



Geometry in nature and Persian architecture

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Abstract

Nature displays profound preference for certain specific ratios to design her life-forms. These are geometric relationships that are transcendent and originated from Sacred Geometry. The view that geometry had a ritual origin is a part of a wider view that civilisation itself had a ritual origin, and therefore the history of utilisation of Sacred Geometry by man goes back to many centuries ago. The Pythagorean tradition, and the Egyptian and Babylonian sciences from which it derived, and Persian mathematics, a part of which reflects a Pythagorean intellectuality, are based on the sacred conception of numbers and their symbolism. In the traditional world, geometry was inseparable from the other sciences of the Pythagorean *Quadrivium*, namely arithmetic (numbers), music and astronomy. Traditional geometry is related to the symbolic configurations of space. Geometric forms such as the triangle, square and various regular polygons, the spiral and the circle are seen in the traditional perspective to be, like traditional numbers, as aspects of the multiplicity of the Unity.

Architecture itself has always had a sacred meaning to all traditional civilisations through millennia, by which means man has tried to provide for himself a manifestation of heavens. Persian architecture always emphasised on Beauty, and by means of Sacred Geometry Persians measured the proportions of heaven and reflected them in the dimensions of buildings on the earth. A comprehensive utilisation of proportions in Persian architecture, such as in the design of plans, elevations, geometric and architectural patterns, and mechanical and structural features, can be proved through geometrical analysis of Persian historical buildings.

In this paper, the sacred conception of geometry and its symbolism in the Pythagorean tradition, and Sacred Geometry and proportions in natural life-forms will be explained. The use of the science of geometry in design of a number of Persian historical buildings will be presented. The geometric factors upon which the design of these buildings, from both architectural and structural viewpoints, is made will be discussed.

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1. Introduction

In Persian and Arabic, the term *muhandis* (engineer) is derived from *hindisah* (the common word for geometry) with the meaning of measuring and it is used for both the sciences of geometry and architecture. The Greek γεωμετρία (geometry), composed of γεω—combination form of γη (earth) and—μετρία (measuring), in etymological sense means *the art of measuring ground*.

The word *architect* is derived from the Greek ἀρχιτέκτων, composed of ἀρχι—(chief, principle, first in authority or order) and τέκτων (builder, craftsman); thus, the word literally means a master-builder or a skilled scholar of the art of building. As an architect skilfully builds up the reflection of Divine Beauty in Universal Order, the word architecture is close in meaning to the Greek κόσμος (cosmos), which means at once the world, order and beauty, and to αἰσθησις (aesthetics), perception by the senses especially by feeling.

The emphasis of Persian architecture was on Beauty. Persians always placed a high value on beauty through

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many centuries; and *scientia geometriae* was the powerful tool of the Persian *muhandis* by which he was able to measure the proportions of heaven and create equilibrium, harmony and beauty on the earth; to put beauty in order. Geometry was both *scientia* (science) and *ars* (art).

The ultimate object of Persian traditional architecture was the Absolute. Architecture was a symbolic language by which Archetypal Ideas could be expressed in patterns which were conceivable by human understanding. As the object of architecture was in the realm of spirit and wisdom, geometry as a tool by which Persian architects built up the shapes of planes and bodies had to be holy in itself.

If the origin of sacred geometry is to be found, it would suffice to return to the oldest known civilisations in which geometry did govern the design of sacred buildings destined to represent the imagined structure of the Universe as the domain of the Absolute, and such a geometry is holy by virtue of its power to please and attract the Divine Nature. The most obvious instance is the structure of the domed temple in its geometric form symbolising the heavenly sphere above embracing the earthly circle or square below.

The Universe, and Nature, created by the Absolute as a rational and therefore mathematical reality, the highest manifestation of Divine Wisdom, are reflected in a mystic and symbolic language by the sacred architectural buildings to express Divine Order, Harmony and Beauty. Certain geometric patterns and their related numbers, as references to cosmological concepts, play a symbolic role in this architectural creation.

2. The universe as geometrisation of Divine Unity

2.1. Geometric progression

Plato (c. 427–347 BC) has great respect to Pythagoras (582?–500? BC) and the *Musica Mundana*, the Music of the Spheres, and in his book *Timaeus* [1] describes the geometric creation of the world.

In the *Timaeus*, he presents the idea that the Creator created the visible world similar to a geometric progression. The *Platonic Solids*, five solids, make up the four elements and heaven.

On geometry he writes in his *Republic* [2], “[geometry is]...persuaded for the sake of the knowledge of what eternally exists, and not of what comes for a moment into existence, and then perishes...[it] must draw the soul towards truth and give the finishing touch to the philosophic spirit”.

