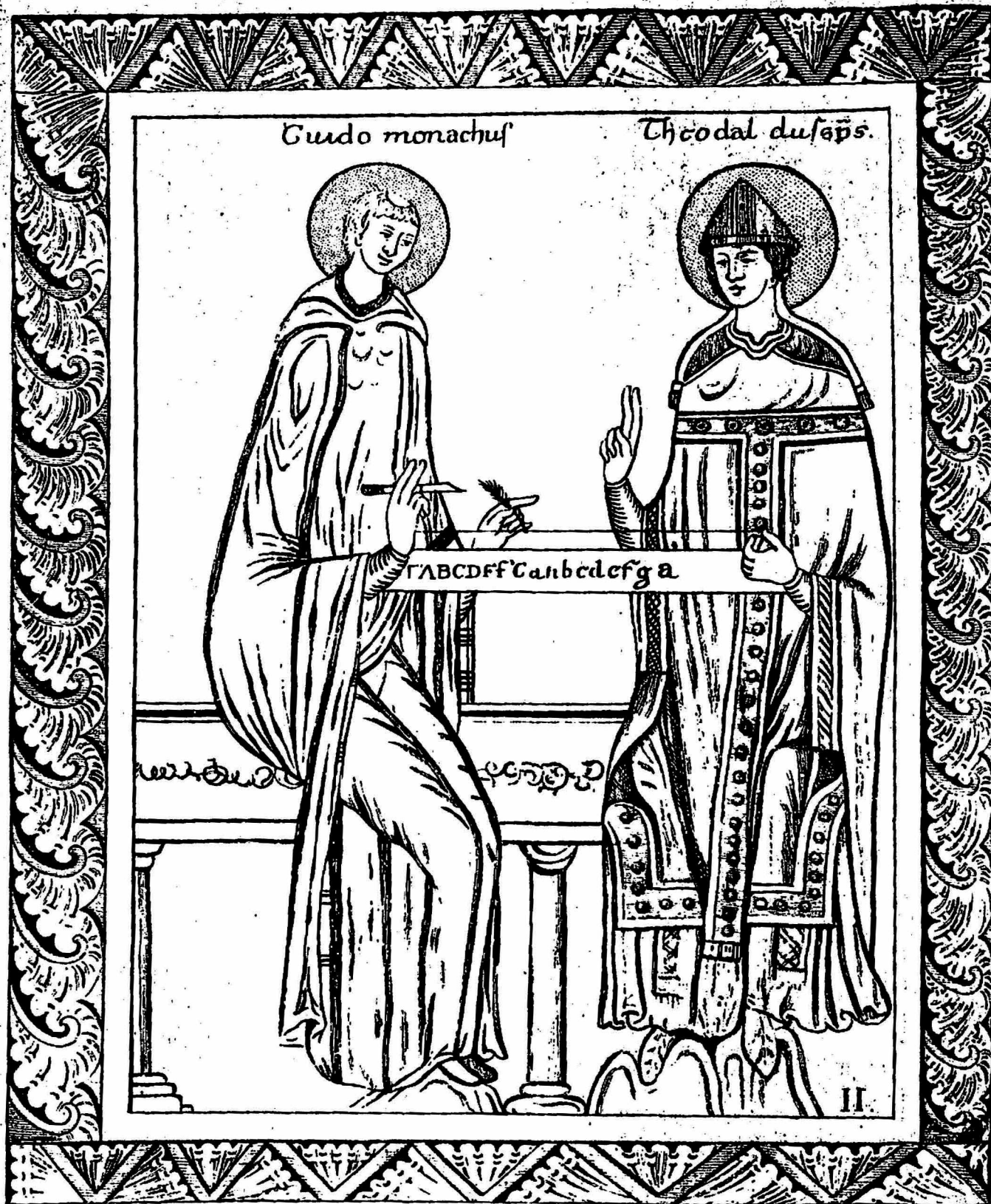
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MONOCHORD, the most effective demonstration of the Pythagorean laws of proportion, consists of a single string stretched on a frame. In this print, from a 12th-century manuscript, the 11th-century theoretician Guido d'Arezzo is explaining the divisions of the

string to Archbishop Theobald. The letters along the side of the instrument are the notes of the scale, beginning with gamma, or G, which represents the total length of the string. The pitch of a note depends on the length of string that has been left free to vibrate.

