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Euclid's *Sectio canonis* and the History of Pythagoreanism

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The treatise which has come down to us as the *Sectio canonis* or *Division of the Canon* consists in an introduction of thirty-three lines [Menge 1916, 158.1–160.4] and twenty interconnected demonstrations articulated in roughly the same way as those in Euclid's *Elements* [cf. Jan 1895, 115–116].¹ Beyond this most everything is in dispute. To begin, scholars debate the authorship of the *Sectio*. Those who deny or qualify the thesis that it derives from Euclid usually proceed by comparing it to treatises more commonly acknowledged to be Euclid's, and by pointing out supposed inconsistencies in the *Sectio* itself which are presumed inappropriate for a mathematician of Euclid's stature [cf., e.g., Menge 1916, xxxviii–xxxix]. None of the arguments, however, are particularly persuasive. In the first place, the critics tend to ignore the variety of logical structure and language evidenced throughout the Euclidean corpus, and to suppose that any ancient author writing treatises in the various sciences of his age would necessarily do so according to the same standards of expository style and precision.² Such an assumption fails when applied to the works in the Ptolemaic corpus, for example [cf. Neugebauer 1946, 112–113]. In the second place, the numerous inconsistencies 'discovered' in this treatise signify, in my view, a failure in scholarship rather than any serious problem in the document itself. Indeed, my main purpose in this chapter is to undercut these claims of inconsistency by setting out a new reading of the introductory part of this treatise.

¹ On the question of Euclid's date, which I put in the third quarter of the third century BC, see Bowen and Goldstein 1991, 246n30 or Bowen and Bowen 1991, section 4.

² For criticism of the case for authenticity based on linguistic data, see Menge 1916, xxxix–xl.

The learned debate about the provenance and nature of the *Sectio canonis* centers on five questions:

- (1) What is the argument of the preface?
- (2) How does the preface bear on the subsequent twenty demonstrations?
- (3) What is the relation of the first nine demonstrations to the next nine?
- (4) Do the last two demonstrations, the very ones describing a division of the *canon*, belong with the preceding eighteen?, and
- (5) Is the treatise complete as it stands?

The first is fundamental, since answers to the others all presuppose an interpretation of the preface. So, in what follows, I will concentrate primarily on the first question, though I will address a few remarks to the last. I will proceed, moreover, by way of a detailed analysis of the sequence of arguments comprising Euclid's preface to the *Sectio*, my aim being to suggest a reading of these arguments which joins them in a coherent, intelligible whole [section 2].³ I emphasize that it is not my intention to argue that all other interpretations of the preface are wrong. For, not only would this be an improper category of criticism in the present case, it would belie my debt to these other interpretations and, in particular, to the nicely argued account offered by Andrew Barker [1981]. Rather, my purpose is to determine the minimum set of assumptions needed to present the preface as a credible, reasoned unity. And, in doing this, I will rely as much as possible on the internal evidence of the preface itself, and adduce assumptions from elsewhere only when necessary.

My basic contention is that the *Sectio canonis* elaborates in harmonic science the ontologically reductive thesis that all is number; and that once this thesis as it appears in the *Sectio* is properly understood, the most serious of the past worries about the structure and meaning of this treatise dissipate. In other words, if, as Barker suggests [1981, 15–16], the *Sectio canonis* shows above all how to analyze music precisely, it does this by displaying in detail how items in a specific domain, musical sound, are to be construed as number.

But, if this is correct, it would seem that we have replaced one set of problems about the *Sectio* with another concerning its alleged Pythagoreanism.

³ Those familiar with this treatise may discern my approach to the second and third questions. The fourth, which is often raised in the context of reports by Proclus and Marinus of a *Musica elementa* by Euclid [cf. Menge 1916, xxxvii–xxxviii], and which was argued in the negative by Paul Tannery [1912, 213–215], has, I think, been well answered by Andrew Barker [1981, 11–13]. As for the fifth, it requires critical study of the entire treatise and introduces questions about technical writing in the various sciences which I must postpone for now.

For, according to Aristotle, one of the basic tenets of early Pythagoreanism is that all is number; and, as I understand him [see Bowen 1992], this means that numbers are what things really are. So, to conclude this chapter I will address the cluster of problems concerning Euclid, the *Sectio canonis*, and Pythagoreanism [section 3].

1. The preface to the *Sectio canonis*

Let us consider, then, how Euclid introduces the twenty demonstrations in the *Sectio canonis*. The Greek text reproduced here is taken from Menge's edition of 1916 with some slight changes in punctuation and the addition of sentence numbers in square brackets to assist textual analysis.

Euclid's *Sectio canonis*, pref. [Menge 1916, 158.1–160.4]

[1] Εἰ ἡσυχία εἴη καὶ ἀκίνησία, σιωπὴ ἂν εἴη· [2] σιωπῆς δὲ οὐσῆς καὶ μὴ δένος κινουμένου οὐδὲν ἂν ἀκούοιτο· [3] εἰ ἄρα μέλλει τι ἀκουσθήσεσθαι, πληγὴν καὶ κίνησιν πρότερον δεῖ γενέσθαι. [4] ὥστε, ἐπειδὴ πάντες οἱ φθόγγοι γίνονται πληγῆς τινος γινομένης, πληγὴν δὲ ἀμήχανον γενέσθαι μὴ οὐχὶ κινήσεως πρότερον γενομένης,—τῶν δὲ κινήσεων αἱ μὲν πυκνότεραι εἰσιν, αἱ δὲ ἀραιότεραι, καὶ αἱ μὲν πυκνότεραι ὀξυτέρους ποιοῦσι τοὺς φθόγγους, αἱ δὲ ἀραιότεραι βαρυτέρους,—ἀναγκαῖον τοὺς μὲν ὀξυτέρους εἶναι, ἐπεὶ περ ἐκ πυκνότερων καὶ πλειόνων σύγκεινται κινήσεων, τοὺς δὲ βαρυτέρους, ἐπεὶ περ ἐξ ἀραιότερων καὶ ἐλασσόνων σύγκεινται κινήσεων, ὥστε τοὺς μὲν ὀξυτέρους τοῦ δέοντος ἀνιεμένους ἀφαιρέσει κινήσεως τυγχάνειν τοῦ δέοντος, τοὺς δὲ βαρυτέρους ἐπιτεινομένους προσθήσει κινήσεως τυγχάνειν τοῦ δέοντος. [5] διόπερ ἐκ μορίων τοὺς φθόγγους συγκείσθαι φατέον, ἐπειδὴ προσθήσει καὶ ἀφαιρέσει τυγχάνουσι τοῦ δέοντος. [6] πάντα δὲ τὰ ἐκ μορίων συγκείμενα ἀριθμοῦ λόγῳ λέγεται πρὸς ἄλληλα, ὥστε καὶ τοὺς φθόγγους ἀναγκαῖον ἐν ἀριθμοῦ λόγῳ λέγεσθαι πρὸς ἀλλήλους· [7] τῶν δὲ ἀριθμῶν οἱ μὲν ἐν πολλαπλασίῳ λόγῳ λέγονται, οἱ δὲ ἐν ἐπιμορίῳ, οἱ δὲ ἐν ἐπιμερεῖ, ὥστε καὶ τοὺς φθόγγους ἀναγκαῖον ἐν τοιοῦτοις λόγοις λέγεσθαι πρὸς ἀλλήλους. [8] τούτων δὲ οἱ μὲν πολλαπλασίοι καὶ ἐπιμόριοι ἐνὶ ὀνόματι λέγονται πρὸς ἀλλήλους. [9] γινώσκουμεν δὲ καὶ τῶν φθόγγων τοὺς μὲν συμφώνους ὄντας, τοὺς δὲ διαφώνους, καὶ τοὺς μὲν συμφώνους μίαν κρᾶσιν ἐξ ἀμφοῖν ποιοῦντας, τοὺς δὲ διαφώνους οὐ. [10] τούτων οὕτως ἐχόντων εἰκὸς τοὺς συμφώνους φθόγγους, ἐπειδὴ μίαν τὴν ἐξ ἀμφοῖν ποιοῦνται κρᾶσιν τῆς φωνῆς, εἶναι τῶν ἐν ἐνὶ ὀνόματι πρὸς ἀλλήλους λεγομένων ἀριθμῶν, ἥτοι πολλαπλασίους ὄντας ἢ ἐπιμορίους.

