

The Bohlen-Pierce Site

On "Pythagorean" Scale Derivation

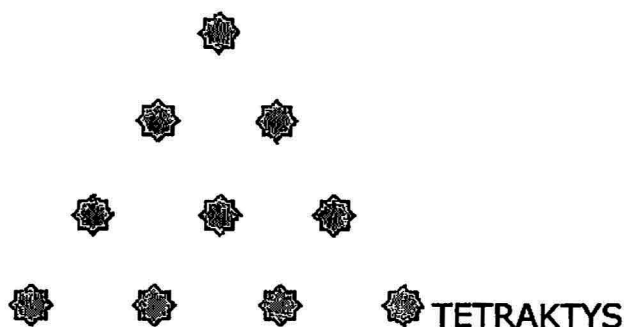
Last update: December 11, 2003

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Bohlen



We assume that the Pythagoreans related their musical scale derivations to their sacred (and secret) symbol, the tetraktys:



(That stars are chosen to mark the elements of the tetraktys has no deeper meaning: it's just convenient when using Power Point to generate these pictures.) The basic intervals of the Pythagorean scales

$$4:3, 3:2, 2:1$$

can indeed be directly read from this figure. But there are other intervals it can yield. If because of 2:1 we consider the octave frame as given, we might attempt to check which intervals can be found from the inversion of the primary intervals, transformed back into the octave. 3: 4, multiplied by 2, reveals nothing new: 3:2. Similarly, 2:3 ends up as 4:3, and only 1:2 adds at least 1:1, the unison, to the system. If we are not discouraged by this meager result, we might investigate, which combinations of these intervals fall into the octave frame. In this manner we find two more candidates: 4:3x4:3 yields 16:9, and 3:4x3:2 leads to 9:8. All other combinations lead to already known intervals, or fall outside the octave frame. Thus we conclude that the tetraktys supports derivation of the following 6 intervals, in ascending order of magnitude:

$$1:1, 9:8, 4:3, 3:2, 16:9, 2:1.$$

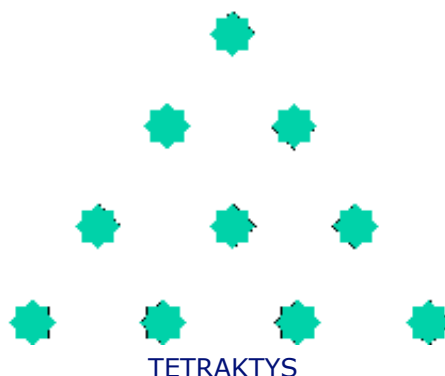
There is still a lot of material missing from a 12-tone scale; if we intend to find it we need to extend

453 pax
BOHLEN, H.
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On "Pythagorean" Scale Derivation A 47-step scale within the double octave

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We assume that the Pythagoreans related their musical scale derivations to their sacred (and secret) symbol, the tetraktys:



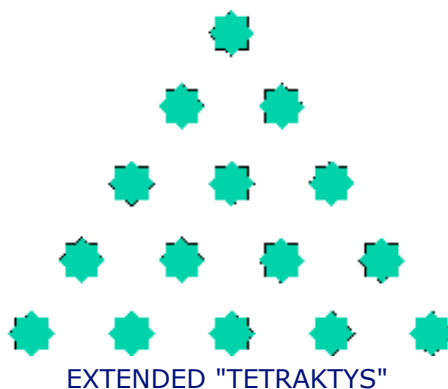
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1:1, 9:8, 4:3, 3:2, 16:9, 2:1.

There is still a lot of material missing from a 12-tone scale; if we intend to find it we need to extend the tetraktys by one further row:



The interval relations we can read directly from this modified symbol (in addition to those derived from the original tetraktys) are **5:4** and **5:3**. Their inversions, transferred into the octave frame, yield **8:5** and **6:5**. The next step, combinations, reveals a wealth of new intervals: **15:8** (3:2x5:4), **10:9** (2:3x5:3), **16:15** (2:3x8:5), **9:5** (3:2x6:5), **25:16** (5:4x5:4), **32:25** (4:5x8:5), **25:24** (5:8x5:3), **25:18** (5:6x5:3), and **36:25** (6:5x6:5). Most of this new material proves to be useful for a 12-tone scale, some is rather "out of tune". Anyway, this is the yield (bold letters indicating a good fit, step numbers bracketed and in italics):

1:1 (0); 25:24; **16:15** (1); 10:9; **9:8** (2); **6:5** (3); **5:4** (4); 32:25; **4:3** (5); 25:18; 36:25; **3:2** (7); 25:16; **8:5** (8); **5:3** (9); **9:5** (10); **15:8** (11); **2:1** (12).