The Vibrating String of the Pythagoreans

The monochord gave rise to far more than Western musical scales. For the Greeks and those who followed them music was number, and the ratios of the scale were manifested in nature and in the arts

by E. Eugene Helm

C₂

Igor Stravinsky once observed that musical form is "far closer to mathematics than to literature—not perhaps to mathematics itself, but certainly to something like mathematical thinking and mathematical relationships." Stravinsky spoke from the viewpoint of a composer who, like many of his colleagues, has consciously made "something like mathematical thinking" an integral element of his music. This element is widely regarded as a hallmark of modern music. What is seldom appreciated, even in the musical world, is that the roots of the relation between music and mathematics extend deep into antiquity. The way in which the two were betrothed and the course the marriage has taken over the centuries make a beguiling and instructive tale.

They were plighted, these two, by the Greeks—more precisely, by the school of Pythagoras in the sixth century B.C., although their courtship certainly began, perhaps even earlier, among the Chaldeans, Egyptians, Babylonians and Chinese. During the next few centuries the Pythagoreans joined music and number irrevocably by means of a vibrating string. Whether the string was actually mounted and measured or was only described in speculative treatises and imaginatively or pedantically discussed, it has vibrated in the conscious and unconscious minds of musicians, mathematicians, philosophers, astronomers, physicists and architects ever since.

In its simplest form the basic Pythagorean doctrine relating number to music can be described as follows. Pluck a stretched string of any length and allow it to vibrate; it will sound a certain pitch. Allow only half of it to vibrate, and the pitch will rise an octave. If two-thirds of the string vibrates, the pitch will rise a fifth above the one produced by the total

length. For instance, if the total length produces C, two-thirds of the string will produce G. (The interval C-G is called a fifth because five lines and spaces on the musical staff are traversed in going from one to the other, counting C and G.) Three-fourths of the string will yield a pitch a fourth higher than the total length (F, if the total yields C); eight-ninths, a whole step, or a second, higher (D if the total yields C), and so on. In time the fractions become more complex and the two notes represented by the resulting intervals become more dissonant if they are sounded together. The fraction for the half-step, for example, is $\frac{243}{256}$.

The Pythagoreans used these musical facts to construct scales. Gradually the relation of the vibrating portion of a string to the entire string came to be expressed in ratios, such as 1 : 2 for the octave, 2 : 3 for the fifth and so forth. Soon it was noted that the most harmonious intervals were those with the simplest ratios. For example, 2 : 3 (the fifth) was much more harmonious than 8 : 9 (the whole step). Both the intellectual appeal of simplicity and the sensuous appeal of harmoniousness made the simple relations come to be regarded as superior to complex ones. Harmonious sounds, said the Pythagoreans, are produced by ratios expressible as whole numbers, and the simpler the ratios—the smaller the whole numbers expressing them—the more consonant the sound.

It followed from these attitudes that the octave, the fifth and the fourth were regarded as musically superior to other intervals. Indeed, since early medieval times the octave, fifth and fourth have been called the "perfect consonances." Until the end of the Middle Ages these intervals were the theoretical basis of almost all Western polyphony.