[1] If there were rest and lack of motion, there would be silence. [2] But, if there were silence and nothing moved, nothing would be heard. [3] Therefore, if anything is going to be heard, there must previously occur striking

and motion. [4] Consequently, since all musical notes occur when there is a certain striking, and since it is impossible that a striking occur unless a motion occurs previously—some motions are closer together but others are less close together; and the ones that are closer together produce notes higher (in pitch); but those less close together, notes that are lower (in pitch)—it is necessary that the former notes be higher (in pitch) because they are composed of motions that are closer together and so more numerous and that the latter notes be lower (in pitch) because they are composed of motions that are less close together and so less numerous; so that notes higher (in pitch) than what is needed reach it when lowered by subtraction of motion, and those lower (in pitch) than what is needed reach it when raised by addition of motion. [5] Wherefore, we should say that musical notes are composed of parts, since they reach what is needed by addition and subtraction. [6] But all things composed of parts are described in relation to one another by a ratio of (whole) number, so that musical notes must also be described in relation to one another by a ratio of (whole) number. [7] But some numbers are said to be in multiple ratio, some in superparticular ratio, and others in superpartient ratio,⁴ so that notes too must be said to be in these sorts of ratio in relation to one another. [8] Of these [*scil.* musical notes] the multiple and superparticular are described in relation to one another by a single term. [9] In fact, we perceive some notes as concordant but others as discordant, and the concords as making a single blend out of a pair (of notes) but the discords as not. [10] Since these things are so, it is appropriate that concordant notes, being either multiple or superparticular, belong to (whole) numbers described in relation to one another by a single term, since they produce a single blend of sound out of a pair (of musical notes).

2. Analysis of the preface to the *Sectio canonis*

This introduction is, in fact, a series of five arguments establishing that

- (a) prior striking and motion are required if anything is to be heard [1]–[3];
- (b) the relative pitch of a musical note varies directly as the relative close-packedness or compactness of the motions constituting it [4];
- (c) musical notes are composed of parts [5];

⁴ If m and n are whole numbers, where $1 < n < m$, then ratios of the form $m:1$, $(m+1):m$, and $(m+n):m$ are multiple, superparticular, and superpartient, respectively.

- (d) two notes may stand in either multiple, superparticular, or superpartient ratio [6]–[7], and
- (e) concordant notes are reasonably said to belong to those whole-number ratios which are predicated by a single term [8]–[10].

I suspect that this is sufficient to highlight the fact that the introduction to the *Sectio canonis* is peculiar. Indeed, the oddity of the locutions in these arguments, their sense, and how they fit together are real puzzles. And there is no way to solve them except by a careful study of what is actually written.

2.1 First argument

- [1] If there were rest and lack of motion, there would be silence.
- [2] But, if there were silence and nothing moved, nothing would be heard.
- [3] Therefore, if anything is going to be heard, there must previously occur striking and motion.

Though its structure is clear, it is not easy to see what this argument is about. Still, as we read on there are, I think, three alternatives to consider in deciding what moves and what is struck. The motion may be that of

- (a) something which strikes a sonant body, a hand plucking the string of a lyre for instance; or
- (b) a sonant body striking the ambient air, for example, the string of the lyre striking the air as it moves back and forth after being plucked; or
- (c) the moving air which has been set in motion by the sonant body and strikes the ear.

2.2 Second argument

- [4] Consequently, since all musical notes ($\phi\theta\acute{o}\gamma\gamma\omicron\iota$) occur when there is a certain striking, and since it is impossible that a striking occur unless a motion occurs previously—some motions are closer together ($\pi\kappa\upsilon\acute{o}\tau\epsilon\rho\alpha\iota$) but others are less close together ($\acute{\alpha}\rho\alpha\iota\acute{o}\tau\epsilon\rho\alpha\iota$); and the ones that are closer together produce notes higher (in pitch); but those less close together, notes that are lower (in pitch)—it is necessary that the former notes be higher (in pitch) because they are composed of motions that are closer together and so ($\kappa\alpha\iota$) more numerous and that the latter notes be lower (in pitch) because they are composed of motions that are less close together and so less

numerous ($\acute{\epsilon}\lambda\alpha\sigma\sigma\acute{o}\nu\omega\nu$); so that notes higher (in pitch) than what is needed reach it when lowered by subtraction of motion, and those lower (in pitch) than what is needed reach it when raised by addition of motion.

Here it is evident that not only must the striking or impact and motion precede the musical note, this motion must also be prior to the striking. In short, if there is to be musical note, there must first be motion which produces an impact which in turn produces the note. Yet, the story is now more complex, given that $\phi\theta\acute{o}\gamma\gamma\omicron\iota$ (which I have rendered by 'musical notes')⁵ are not only produced by motions, they are composed of them. In any case, if the $\phi\theta\acute{o}\gamma\gamma\omicron\iota$ are to be composed of motions, it would seem unlikely that Euclid means to claim that the motions in question are (a) those of something which strikes a sonant body, like the hand's motion in plucking the string of a lyre, or (b) the motions of the sonant body, such as those of the sonant string to and fro. So, by elimination, it seems that the first argument concerns the motion of air as it strikes the ear. But this still leaves a problem: if the motions constitute the $\phi\theta\acute{o}\gamma\gamma\omicron\iota$ or musical notes, it is difficult to see how the motions are to precede them.