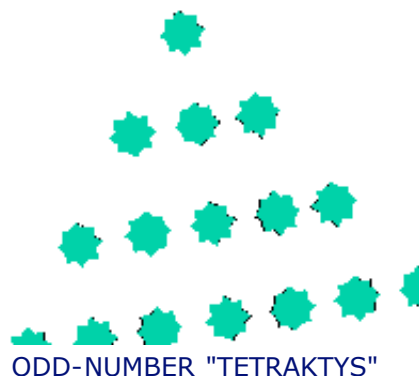
With the exception of a useful sixth step (the problem of the tritonus persists also here) the procedure produces the complete 12-tone scale in just tuning.

Is this procedure expandable? Seen from a mathematical view point, the tetraktys is (or rather seems to be) a most trivial thing: starting with one element in the first row, each following row carries one element more than the preceding one. We can describe this with a simple equation, containing a small twist, however, that takes care of a possible generalization:

$$r = N R - N + 1$$

where r is considered to be the number of elements in a row, R is the number of the row ($R = 1, 2, 3, \dots$), and N is an order ($N = 0, 1, 2, 3, \dots$). Seen this way, the tetraktys or its extension are nothing else than the **first order series** ($N = 1$) of our little equation: $r = R$.

This rises some expectations: What about the **second order series**, $r = 2R - 1$? We can create a symbol analog to the tetraktys by using the first four members of the series again. It turns out to contain only odd numbers (**1, 3, 5, 7**):



Playing the game again that we played with the original tetraktys, we observe that the octave (2:1) has to be replaced by the twelfth (**3:1**) in this case, and that we can derive the following intervals at first sight: **7:5**, **5:3**, **7:3**. Inversions, transferred back into the twelfth, yield **1:1**, **15:7**, **9:5**, **9:7**.

The third step, combinations, results in

- **27:25** (3:5x9:5)
- **25:21** (5:7x5:3)
- **75:49** (5:7x15:7)
- **49:25** (7:5x7:5)
- **63:25** (7:5x9:5)
- **25:9** (5:3x5:3)
- All other combinations lead to already known intervals, or fall outside the 3:1 frame.

Arranging all intervals found this way in ascending order reveals the complete chromatic Bohlen–Pierce (BP) scale, nothing missing, nothing added:

1/1, 27/25, 25/21, 9/7, 7/5, 75/49, 5/3, 9/5, 49/25, 15/7, 7/3, 63/25, 25/9, 3/1.

Thus the duality that reigns most relationships between the traditional Western scale and the BP scale is confirmed once more: the **first order series** of our equation **leads to the traditional scale**, the **second order yields BP**.

That arouses curiosity: What may be hidden behind the **third order series**, $r = 3R - 2$? It leads to a "tetraktys" in which each row contains 3 more elements than its predecessor, e.g. **1, 4, 7, 10**.

The frame interval here is **4:1**, the double octave. Other directly readable intervals are **7:4**, **5:2** (10:4) and **10:7**. The inversions yield **1:1**, **16:7**, **8:5** and **14:5** (28:10). Out of the combinations, the following fit into the double octave: **49:16** (7:4x7:4), **49:40** (7:4x7:10), **100:49** (10:7x10:7), and **25:7** (10:7x5:2). Extending the series by one member (**13**) contributes a number of 13-limit intervals: **13:10**, **13:7**, **13:4** with **40:13**, **28:13** and **16:13** resulting from their inversions, and a major number of combinations. This altogether seems odd at first sight, but soon begins to make sense in a surprising manner: all these intervals are members of a **47-step, non-octave scale within the framework of the double octave!** The following table shows this by comparing the intervals found with the intervals of a 47-step equal-tempered double octave. Several of the intervals are enharmonic to others; this is indicated by bracketing the step number. The span of a step is approximately a quarter tone; in equal temperament it counts 51.06 cents.