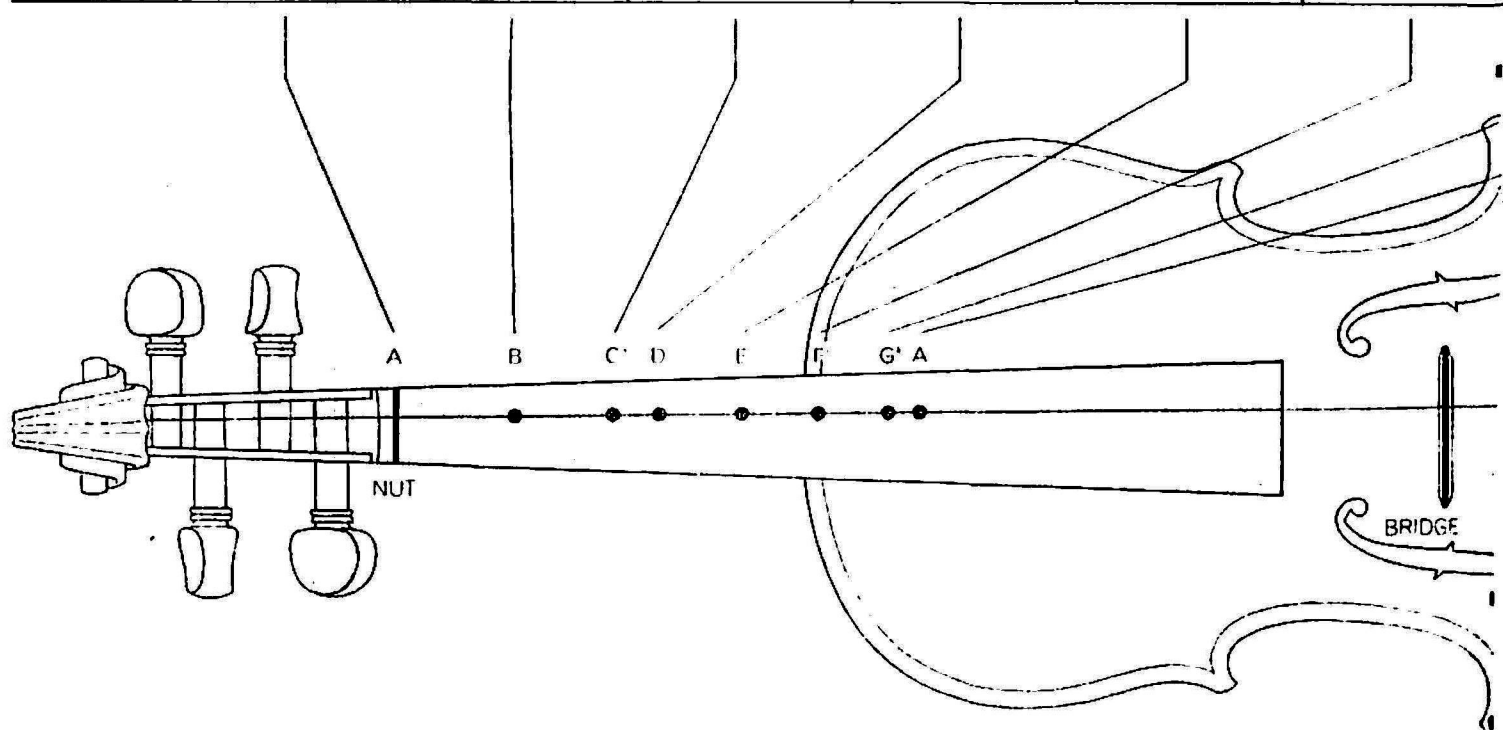
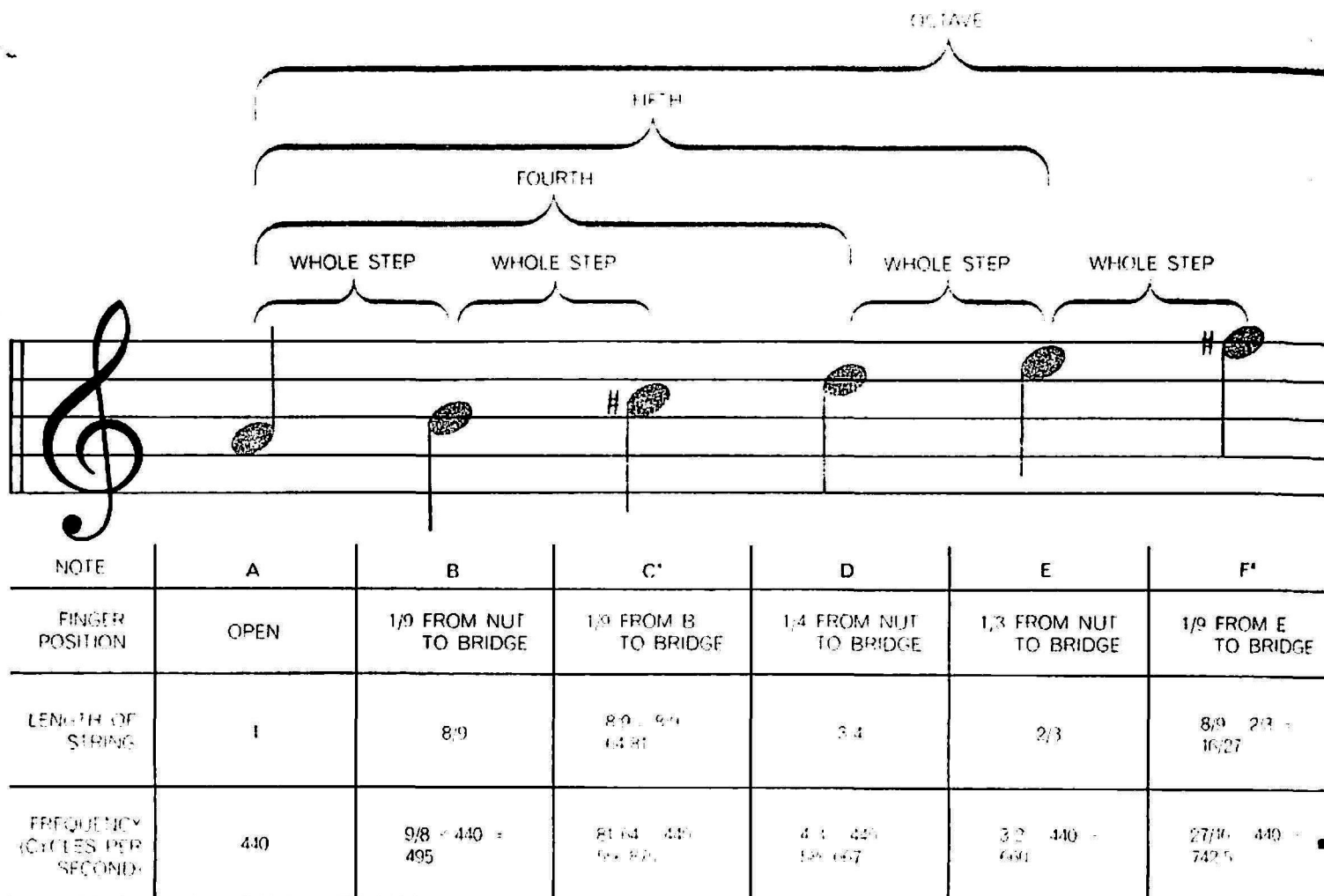
To the fascination of the Pythagoreans