This problem, however, is not insuperable. As our first hypothesis, let us grant Euclid a distinction between musical sound as heard (phenomenal

⁵ The noun, $\phi\theta\acute{o}\gamma\gamma\omicron\varsigma$, has a variety of attested meanings which include any clear, distinct sound—especially vocal sound, where this was primarily that of (male) voices and later extended to cover sound produced by any animal with lungs—as well as speech, musical sound, and sound in general. The tendency among scholars who have studied the *Sectio canonis* is to suppose that it here means 'a sound in general' and that the preface draws on ancient acoustical physics. I reject this for two reasons. First, I no longer see the point [cf. Bowen 1982] in elevating the sort of remark made in texts in harmonic science like the *Sectio* (or in others which attempt, for example, to explain hearing in terms of some philosophical theory of change and motion) to the status of an independent, acoustical physics: for, to do this without proper regard for the context of these remarks is to risk abstracting a domain of technical discourse which did not exist in ancient times, and to confound efforts to determine the sense and the history of the texts in question. In truth, regarding every discussion of sound as belonging to an acoustical theory makes as much sense as treating liver omens as part of some ancient veterinary science. Second, those who take $\phi\theta\acute{o}\gamma\gamma\omicron\varsigma$ to mean 'sound' arbitrarily introduce difficulties in explaining how the preface to the *Sectio* bears on the subsequent demonstrations—it is not surprising that they isolate the preface on the ground that it concerns sound in general, and view the first nine demonstrations as establishing truths about ratios without regard for musical phenomena [cf., e.g., Ruelle 1906; Mathiesen 1975, 237; Barker 1981, 1–3; Fowler 1987, 146]. Thus, to counter what I see as a gratuitous balkanization of the treatise, I propose, with equal justification *prima facie*, to start differently and to render $\phi\theta\acute{o}\gamma\gamma\omicron\varsigma$ as 'a musical note'.

musical sound) and musical sound as constituted of motions (objective musical sound), and let us suppose accordingly that the argument in sentences [1]–[3] of the preface is about the former. In other words, let us take the first argument to focus on conditions needed for the occurrence of musical sound as heard.

This hypothesis is plausible. In my view [cf. Bowen 1982], such a reduction of what is heard to objective, quantifiable conditions underlies the sequence of illustrations and observations made in the fragment from Archytas of Tarentum (who was active during the late 5th and early 4th centuries); and, indeed, this fragment bears interesting parallels to Euclid's preface.⁶ More compelling is the fact that the notion of phenomenal musical sound is essential to the distinction of concords and discords in sentence [9] (note γινώσκωμεν with the present participle construction), and that sentence [10] as a whole plays on the relation of phenomenal and objective musical sound [see section 2.5, below].

So far, then, it would appear that the motion mentioned in [1]–[3] and in the first two premisses of [4] occurs between the sonant body and the ear, and that this motion produces the musical note we hear by striking the ear. The second argument continues by way of an interjection, in which it is evident that the motion responsible for producing what we hear as a single musical sound is really a series of consecutive, discrete motions; and that the relative pitch of two musical notes as heard varies directly as how closely the motions in each series follow upon one another, that is, as the relative compactness (πυκνότης) of the series. Next, and most important, comes the conclusion that what we hear as a single note is in fact just the series of motions which produces it.⁷ This is the force of 'because they (*scil.* the musical sounds as heard) are composed of (σύνκεινται ἐκ with the genitive) motions... '.

Several features of the argument in sentence [4] merit comment. First, that feature of phenomenal musical sound which most concerns Euclid is its pitch. The isolation of this feature is important. Though Euclid mentions the perception of a blending of concordant musical notes later in sentence [9], it is clear that he intends a blending of pitch. In short, this treatise prescind from any other features of phenomenal musical sounds which one

⁶ Curiously enough, Jan [1895, 132, 135, 146] adduces this same fragment in contending that Euclid's preface concerns the motions of a sonant body striking the air. Jan, however, neglects the claim that the φθόγγοι are composed of motions.

⁷ For discussion of Boethius' treatment in his translation [Friedlein 1867, 301.12–308.15] of the *Sectio canonis*, of this reduction of phenomenal musical sound to a series of motions striking the ear, see Bowen and Bowen 1991, section 4.

might be disposed to view as contributing to their musicality (e.g., volume, rhythm, and timbre).

Moreover, given our hypothesis about Euclid's distinction of phenomenal and objective musical sound, it seems that not only does he focus on but one of the many salient characteristics of phenomenal musical sound, pitch, he takes this in turn to be nothing more than a series of motions that strike the ear. This is admittedly peculiar, but still intelligible. As I will explain more fully when we come to the problem of the relation between musical intervals and numerical ratios [see section 2.4, below], what we have here is the initial step in a reductive analysis of music as heard to relative number.

Next, it appears that, for Euclid, pitch is a relative phenomenon—he neither gives any hint that the pitch of a note is to be understood absolutely, nor, I maintain, is it necessary to the sense of this passage that pitch be construed as absolute. This is, of course, consistent with the absence of evidence from other sources that the ancient Greeks conceived of an absolute standard of pitch or that they possessed the means of measuring time so as to define one. Perhaps, the predilection evident in Greek scientific and philosophical documents for defining ratios only between quantities of the same kind explains this [cf. Euclid, *Elem.* v defs. 3 and 4]. In any case, it follows immediately that the numerosity or compactness of motions is not equivalent to frequency. In other words, Euclid does not assume here a vibrational theory of how sound propagates. For, although the various series of motions occur in time and are differentiated by the lapse of time between elements in each series, the series themselves are not to be quantified in relation to some unit of time. Thus, the compactness of motions is not the same as some number of motions per second, as Tannery [1912, 217] and Barker [1981, 8], for example, would appear to suppose.

If the compactness or close-packedness of each musical note is relative and not measured in relation to time, how then is it to be quantified? In the preface to the *Sectio*, it is clear that the higher pitch is assigned the greater number in the ratio of the musical notes, since the higher-pitched note is constituted of more motions. In order to quantify this, all one would need to know is that pitch varies inversely as the length of a sonant string or pipe. But this very assumption figures prominently in the last two demonstrations of the *Sectio canonis* [see, e.g., Menge 1916, 178.14–18]. So, for Euclid, it seems, if the compactness of the motions constituting a musical note at a certain pitch varies inversely as the length of the sonant string or pipe producing it, then, to quantify the relation between two musical notes *qua* pitches, one must measure the relative lengths of the strings or pipes producing them.

2.3 Third argument

[5] Wherefore, we should say that musical notes are composed of parts, since they reach what is needed (τοῦ δέοντος) by addition and subtraction.

Again we have the eliminative reduction of phenomenal to objective musical sound, that is, the musical note as heard *qua* pitch to the series of motions that strike the ear. And as before, just as *relative* pitch is taken to be the primary or defining quality of phenomenal musical sound, *relative* compactness or close-packedness is to be the main characteristic of objective sound. What is added is the claim that each musical note so understood has parts, since it is constituted of motions to which motions may be added or subtracted. What sort of rationale might there be for this?

Consider the behaviour of a sonant string on a lyre. According to Archytas [cf. Bowen 1982], such a string strikes the air with each motion back and forth and sets the ambient air in motion like a projectile which strikes the ear causing one to hear a single sound at a pitch that varies inversely as the effective length of the string. It would, of course, be easy to elaborate this (in a way Archytas did not) by supposing that the pitch of the sound heard is determined proximately by the rate of the string's motion to and fro, and that this is inversely dependent on the string's effective length.⁸ Since the string's motions to and fro are seemingly consecutive and discrete, it would seem plausible that the series of airy projectiles moving from the string to the ear is likewise consecutive and discrete, and that the relative numerosity or close-packedness of this series depends directly on the rate of the string's motions. Moreover, given that the pitch of the note heard varies directly as the rate of the string's motion back and forth, it would follow that one may adjust the pitch by increasing or decreasing the rate of the string's motion. And, of course, one would do this by decreasing or increasing the effective length of the string itself. Thus, by identifying pitch as heard with the series of airy projectiles striking the ear, one gets the result that each musical note consists of discrete, consecutive parts subject to additive increase or decrease by decreasing or increasing the effective length of the string.