Step	Just interval (span)	Just interval (cents)	Equal temp. (cents)	Defect (cents)	Remarks
0	1:1	0	0	0	
2	52:49	102.9	102.1	-0.8	13:7x4:7
3	100:91	163.3	153.2	-10.1	10:7x10:13
7	16:13	359.5	357.4	-2.1	
(7)	49:40	351.3	357.4	6.3	7:4x7:10
9	13:10	454.2	459.6	5.4	
10	35:26	514.6	510.6	-4.0	7:4x10:13
12	10:7	617.5	612.8	-4.7	
16	8:5	813.7	817.0	3.3	
18	169:100	908.4	919.1	10.7	13:10x13:10
19	7:4	968.8	970.2	1.4	
21	13:7	1071.7	1072.3	0.6	
22	25:13	1132.1	1123.4	-8.7	5:2x10:13
24	100:49	1235.0	1225.5	-9.5	10:7x10:7
26	28:13	1328.3	1327.7	-0.6	
28	16:7	1431.2	1429.8	-1.4	
(28)	91:40	1423.0	1429.8	6.8	7:4x13:10
30	169:70	1525.9	1531.9	6.0	13:10x13:7
31	5:2	1586.3	1583.0	-3.3	
33	130:49	1689.2	1685.1	-4.1	10:7x13:7
35	14:5	1782.5	1787.2	4.7	
38	40:13	1945.8	1940.4	-5.4	
(38)	49:16	1937.7	1940.4	2.7	7:4x7:4
40	13:4	2040.5	2042.5	2.0	
42	169:49	2143.4	2144.6	1.2	13:7x13:7
43	25:7	2203.8	2195.7	-8.1	10:7x5:2
47	4:1	2400	2400	0	Double oct.

The gaps between the already existing steps can easily be filled by extending the series by two

members (16 and **19**). The resulting new intervals fit in quite well, as for instance **19:16** (step 6), **19:13** (step 13), **19:10** (step 22), **19:7** (step 34). In first line, however, it deserves to be noted that the scale accommodates 7- and 13-limit exceptionally well.

Picking just the best fits (defect < 6 cents) and adding their complementary intervals leads to the following selection of 22 tones in the double octave. The pattern is symmetric (2322122232 5 2322212232) with (2) being close to a half tone. This scale appears melodically viable through a multitude of half tone steps, and harmonically challenging due to many 7- and 13-limit intervals.

Step	Just interval (span)	Just interval (cents)	Equal temp. (cents)	Defect (cents)	Remarks
0	1:1	0	0	0	
2	52:49	102.9	102.1	-0.8	
5	196:169	256.6	255.3	-1.3	
7	16:13	359.5	357.4	-2.1	tridecimal neutral third
9	13:10	454.2	459.6	5.4	
(9)	64:49	462.3	459.6	-2.7	
10	35:26	514.6	510.6	-4.0	
12	10:7	617.5	612.8	-4.7	septimal augm. fourth
14	98:65	710.8	714.9	4.1	
16	8:5	813.7	817.0	3.3	minor sixth
19	7:4	968.8	970.2	1.4	harmonic seventh
21	13:7	1071.7	1072.3	0.6	"16/3-tone"
26	28:13	1328.3	1327.7	-0.6	
28	16:7	1431.2	1429.8	-1.4	
31	5:2	1586.3	1583.0	-3.3	just major tenth
33	130:49	1689.2	1685.1	-4.1	
35	14:5	1782.5	1787.2	4.7	
37	104:35	1885.4	1889.4	4.0	
38	49:16	1937.7	1940.4	2.7	
(38)	40:13	1945.8	1940.4	-5.4	
40	13:4	2040.5	2042.6	2.1	
42	169:49	2143.4	2144.6	1.2	
45	49:13	2297.1	2297.9	0.8	
47	4:1	2400	2400	0	double octave

A closer inspection reveals that step 14 is almost a just fifth 3:2 (deviation 12.9 cents), and step 37 is nearly a just twelfth 3:1 (deviation 12.6 cents). Beyond that step 26 represents the just interval 15:7 quite well (deviation 8.3 cents), and step 33 approaches 8:3 (deviation 12.9 cents), as does step 45 with 15:4 (deviation 9.6 cents).

The author likes to derive scales on the basis of certain triads. He admits, however, that in this case it was only an afterthought that the scale harbors the following triads and tetrads in multiple ways:

$$\begin{aligned}
 & \mathbf{1:1 - 7:4 - 5:2 - 13:4} \text{ (4 - 7 - 10 - 13)} \\
 & \mathbf{1:1 - 10:7 - 13:7 - 16:7} \text{ (7 - 10 - 13 - 16)} \\
 & \mathbf{1:1 - 13:10 - 8:5} \text{ (10 - 13 - 16)}
 \end{aligned}$$

It should be noted that steps 7, 12, 19, 31, and 38 are almost identical to steps 4, 7, 11, 18, and 22 of Gary Morrison's 88 CET scale. Both scales therefore share the 4 - 7 - 10 triad.

Heinz Bohlen
December 1998



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