the same sets of numbers seemed to keep cropping up in other matters. A single order, expressible in number and ratio, evidently governed all the rhythms of nature: the four seasons, the two tides, the pendulum-like fluctuations of human affairs—in fact, all the machinery of the universe from microcosm to macrocosm. The most succinct expressions of the numbers and ratios governing the orderly events of nature seemed to be precisely those that applied also to music. It is small wonder, then, that Pythagorean thinkers came to regard music as a key to all being.

A notable example of their thinking is the myth of the "music of the spheres." The Pythagoreans believed that bodies moving in space produced sounds unheard by ordinary mortals, and that bodies moving rapidly produced sounds of higher pitch than the sounds produced by bodies moving slowly. Greek astronomy of the sixth century B.C. held that the greater the distance of a planet from the earth, the more rapidly the planet moved. Moreover, the distances between planets and the ratios between speeds of planets (relative to the earth) were both thought to be harmonically determined, that is, expressible in whole-number ratios. Therefore the sounds produced by the planets harmonized with one another.

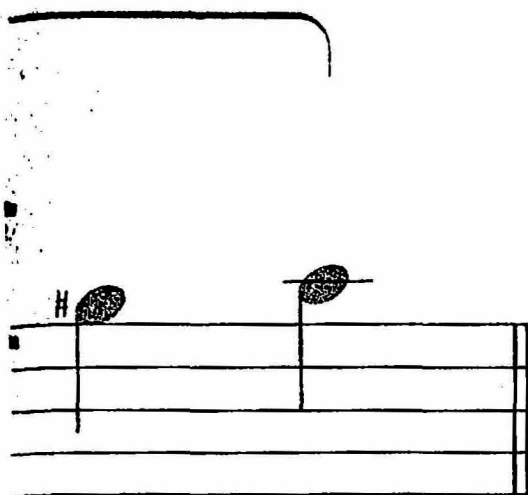
"Music of the spheres" also meant the mathematical ordering of the physical universe: the regulation of the world by laws of musical proportion. In this sense music was mathematics. Thus it was primarily the Pythagoreans who established music as a mathematical discipline, causing it eventually to be put into the quadrivium—the standard three-year course of study in medieval universities—along with the other mathematical studies of arithmetic, geometry and astronomy.