⁸ Cf. the analyses offered by Adrastus in Theon [Miller 1878, 50.11–21], Nicomachus [Jan 1895, 243.17–244.1; 254.5–22], and Porphyry's version of Heracleides' report of Xenocrates' remarks about Pythagoras [Düring 1932, 30.9–31.21].

Such an account of what underlies sentence [5] is admittedly conjectural. Its main advantage is that it adheres to, and is consistent with, what is actually written in the preface to the *Sectio*, and that it enables an intelligible transition from sentence [4] to sentence [6]. In any case, it is important to see that, were the relative numerosity or close-packedness of the series of motions constituting two musical notes to be understood and quantified in this way, there would be no need in addition to worry about the relative incidence of the component motions in pairs of series at the ear. Granted, one might well choose to develop some account of this for independent reasons; but it remains the fact that Euclid's writing of consecutive series of motions constituting musical notes as heard is by itself no warrant to suppose that the *Sectio* entails any views at all about how pairs of series impinge on the ear in relation to one another.

2.4 Fourth argument

[6] But all things composed of parts are described in relation to one another by a ratio of (whole) number (ἀριθμός), so that musical notes must also be described in relation to one another by (ἐν) a ratio of (whole) number. [7] But some numbers are said to be in multiple ratio, some in superparticular ratio, and others in superpartient ratio, so that notes too must be said to be in these sorts of ratio in relation to one another.

From the conclusion that musical notes are composed of parts, Euclid now argues that such notes must stand to one another in whole-number ratios. Tannery [1912, 215–216; cf. Fowler 1987, 146] objects to the argument on the ground that it is simply not true that any two objects composed of parts need manifest a numerical ratio, and he concludes that a geometer like Euclid could scarcely have written this. Now, whether we should expect that Euclid would have written 'things composed of *discrete* parts' (i.e., 'pluralities') is a nice question. In any case, if I am right about the sense of the preceding sentences, this is what πάντα δὲ τὰ ἐκ μορίων in fact means; and so there is no real difficulty. Indeed, I suspect that Tannery is wrong to abstract this sentence from its context and to criticize it as though it were a universal proposition. As for what may be Tannery's assumption that an ancient author who writes in one scientific field will necessarily write with the same degree of precision on the same topics in another, I have indicated that this is not true of Ptolemy. Further, we should recall that the degree of articulation in the deductive structure of Euclid's *Elements* results in great part from its focus on problems of incommensurability, problems

which require precise definitions for solution [see Neugebauer 1941, 25–26]; and we should realize that the varying sophistication in explanatory structure of the other sciences may likewise depend on the nature of their problems. In my view, given that the *Sectio* ostensibly presents a science of relations among the pitches of musical notes and is limited to the domain of incommensurable magnitudes, Tannery's objection to this sentence in the preface is more captious than substantive: it is certainly no reason to deny Euclid's authorship of the *Sectio canonis*.

That the sort of harmonic science presented in the *Sectio canonis* is indeed limited to ratios of whole numbers follows immediately from two considerations already mentioned. The first is that pitch is to be understood relatively, that musical pitches are conceived only in relation to one another. The second is that in the *Sectio* one is apparently to quantify pitch by measuring string-lengths according to a common unit [cf. *dem.* 19–20: Menge 1916, 178.11–180.31]: such measurement by a common unit is an empirical process and will inevitably yield a ratio of whole numbers [cf. Bowen 1982, 96].⁹

As for the reference to multiple, superparticular, and superpartient ratios, there is no need in either grammar or sense to take this as an exhaustive tripartition. Were one moved to do this, however, it would follow that these three kinds of whole-number ratio are fundamental or basic, that the multiple superparticular and multiple superpartient ratios evident especially in the last two demonstrations are therefore derivative.¹⁰ For my part, I prefer to suppose that Euclid mentions the three kinds of whole-number ratios he does and passes over the others because they are not germane to the purpose of the preface [see section 2.5, below].

⁹ If this is correct, we have an explanation for the fact that, when Greek theorists relied on ratios to analyze musical relations, they confined their attention to ratios of whole numbers. In a sense, then, this limitation is not arbitrary, though it is clear that not all the ancients understood it and that they may even have viewed it as a matter of convention. Adrastus [Hiller 1878, 50.14–16], for example, mentions ratios of incommensurable magnitudes and relegates these to noises or non-musical sounds. But I take this to be symptomatic of a somewhat specious logical completeness characteristic of much Peripatetic writing. For, according to Adrastus, if one assigns musical notes to ratios of whole numbers by quantifying speeds using some unit as a common measure, then one may assign noises to ratios of incommensurable speeds (presumably) by quantifying speeds using geometrical techniques and not a common measure. But see Barker 1984–1989, ii 214n16.

¹⁰ If m , n , and p are whole numbers, where $1 < n < m$ and $1 < p$, then ratios of the form $(mp + 1):m$, and $(mp + n):m$ are multiple superparticular and multiple superpartient, respectively.

Now, if the pitch of a musical note and the compactness of the series of motions that strike the ear are both relative, it follows that the fundamental musical phenomenon according to the *Sectio canonis* is the interval or separation (*διάστημα*) defined by two distinct pitches. In short, the phenomenon of music is not so much a sequence of pitches, as a sequence of separations defined by pitches.¹¹ Moreover, given the hypothesis of the

¹¹ Fowler [1987, 148] reiterates Szabó's claim [1978, 99–144: cf. Barker 1981, 13] that in the texts like the *Sectio* *διάστημα* signifies a “distance between” or “interval” in a very general sense [cf. Bowen 1984, 337–341], a claim which is perhaps one reason why he does not see that in the *Sectio* whole-number *λόγος* (ratios) are what *διαστήματα* really are [cf. Bowen and Bowen 1991, section 4].

In any case, Szabó's claim rests on poor philology. As I have argued elsewhere [Bowen 1984, 340–341: cf. 1982, 95 and nn81–83], the root sense of *διάστημα* is ‘separation’. Of course, the challenge is to characterize this separation and one way is to view it as a linear difference between pitches. But there are others and none is intuitively more correct. Indeed, Porphyry [Düring 1932, 90.24–95.23] suggests that the schools of harmonic science all start from the assumption that an interval is the separation of pitches, but differ as to how this separation is conceived. In particular, he reports that some think of musical intervals as differences (*διαφοραί, ὑπεροχαί*), whereas others say that they are whole-number ratios, and still others that they are continuous ranges of pitch defining *τόνοι* (regions). Let us consider this further.

Pitch is a magnitude admitting a more and a less. The difference between two pitches may be likened to the separation of the endpoints of two line-segments which coincide and share a common origin. Now, there are three ways to describe this separation and each was adopted by some school of harmonic science. Some took the separation as the whole-number ratio specified by the magnitudes of the two line-segments: among these were the Pythagoreans and Euclid. Others defined the separation as the numerical excess of the greater line-segment over the less. Aristoxenus, who views theorists of this sort as his predecessors, calls them *ἀρμονικοί*; and for want of a better term we may follow him, though I must add that his use of the term may well be partisan—Aristoxenus so opposes Pythagorean theory that he denies it status as harmonic science and refuses to name any Pythagorean a *ἀρμονικός* or to allow that any was his predecessor [see Barker 1978a]. In any case, Euclid, the Pythagoreans, and the *ἀρμονικοί* all define the separation of two pitches by reference to their magnitude, the first two taking it as a ratio and the third as a numerical difference or excess.