Plato, under the influence of Pythagorean

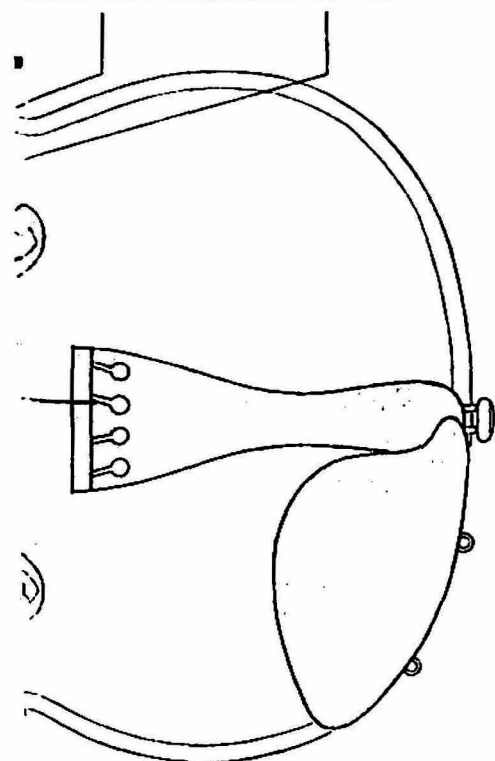


PYTHAGOREAN SCALE (not quite the same as the modern "equal-tempered" scale) is derived by dividing a string, as illustrated here with the A string of the violin. Sounding the total length from nut to bridge produces an A, with a frequency of 440 cycles

per second. (Frequencies are a modern concept, unknown to the Pythagoreans.) Stopping the string by placing the finger 1/9 of the way from the nut to the bridge leaves 8/9 of the string open, producing a B, and so on. The only intervals used to construct the Py-



G'	A
9 FROM F' TO BRIDGE	1:2 FROM NUT TO BRIDGE
$8/9 \cdot 16/27 = 128/243$	$1/2$
$243/128 \cdot 440 = 835.312$	$2 \times 440 = 880$



Pythagorean scale are the octave, fifth, fourth and whole step; the other intervals, such as the third (A to C-sharp) have complex ratios that must be derived from the simpler ones.

reanism, probably did as much to raise music to this level as the Pythagoreans themselves did. He gives a description of the music of the spheres in Book X of the *Republic*. His classic view of a world created according to musical proportion appears in the *Timaeus*.

Pythagorean and Platonic doctrines of music as number and proportion and music as part of science and mathematics were passed on almost intact into the Christian era. It is no coincidence that Ptolemy, the great astronomer of the second century, was also a leading music theorist. Nor is it surprising that the medieval thinkers who followed him could hardly begin to scrutinize the universe quantitatively without first taking a long look at music.

The chief medieval transmitter of the ancients' ideas on musical proportion was Boethius, the sixth-century Roman philosopher and mathematician. He classified the proportions and helped to establish a new terminology expressing them in words: *proportio dupla* for 1 : 2, *proportio sesquialtera* for 2 : 3, *proportio sesquitercia* for 3 : 4 and so on. His name is most remembered, however, for another refinement of the old doctrines: the division of music into three types. One was *musica mundana*, the harmony of the spheres; the second was *musica humana*, the harmony of the human soul and body, and the third was *musica instrumentalis*, the actual playing and singing of music. The third category, which is now considered the main aspect of music, was in the time of Boethius regarded as the lowest of the three forms. Playing and singing were only tiny reflections of the greater harmonies embodied in the other categories. The true musician was the man who knew music as part of a celestial order; the mere performer was only a servant. As Boethius put it, the performer was "separated from music" because "physical skill obeys like a handmaid while reason rules like a mistress."

Medieval treatises on music were often slavish restatements of the old doctrines as Boethius had defined them. Philosophers and mathematicians, as well as music theorists and composers, commonly felt obliged to pay their respects to music as a traditional mathematical discipline before proceeding to more contemporary matters. Standard operating procedure seemed to be to quote respectfully from Boethius (or to steal his ideas without acknowledgment) and to include a drawing of Pythagoras striking sonorous bodies or adjusting strings of various lengths. An instrument called

the monochord became widely used as a means of illustrating the ancient musical laws. This instrument, dating back to Greek times, consisted of a single string mounted on a flat surface and was provided with movable bridges and an accurate means of marking off certain lengths of the string.

Throughout the Middle Ages this speculative side of the musical art ran parallel to music making. Although folk music and the music of such practical artists as troubadours and minnesingers were relatively untouched by speculation, the all-pervading ratios and the music of the spheres were not far distant. The great monasteries, and later the cathedral schools and the universities, kept the old mathematical traditions alive and flourishing. For example, one of the writings of Pietro d'Abano, who in 1315 was Distinguished Professor of Medicine, Philosophy and Astrology at the University of Padua, was a treatise entitled "Is Musical Consonance Found in the Pulse?" Occasionally the mathematical way of thinking affected music in the most direct way. Philippe de Vitry's treatise *Ars Nova*, which dates from about 1320, presented a revolutionary system of metrical organization that inaugurated a new musical age and that had to be justified in traditional mathematical fashion. By the end of the 14th century French secular music had arrived at a degree of rhythmic complexity undreamed of by De Vitry and hardly equaled in the 20th century. (This music, incidentally, has recently been widely admired by a number of contemporary composers and avant-garde musicians.)

One would be remiss to discuss the laws of musical consonance as they applied to music alone in the Middle Ages. Musical proportions appear almost everywhere in the thought and art of medieval Europe. To St. Augustine the octave seemed to be rooted in the very being of even the most untutored man and therefore must have been implanted in man's nature by God himself as a means of conveying to human ears the meaning of redemption. To the School of Chartres in the 12th century the garbled fragment of the *Timaeus* then available had almost the stature of Holy Writ. The scholars of Chartres discovered musical proportions in the dimensions of Solomon's temple and built such proportions into their own great cathedral.

Indeed, Otto von Simson has shown in his recent book *The Gothic Cathedral* that not only the cathedral of Chartres but also nearly all the great Gothic cathedrals were conceived as "music in

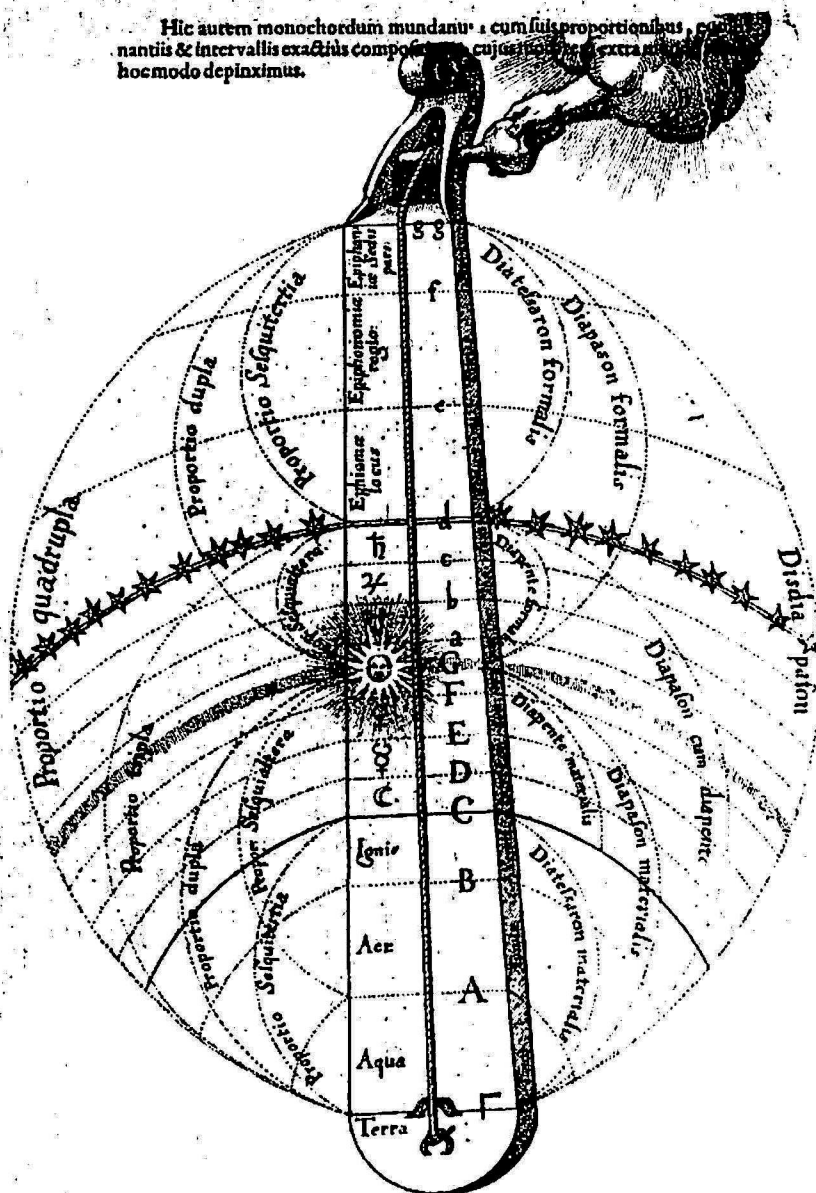
Kepler associated the varying angular velocity of each planet with a musical