But there is yet another way of looking at the separation of the endpoints of our two line-segments. Aristoxenus and his followers define an interval as the range of pitch between two pitches and stipulate that the identity of an interval is preserved as the magnitude of this range varies within boundaries which the ear, by attending to the melodic function (*δύναμις*) of the pitches, determines to be the limits of that interval. To apply this to our line-segments, then, the Aristoxenians think that the separation of endpoints is the range between them and hold that such a separation may preserve identity when this range increases or decreases in magnitude between certain limits determined on qualitative grounds.

eliminative reduction of phenomenal to objective musical sound, it also follows that each interval or separation is to be identified as a ratio of whole numbers. In effect, we have here what I have called an eliminative, reductive analysis of music *qua* system of relative pitches (intervals) to relative number (ratios).

Accordingly, it is, a mistake to suppose that Euclid's talk of adding and subtracting motions in sentence [4] means that musical notes are numbers and that the musical intervals defined by pairs of pitches are numerical differences. Hence, Düring [1934, 177] is, I think, wrong to maintain that 'Theophrastus' criticism of Pythagoreans for treating musical intervals as numbers [Düring 1932, 62.5–10: see Barker 1977, 3–5 for text and explanation], that is, for confusing a ratio of two numbers with their difference [cf. Thrasyllus in Düring 1932, 91.14–92.8], should be read as directed against the Pythagorean tradition which (Düring thinks) the *Sectio canonis* retails.¹² For, not only does this misconstrue the *Sectio*, a document which may well not be Pythagorean, there is, so far as I am aware, no good evidence that any early Pythagorean was so benighted as to confuse notes evaluated relatively with those specified independently or absolutely. As I see it, Theophrastus' criticism is not directed against any real Pythagoreans at all: given Aristotle's scattered remarks about Pythagoreanism and the few fragments remaining of Philolaus' remarks concerning musical theory, I would say instead that Theophrastus' criticism is an assault on a straw man contrived on the basis of a literal reading of passages in Aristotle's *Metaphysics*.

2.5 Fifth argument

[8] Of these (τούτων) the multiple and superparticular are described in relation to one another by a single term (ἐν ὀνόματι). [9] In fact, we perceive some notes as concordant but others as discordant, and the concords as making a single blend out of a pair (of notes) but the discords as not. [10] Since these things are so, it is appropriate that concordant notes, being either multiple or superparticular, belong to (whole) numbers described in relation to one another by a single term (ἐν ὀνόματι), since they produce a single blend of sound (φωῆς) out of a pair (of musical notes).

¹² Ruelle [1906, 319] wrongly supposes that διόστημα in dems. 1–9 signifies a numerical difference: cf. Bowen and Bowen 1991, section 4.

These three sentences constitute a single argument which is in fact the culmination of the preface. But, though most will admit this, there is little agreement about what the argument really is.

The controversy begins with sentence [8]. What is the referent of the demonstrative in 'of these' (τούτων)? Some [e.g., Burkert 1972, 383–384; Barker 1981, 2–3; Fowler 1987, 144] think that it is 'numbers', i.e., that multiple and superparticular numbers are to be designated by a single term. Others [e.g., Jan 1895, 117–118; Tannery 1912, 218–219] suggest that the referent is 'ratios'. These views are in fact equivalent, since it is the same thing to talk of a multiple ratio and to speak of one number as a multiple of another; that is, λόγοι τοῦ ἀριθμοῦ are the same as ἀριθμοὶ πρὸς ἀλλήλους.¹³ And so on either view, the problem is to discover what this single term is, because none is given in the text.

Jan [1895, 118] consults Porphyry [Düring 1932, 98.3–6] and proposes that the multiple and superparticular ratios are *potiores* or possessed of greater power (κρείττονες), because such ratios are simpler relations than the superpartient. Barker [1981, 2–3],¹⁴ however, argues that there is in fact no general term for these ratios or numerical relations. Instead, he suggests that what Euclid alludes to is the linguistic fact that the Greeks expressed each multiple and superparticular ratio by a single term but used phrases for each superpartient. Such a thesis has the obvious advantage of explaining why no single term is given explicitly in the *Sectio canonis*—a problem which moved Jan [1895, 118–119] to posit a lacuna in the text—but like Jan's version, the resultant argument is not very convincing. After all, there is no compelling reason to connect the simplicity of multiple and superparticular ratios and the unity of concordance, or to connect linguistic practice (νόμος) in naming these ratios and the nature (φύσις) of concordant sound.

¹³ Mathematically the same, that is: there is a difference between the two locutions which raises epistemological and ontological questions about the status of relations *vis à vis* their *relata*. When one says that some number is a multiple of another, one *relatum* may be treated as subject and the other as part of a complex predicate: e.g., *p* is a-multiple-of-*q*. In this account, the relation of *p* and *q* is to be seen as a property belonging to one *relatum* and specified in terms of the other. But, when one says that the ratio, *p:q*, is multiple, the relation of *p* and *q* is characterized first as a ratio, and then this ratio is qualified by the predicate 'multiple'. Hence, the relation is at least conceived apart from the *relata* exhibiting it.

More narrowly, the difference between the two locutions is that between treating music as sequence of musical notes and as a sequence of melodic intervals.

¹⁴ Cf. Tannery 1912, 218–219; Ruelle 1906, 319; Burkert 1972, 383–384; Fowler 1987, 146–147.

There is not much to choose between these alternative accounts of the single term: where Jan focuses on the relative simplicity of the relation between the terms in multiple and superparticular ratios, the others adduce its manifestation in language. And were there no other possibilities, we would have to leave the matter here and content ourselves with a *Sectio* that simply falls apart just as it reaches its conclusion.

But let us look more closely at this final argument. As a matter of grammar, the referent of τούτων in [8] may, in fact, not be numbers or ratios but musical notes (φθόγγοι) [cf. Ruelle 1906, 319; Mathiesen 1975, 254n12]. So, though it is admittedly possible at first glance that the demonstrative τούτων refers to numbers [cf. [7]: τῶν δὲ ἀριθμῶν] or ratios [cf. [7]: ἐν τοιοῦτοις λόγοις],¹⁵ let us suppose that it picks up the subject of the immediately preceding resultative clause (ὥστε τοὺς φθόγγους . . . ἀλλήλους). Accordingly, sentence [8] would mean that multiple and superparticular musical notes (that is, musical pitches qua series of consecutive motions) when taken in relation to one another form a single class of musical sounds.

Granted, this does entail that such musical notes belong to a special class of multiple and superparticular ratios. But it would now seem possible that the term for this class is musical and not necessarily some predicate appropriate to whole-number ratios as such. In other words, the *analysantia*, certain whole-number ratios, may have a predicate appropriate in the first instance to the *analysanda*, certain musical notes as heard.

Sentence [8] thus poses the question, What is this single term for multiple and superparticular notes? Since none is given explicitly in the text, there would seem to be two ways of seeking an answer. The first is to look elsewhere in other texts for a term satisfying the requirements of the argument in sentences [9] and [10]. This is the sort of approach taken by Jan and Barker, for example. The second is to consider the train of thought leading from sentence [8] to sentences [9] and [10] in order to see whether the term figures implicitly in the argument. (Of course, it is entirely possible that the term is simply unrecoverable, that there is an unbridgeable gap at this point in the logic of the preface.¹⁶)

In considering the transition from sentence [8] to sentence [9], let us not forget that [8], as I construe it, is about objective musical sound. Suppose, then, that one musical note (qua series of consecutive motions) is 'taken

¹⁵ Mathiesen [1975, 254n12] dismisses out of hand the possibility that Euclid is thinking of the linguistic fact that the Greeks use single terms to designate multiple and superparticular ratios.