Kepler's third law is usually stated as a mathematical formula: $T^2/D^3 = K$, where T is a planet's period of revolution, D is its average distance from the sun and K is a constant. The values of T and D are known for the earth: T is one year and D is 93 million miles. Therefore K can be computed, so that one can compute any other planet's average distance from the sun if the period of revolution is known or the period of revolution if the average distance is known. The formula looks antiseptic, but like so much of mathematics it springs from the aesthetic power of natural order. Essentially it is this same aesthetic power that lies at the root of music's relation to number.

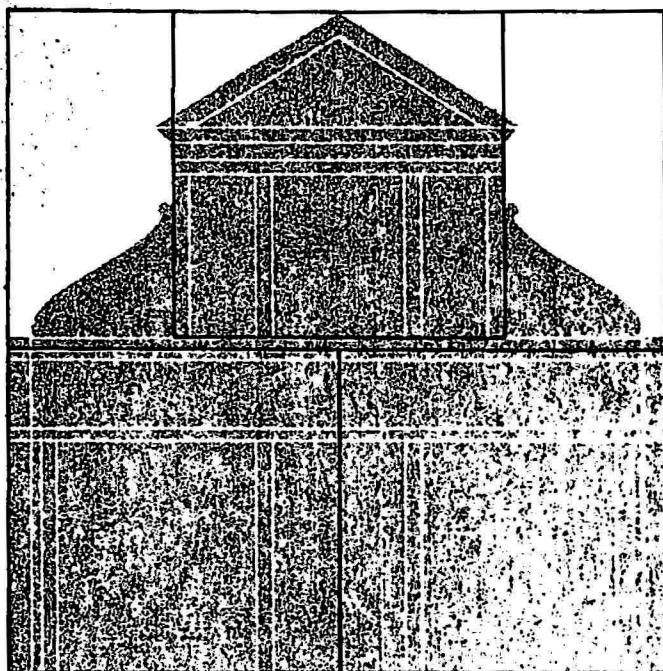
As "duality" is the word for the attitude of thinkers during and immediately after the Renaissance toward the relation of music and number, so "reconciliation" is the word for the period from 1600 to 1750. This period, the age of the Baroque, is one in which feeling seems at first to have predominated over mathematical tradition. Below the surface of Monteverdi's operas, Corelli's sonatas and Bach's fugues, however, is a continuing and fundamental respect for number as a basis of music.

The reconciliation was particularly evident in the work of the French theorist and composer Jean Philippe Rameau, who stands as perhaps the foremost theorist in the history of Western music. In the first half of the 18th century he produced a series of treatises based on what he called "natural principles." These principles are expressions of the age-old laws of proportion. Rameau's explanations of such basic harmonic concepts as the progression of chords, the invertibility of chords, chord-building by thirds and the relation of chords and melody make him the creator of the modern science of harmony. Yet his writings are based on and reconciled with the previous discoveries of such theorists as Zarlino—thinkers who walked the well-worn path leading back to antiquity.

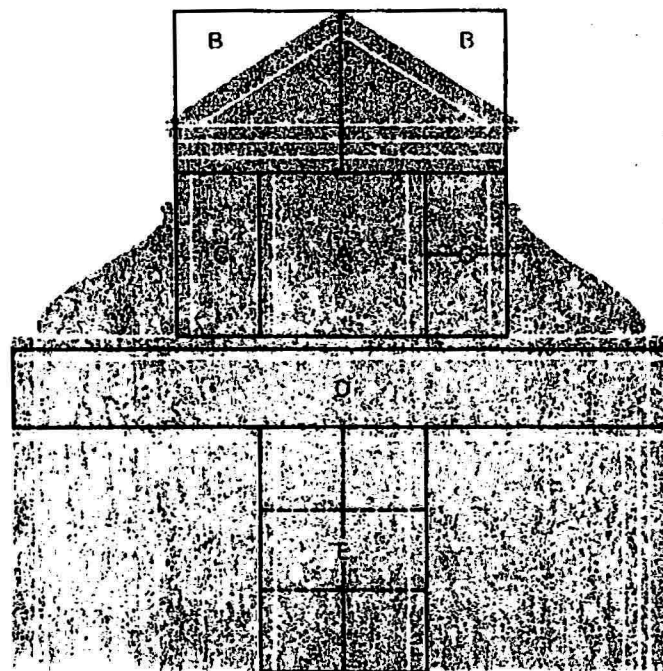
It was also in the early 18th century



GOD'S HAND tunes the "monochord of the world" in an illustration from a book published in 1617. The classical elements and the planets appear, along with the musical ratios.



SIMPLE RATIOS were employed by the Italian architect Alberti in designing the facade of Santa Maria Novella in Florence, as shown by Rudolf Wittkower of Columbia University. The black lines indicate how the facade is made up of squares related in the ratio 1 : 2, as in an octave (*left*). The same ratio recurs in the sub-



units (*right*). The sides of the central bay of the upper story (*A*) form a square half the width of the full story. Squares of the same size enclose the pediment and entablature (*B*); these are twice the width of the side bays (*C*) and twice the height of the attic (*D*). The ratio of the width to the height of the entrance bay (*E*) is 2 : 3.

that the French physicist Joseph Sauveur achieved perhaps the most concrete theoretical reconciliation of old and new with his discovery of overtones: the fact that a vibrating string sounds not only a fundamental note but also a subsidiary series of higher notes that are integral multiples (as defined by speed of vibration) of the fundamental one, with the most important of these notes lying an octave, a fifth and a fourth apart. The phenomenon is also known as the harmonic series and has been called the "chord of nature."

Sauveur's discovery marks a major division in the history of musical thought. Before his time the power of music was largely explained through mathematical flights of reason and fancy. Since his time most theorists have tried to discover the secret of music in acoustical laws, such as the law of overtones. The name of Pythagoras still figures importantly in most books on acoustics, but to regard the modern science of acoustics as a continuation of Pythagoreanism would be to equate two different modes of thought.

Against the historical background that I have sketched let us consider further the role of mathematics from the viewpoint of the composer. The fact is that the string of the Pythagoreans has always vibrated more in minds than in ears, because the composer has seldom been able to put the proportions to direct use during the process of musical

composition. At face value they seem too simple; when they are applied subliminally or "in principle," they seem to disappear altogether. To put the matter another way, composers usually realize the mathematical bases of composition only after the composing is done.

Nonetheless, there have been few periods in Western history when most composers did not perceive mathematical order in the structure of music and strenuously try to find a meeting ground between the old mathematical laws and the creation of new music. The attempts have become more subtle, but the question remains: Are there any valid mathematical approaches to composition?

Among the groups of composers who would answer the question affirmatively are the serialists, whose "12-tone system" is based on the harmonic and melodic equivalence of the 12 tones of the harmonic scale (the white and black keys from one C to the next on a piano). Serialism was invented some 50 years ago by Arnold Schoenberg, who saw the technique as a unifying device for atonal music, or music without a key. In this technique, which has since been extended, the 12 notes are arranged in some specific order that is then used as the basis for a composition. The series of tones can be represented as a series of numbers.

A recent chamber work indicates how far the serialist technique has moved. It is *Sestina*, by Ernst Křenek. This com-

position is serially determined not only in pitch but also in rhythm and in density (the number of instruments playing at any given moment). It is not only a model of the uncompromisingly mathematical in music but also a manifesto: one of the lines of its poetry, which was written by Křenek, states "What looks ahead subordinates itself to number."

For the most part, however, mathematics swims seductively just below the surface of music. It is a naiad gazing at the composer, seemingly within reach but actually unreachable. Its presence is explained by the unshakable principle that music, like all other arts, must be founded on some kind of order.

Considering the seductiveness of the naiad, is it any wonder that Athanasius Kircher, the 17th-century German mathematician and self-styled musical encyclopedist, built a composing machine whose numerical combinations would dictate pitch, rhythm and tempo to any composer willing to use it? (Nobody seemed willing.) Is it presumptuous to believe that Mozart, Haydn, Handel and others may not have had their tongues completely in their cheeks when they wrote music dictated by the throw of dice? Can we begin to understand the mathematical preoccupations of such contemporary composers as Křenek, Karlheinz Stockhausen, Milton Babbitt and Pierre Boulez? Should we be more tolerant of music generated by computer?