¹⁶ Such a break in logic, however, need not signify a lacuna in the text as it stands.

in relation to' a second. This means that these two notes manifest a whole-number ratio. When one regards the same two notes phenomenally, this whole-number ratio turns out to be the reality of the separation or interval (διάστημα) heard between the notes [cf. section 2.4, above]. In other words, the phenomenal counterpart of the claim that multiple and superparticular notes (qua series of consecutive motions) are described in relation to one another by a single term is that the intervals defined by these notes are determined by a single class of multiple and superparticular ratios. So the question about the single term is at the same time a question about a class of intervals or notes as heard.

Now, sentence [9] presents a distinction among phenomenal musical notes: those perceived as concords make a single blend of sound, whereas those perceived as discords do not. I emphasize that this distinction is not necessarily a dichotomy: contrary to the usual understanding of this passage, the text actually leaves open the possibility (a) that some melodic notes are neither concordant nor discordant, (b) that not every pair of notes perceived as a single blend of sound is a concord, and (c) that not every pair or notes *not* heard as a unified sound is a discord. (Note that those who assume a dichotomy quickly encounter difficulties in other parts of the *Sectio* which often they then use to impugn it [cf. n18, below].) Further, given that Euclid identifies phenomenal and objective musical sound, it would appear that the pairs of multiple and of superparticular musical notes (qua series of consecutive motions) mentioned in sentence [8] may either be concordant or discordant, that the single term said to designate these notes (objectively construed) may either be 'concordant' or 'discordant'.¹⁷

Sentence [10] continues as an inference from sentences [8] and [9]—as the phrase 'since these things are so' indicates—supplemented by way of two subordinating constructions. In effect, the inference in [10] is:

- (p₁) since (pairs of) concordant musical notes are either multiple or superparticular
- (p₂) since (pairs of) concordant notes are heard as a single blend of sound

- (P) it is appropriate that (pairs of) concordant notes belong to ratios designated by a single term.

¹⁷ Mathiesen [1975, 254n12] maintains that the term in question is 'concordant' and cites the same passage from Porphyry [Düring 1932, 98.3–6] which Jan adduces to show that it is κρείττων.

To unpack this and the final argument as a whole we need to determine the relation between sentences [8] and [9], and the subordinating constructions in [10] represented as premisses p_1 and p_2 . It is obvious that p_2 recasts sentence [9]. So, does p_1 reformulate [8]? If we suppose it does, we do get the result that the term for multiple and superparticular notes (*qua* series of consecutive motions) is 'concordant'. Unfortunately, we also get an unproven conversion: saying that multiple and superparticular musical notes (*qua* series of consecutive motions) are concords (so sentence [8]) is not the same as saying that concords are multiple or superparticular. Thus, we should allow the phrase, 'since these things are so', some real significance and treat [8] as an independent premiss in the final argument of the *Sectio*. Accordingly, let us combine sentence [8] and p_1 as

- (p_3) any pair of musical notes (*qua* series of consecutive motions) is designated by a single term, 'concordant', if and only if one is a multiple or superparticular of the other.¹⁸

¹⁸ There are several features of this premiss to observe here. First is that the scope of p_3 is limited to the domain of phenomenal musical sound: there is no reason to suppose that Euclid countenances an unlimited plurality of concords on the ground that there is an unlimited number of multiple and superparticular ratios. For a clear statement of the issue and of the various positions discerned by Adrastus (who is much cited by Theon of Smyrna), see Hiller 1878, 64.1–65.9.

Next, there is the claim by Aristoxenus and later writers that the interval of an octave and a fourth (8:3) is a concord. But do such claims indicate that p_3 is false? Barker [1981, 9–10] maintains that they do, on the ground of p_2 . In other words, he takes it for granted that Aristoxenus' assertion [Da Rios 1954, 25.17–26.1; 56.10–18] that the addition of an octave to any concord yields an interval which will be heard as a concord, is an accurate report of what Aristoxenus' contemporaries actually heard; and concludes that the *Sectio*, by virtue of p_2 , is obliged to allow for this. But I think this concedes and requires far too much. To begin, unlike Barker I do not think that p_2 entails that every sound heard as a single blend is a concord: so, even if the interval of the octave and a fourth was heard as a single blend by Aristoxenus and his contemporaries, it does not follow for Euclid that it is a concord. (Nor, given that [9] does not state a dichotomy of intervals into concords and discords, does it then follow that it is a discord.) Further, Aristoxenus himself provides evidence [Da Rios 1954, 29.5–30.9] of disagreement in matters of musical hearing and of a tendency to extol music others find disagreeable. Indeed, my suspicion about Aristoxenus' claim regarding the interval in question is that it may well be a conclusion drawn from the (rather abstractly stated) principle that any octave added to a concord produces a concord. And, if this suspicion is right, then the claim about the interval of the octave and a fourth may in fact be wholly polemical. In any case, the real problem here is the use of Aristoxenus' testimony and, more generally, determining the relation between ancient harmonic science and musical practice. Solving this problem will, in the present instance, be extremely difficult: for, not

Next, there is the problem of the role of p_2 in sentence [10]. If the conclusion, P , is about objective sound, the inference in [10] becomes very puzzling, since p_2 concerns phenomenal sound. But, if P is about phenomenal sound, p_2 is essential.

I propose, then, to recast the final argument in sentences [8]–[10] (with redundancies) as follows:

- (p_2) since pairs of concordant notes are heard as a single blend of sound [cf. [9]]
 (p_3) given that any pair of musical notes (*qua* series of consecutive motions) is designated by a single term, 'concordant', if and only if one is a multiple or superparticular of the other
-
- (P) it is appropriate that pairs of concordant notes heard as a blend of sound and being multiple and superparticular (*qua* series of consecutive motions) belong to (*scil.* are in reality) pairs of (multiple and superparticular) numbers designated in relation to one another by a single term.

The reader will notice that I have elaborated the conclusion, P , by spelling out (a) that the numbers to which multiple and superparticular notes belong are, in the first instance, themselves multiple and superparticular; and (b) that pairs of notes belong to pairs of whole-numbers in the sense that the latter are the reality with which the former are identified through reductive analysis.¹⁹ Yet, this is not enough. The argument still needs an additional premiss,

only is there no independent evidence confirming that the Greeks of Euclid's time heard the octave and a fourth as a concord, it is clear that the harmonic science he presents is not intended to accommodate all of musical perception [cf. section 2.2, on pitch].

Finally, it seems to follow from p_3 that the tonic interval (9:8) is a concord, though, as is well known, Aristoxenus classifies this interval as a discord [cf., e.g., Da Rios 1954, 25.11–15, 55.12–56.5]. Whether Aristoxenus' views on the matter are a suitable basis for interpreting or criticizing the *Sectio* is a question that arises here too. Euclid, in any case, does not explicitly call this interval a discord, though the closing lines of dem. 12 [Menge 1916, 174.6–7]—if they are not interpolated—may be a problem, since they suggest that it is not a concord. Cf., e.g., Adrastus in Hiller 1878, 50.16–21.

¹⁹ That the 'belonging to' locution signifies the final step in the reductive analysis is clearer given the language of dem. 1: cf. Bowen and Bowen 1991, section 4. See also section 3, below.

- (p_4) characteristics of musical pitches uniquely determined by relations among musical notes qua series of consecutive motions derive from characteristics of the numerical relations which are the reality of what is heard.

Though this premiss does not appear in the text itself, it (or something like it) is certainly necessary on my interpretation of the sentences [8]–[10]; so, I introduce it here as my second hypothesis. p_4 is an adjunct of the eliminative ontological reduction that is essential (again, on my reading) to the *Sectio canonis*. In effect, p_4 isolates a subset of the predicates applied to music as heard (the *analysanda*) and asserts that these predicates hold because they apply above all to the numerical relations (*analysantia*) constituting what the sensible musical relations really are. Thus, sentences [8]–[10] set forth the argument that the musical notes we hear as concordant are, qua series of consecutive motions, multiple or superparticular and so are in reality multiple and superparticular ratios designated by a single term. And, given this much, it seems simplest to conclude that this single term is ‘concordant’ as well—and so I follow all who assume that the single term mentioned in sentences [8] and [10] is the same.

On this reading, then, the upshot of the final argument in sentences [8]–[10] is a justification of the thesis that concords belong to concordant numbers. This a result quite different from the usual claim that the point of the preface is to explain why the notes we hear as concordant are either multiple or superparticular [cf., e.g., Tannery 1912, 218–219; Ruelle 1906, 318; Barker 1981, 3], or to show that the study of musical notes ‘should be assimilated into mathematics’ [Fowler 1987, 146]. As I see it, the preface answers the question, Why are concordant notes concordant?, by proposing that concordant notes are heard as concordant because they are in reality concordant numerical ratios.

But what is the context for such a question and answer? Clearly, it is not Academic [but see Tannery 1912, 218]—at least, not as one might surmise given the question raised in Plato, *Resp.* 531c1–4, when Socrates asks which numbers are concordant and which are not and why in each case. Yet without some sense of the context, it is virtually impossible to assess the importance of the question or the adequacy of the answer, beyond determining the role of the preface in the subsequent theorems. So, since I have postponed the latter project to another occasion, I will turn now to the question of the context of the *Sectio canonis*.

3. Euclid's *Sectio canonis* and Pythagoreanism

While debate about the authorship of the *Sectio canonis* still continues, there is, in contrast, a consensus that this work is in the intellectual tradition we call Pythagorean [cf., e.g., Heath 1921, ii 444–445; Barbera 1984; Fowler 1987, 144]. The broad similarity between the preface to this treatise and a fragment [cf. Bowen 1982] of a work by Archytas on music is obvious and, though one may well doubt Jan's claim [1895, 146] that the source for the bulk of the treatise is Archytas, there is no denying that the *Sectio* retains one proof [cf. *dem.* 3: Menge 1916, 162.6–26] which is attributed to Archytas by Boethius (AD 480–524) in his *De institutione musica* [Friedlein 1867, 285.9–286.4]. Moreover, as Düring suggests [1934, 176–177], the opening sentence of the preface to the *Sectio* compares favourably with what Heraclides Ponticus (late 4th cent. BC) may be ascribing to Pythagoras in the first few lines of the fragment of his *Harmonica introductio* [Düring 1932, 30.7–8] preserved by Porphyry (AD 232–ca. 305).

But, regrettably, just as the debate about the authorship of the *Sectio canonis* may be ill-founded, so may this consensus about its philosophical character. There are, for instance, significant differences between the musical analysis in this treatise and the theory we may attribute to Archytas and which we find repeated in the works of Theon of Smyrna, Nicomachus of Gerasa (both second century AD), and of Boethius, for example. First, as I have already noted, whereas Euclid proposes to justify quantifying musical notes by means of the premiss that each pitch depends on (is) the relative numerosity of the series of consecutive, airy projectiles which strike the ear and produce what is heard as one sound, Archytas [Bowen 1982] maintains that pitch is determined by the relative speed/force of the airy projectile [cf. Archytas, Fragment 1.45–46]. Now, the same view as Archytas' (without the reference to force) is found in Nicomachus' *Harm. man.* [Jan 1895, 242.20–243.10] and in Theon's *Expositio* [Hiller 1878, 60.17–61.11]. Moreover, in Boethius' *De inst. mus.*, there is in book 1 an account fashioned after the preface of the *Sectio* [cf. Friedlein 1867, 189.15–191.4] which adapts it to Archytas' view, and in book 4 a translation of the *Sectio* that departs from the original on this very point [cf. Friedlein 1867, 301.17–18; Bowen and Bowen 1991, section 4]. What this all means is difficult to say.

Though Nicomachus is a Pythagorean and Boethius follows him in harmonic science,²⁰ and though Adrastus (according to Theon [Hiller 1878, 50.4–21]) attributes the sort of account found in the fragment from Archytas to the Pythagoreans, one should hesitate to say that is Pythagorean, if only because Adrastus [cf. Hiller 187, 61.11–17] also ascribes the same view to Eudoxus, who was not, so far as I am aware, regarded as a Pythagorean at any point in antiquity, and because Theon seems to be a Platonist. Indeed, the story is even more complicated.

Consider Fowler's proposal [1987, 145–146] to assimilate the *Sectio* to the Lyceum on the strength of *Prob.* xix 39 and a passage from Porphyry, *In harm.* attributed to Aristotle [Düring 1932, 75.14–27: cf. Barker 1984–1989, ii 98]. Now, in the passage from Porphyry, pitch is correlated with the speed of the motions striking the ear, whereas, in the *Sectio*, pitch is identified with the relative numerosity of these motions [cf. Barker 1984–1989, ii 98, 107n40]—as it is in *Prob.* xix 39 [cf. Barker 1984–1989, i 200–201]. So, it would seem that the thesis of the dependence of pitch on the speed of the motion striking the ear may not be peculiar to the Pythagoreans. In any case, Euclid and the author of *Prob.* xix 39—who is no longer thought to be Aristotle—are the odd men out in this group. Yet this hardly puts Euclid in the Hellenistic Lyceum. Not only is there no good evidence about the provenance of the compilation known as the *Problemata*, the preface of the *Sectio* only requires that relative pitch depend on (be) the relative numerosity of pairs of series of consecutive motions, a thesis which is intelligible and quantifiable as I have indicated, and which does not suppose or need the sort of talk found in *Prob.* xix 39 about the incidence of the pairs of series on the ear.

Furthermore, according to Aristotle, the Pythagoreans thought that all things are number and did not make the sort of ontological separation between appearance and reality found in the Platonic corpus. But, if the Pythagoreans maintained that number and numerical relations constitute the reality of all there is, then, it is interesting to observe that, for Euclid, though phenomenal musical notes are composed of series of consecutive motions which (therefore) stand to one another in numerical ratios, and though these series are said to belong to numbers, they are not said to be composed of numbers. In other words, Euclid appears to regard numbers as the reality of musical sound but—so far as I can tell from his language—he

²⁰ This is an inference based on the general character of the *De inst. mus.*, on the nature of Boethius' references to and treatment of Pythagoras and the Pythagoreans, and on how claims Boethius makes in his own voice (usually in the first person plural) compare with what he says of the Pythagoreans: cf. *De inst. mus.* i 9, ii 21–27, v 8.

does not treat them as a reality *constituting* what is heard. Indeed, Euclid leaves open the possibility of a different account of the relation between appearance (what we hear) and reality. Moreover, in the fragment from Archytas, the pitch of the sound is said only to vary as the speed/force of the motion producing sound at that pitch; it is not claimed that the pitch is composed of motion at this speed/force. Likewise, in the treatises by Nicomachus, Theon, and Boethius—all of which agree with Archytas in correlating pitch and speed—there is no such reduction of sound as heard to the speed of motion. Thus, again, Euclid stands alone: his account fits neither Aristotle's outline of Pythagorean analysis nor the accounts given by such Pythagoreans as Archytas and the others.

Now I admit that such differences may only signify a divergence between rivals schools of the Pythagorean family. But, in the absence of independent evidence confirming this, we should not ignore the possibility that the *Sectio canonis* analyzes music from a standpoint, and for purposes, alien to Pythagoreanism. This means that we should resist the temptation to minimize these differences by carelessly lumping this treatise with other Pythagorean writings and, even worse, by interpreting all these texts in terms of one another.

The deeper problem in addressing the question of Euclid's philosophical allegiances such as they appear in the *Sectio canonis*, however, is that the modern, scholarly category of Pythagoreanism is not well defined in harmonic science, no doubt in part because the Pythagorean version of the science itself still eludes satisfactory interpretation. Most of the criteria currently used to classify a theory as Pythagorean are based upon ancient descriptions of the intellectual schools of thought. Unfortunately, when the ancient musical theorists do make remarks about their predecessors and contemporaries, they do not write as historians following the rules of evidence and interpretation which we now take for granted. Indeed, the most one should concede at the outset is that their classifications and criticisms of intellectual trends and so on may hold at best of the period and cultural context in which they were writing. Thus, for example, Andrew Barker [1978a] has argued that Ptolemy's characterization in the *Harmonica* of the controversy dividing the Pythagorean and Aristoxenian schools of musical theory does not hold of the fourth century BC. Yet, Barker [1978a, 1] still takes it for granted that 'a solid amount of what is attributed to these schools by such writers as Ptolemy and Porphyry quite genuinely goes back to the fourth century, to Aristoxenus on the one hand, and perhaps to Archytas and his followers on the other,' though this should be a matter for argument and proof if we are ever to get an accurate account of Greek harmonic science.

But surely, one may ask, can we not follow the ancients and suppose [cf., e.g., Barker 1981, 3; Fowler 1987, 144] that a theory is Pythagorean if it analyzes music by means of whole-number ratios and prefers reason to hearing in determining what is musical? Granted, these criteria appear to be adequate to the fifth and fourth centuries BC (albeit perhaps because we have so little clear, direct evidence of Pythagorean musical theory from this period). But, on the basis of these criteria, one might also conclude that the *Harmonica* by Ptolemy (ca. AD 150) is a Pythagorean text [cf. Barker 1984–1989, ii 270–271]. And this certainly does no good. For, not only does it conceal the profound differences in epistemology and argumentation which exist between the *Harmonica* and, say, the roughly contemporary *Harmonices manuale* by Nicomachus of Gerasa [cf. Bowen and Bowen 1991, section 3], it also ignores the fact that much of the material in Nicomachus' treatise may also be found in Theon's *Expositio*, a treatise which draws from Peripatetic sources (especially, Adrastus [cf. Hiller 1878, 49.6]) *inter alia* in order to elaborate what is needed to understand Plato. In short, these two criteria quickly prove inadequate to the complexity of relations between the ancient documents concerning music which we do possess.

Likewise, I see no reason to pursue Barbera's contention [1984] that the proper context for interpreting the *Sectio canonis* is the Pythagorean tradition which he thinks is defined by Theon and Nicomachus. Indeed, it begs the question. For, though Nicomachus presents his own work as Pythagorean, Theon makes little mention of the Pythagoreans except to point out where they agree with views he has already stated, and he introduces many of the same points as Nicomachus but as part of a general learning (some of it drawn from Peripatetic sources) that is propaedeutic to the study of Plato's writings. Thus, on what basis and how are we to decide whether the doctrine in question is Pythagorean? But this is the very question we started with. Further, if we follow Nicomachus and regard the doctrine as Pythagorean, should we also follow Theon and suppose that it was generally viewed as propaedeutic to Platonic philosophy? And what antiquity are we entitled to assign this doctrine in any case? But, until these questions, as well as others pertaining to the schools of harmonic science in the second century AD, are answered satisfactorily, there is little to be gained by using Nicomachus and Theon as authorities in interpreting a treatise written perhaps some four hundred years earlier.

In sum, the claim that the *Sectio canonis* is Pythagorean is, by rights, not a starting point but a conclusion; and the same holds of the too often repeated assertion that Euclid was a Pythagorean [cf., e.g., Menge 1916, xxxviii]. Moreover, the argument leading to this conclusion about the *Sectio* will be very arduous indeed. For, not only will it have to deal with

this treatise itself, it will have to uncover plausible criteria of Pythagoreanism in harmonic science, criteria which may well differ from period to period. As matters stand now, we are not sufficiently informed to locate the *Sectio* in a Pythagorean context. But, until we are, we must resist the temptation to speculate by using it, for example, to elaborate the criticism of the Pythagoreans found in book 7 of Plato's *Republic* [cf. Barker 1978b].

Conclusion

The preface to Euclid's *Sectio canonis* has puzzled readers for more than two millennia. Even the ancients found it difficult, if the versions offered by Porphyry [Düring 1932, 90.7–23] and Boethius [Friedlein 1867, 301.7–302.6] are any indication: both Porphyry and Boethius omit the last argument. The main reason, as I interpret the treatise, is that by compressing the reductive, eliminative analysis at its core to the requirements of a deductive or inferential expository style, Euclid obscured his point. This is not, however, a criticism. It is very difficult to present an argument involving an eliminative, ontological reduction, when this reduction necessitates systematic ambiguity in the use of key terms (e.g., $\phi\theta\acute{o}\gamma\gamma\omicron\varsigma$ as 'the musical note or pitch heard' and as 'the series of consecutive motions that strike the ear producing a note at that pitch').

But if so, then harmonic science raised problems for Euclid not found in arithmetic and geometry. One has to be careful, then, in assessing criticisms of the *Sectio canonis* which take the *Elements* as a paradigm of style. As for completeness, let us observe that there are no hints in the manuscript tradition that the preface to the *Sectio* is part of a larger introduction. So, in this limited sense at least, what we have is complete. Yet, is the preface incomplete because it lacks the preliminary suite of definitions and so on that one would expect given the *Elements*? On balance, I would say that even in this sense the preface is complete. For, though the question itself, Why are concords concordant?, is unstated and the single term is implicit, what is written does constitute a very economical, compressed, and coherent answer; and to require a more elaborate account in which all is spelled out (for our benefit) seems unwarranted. Still, the contention that the preface is complete will not be demonstrated satisfactorily in the absence of a reading of the entire treatise showing its unity and coherence, or without a thorough study of the other Euclidean treatises and of the corpus of texts in harmonic science that aims to discover the relevant criteria of exposition and argumentation.