Over his academy door were the words ἀγεωμέρητος μηδεὶς εἰσὶτω meaning, “let no one enter who is lacking in geometry”.

In the *Timaeus*, Plato describes the need for the four elements. Firstly, *fire* to make the world visible, and *earth* to make it resistant to touch. Fire belonging to heaven and earth to earth, these are the two *extreme* elements. He writes, “...it is necessary that nature should be visible and tangible ...and nothing can be visible without fire or tangible without earth...”. Secondly, they need a third as a bond to be connected together, “...but it is impossible for two things to cohere without the intervention of a third...”. Thirdly, the best bond is the geometric proportion, “...[and] the most beautiful analogy is when in three numbers, *the middle is to the last as the first to the middle*, ...they become the same as to relation to each other”. Fourthly, the primary bodies are solids, and must be represented by solid numbers (cubes). One mean is enough to connect two plane numbers (squares), but two means are required to connect two solid numbers, “but if the universe were to have no depth, one medium would suffice to bind all the natures it contains. But...the world should be a solid, and solids are never harmonised by one, but always by two mediums”. Therefore, the Creator put *water* and *air* in the middle of fire and earth, making them in the same ratio to each other; so that *fire might be to air as air is to water and that water is to earth*.

$$\text{fire/air} = \text{air/water} = \text{water/earth.}$$

As the ratio between successive elements is constant, it gives a *geometric progression*.

In the *Timaeus*, Plato describes that all that exists is a Unity, “God, purposing to make the universe most nearly like the every way perfect and fairest of intelligible beings, created one visible living being, containing within itself all living beings of the same natural order”. For Plato the harmony and proportion throughout the creation, be it architecture, art or music, make the multiplicity of things as a single reality. The harmoniously interdependent relationship of parts within the visible world is itself a reflection of the same harmonious relationship within invisible world and also between the visible and invisible worlds. Harmony and proportion have close relationship with Universal Order and thus lead to the concept of Aesthetics, Beauty, and of Cosmos. The Creation is Beautiful because it *harmoniously* and *proportionally* reflects Divine Beauty who made it according to that Beauty.

2.2. Systems of proportions

The selection and use of systems of proportions has always been an important concern for artists and architects. There were not only specific ratios used, but also some *systems of proportions* were preferred. Some systems of proportions were based on the musical intervals, the human body, and the Golden Ratio.

Proportion in geometry, architecture, music and art can be said to be “an harmonious relationship between the parts, with and within the whole”. Vitruvius (70?–25 BC), a Roman architect and engineer, writes in his *Ten Books on Architecture* [3], *De Architectura*, that is the oldest surviving work on the subject, “symmetry is a proper agreement between the members of the work itself, and relation between the different parts and the whole general scheme, in accordance with a certain part selected as standard”. And later, “therefore since nature has proportioned the human body so that its members are duly proportioned to the frame as a whole, ...in perfect buildings the different members must be in exact *symmetrical relations* to the whole general scheme”. By *symmetrical relations* Vitruvius means *the same proportions*. It is through the system of proportions that all parts are harmoniously interrelated with and within the whole; therefore providing a pleasing and functioning design.

2.3. The Quadrivium

The division of mathematics into four groups dates back to the Pythagoreans. The *Quadrivium*, Arithmetic (number), Geometry (as number in space), Music (or Harmony as number in time), and Astronomy (or Cosmology as number in time and space), as Plato points out, were as means for studying σοφία, the highest kind of knowledge; Wisdom. The practice of the *Quadrivium* is the practice of number, of figure, of sound and of motions of the heavens.

2.4. Music of the Spheres

In the *Timaeus*, Plato introduces the idea of the *world soul* being the synthesis and intermediary between the unchanging Essence of the universe and the changing existence of the physical universe itself. This *soul*, the intermediate existence, has been divided into harmoniously proportional subdivisions and formed into a long strip by the Creator. The strip was then marked off into intervals.

First [the Creator] marked off a section of the whole	1
And then another twice the size of the first	2
A third three times the first	3
A fourth twice the size of the second	4
A fifth three times the third	9
A sixth eight times the first	8
A seventh 27 times the first	27

The obtained seven integers; 1, 2, 3, 4, 8, 9 and 27, are composed of the monad, source of all numbers, the first even and first odd, and their squares and cubes. They can be arranged as two progressions; the geometric

progression by 2 (left) and the geometric progression by 3 (right):

Monad	1	Point
First even and odd	2 3	Line
Squares	4 9	Plane
Cubes	8 27	Solid

This is called *Plato's Lambda*, as its shape looks like the Greek letter λ .

Directly after his description of the *Lambda*, Plato shows that the multiplication of 2 and of 3 gives all the numbers for the Pythagorean music system by successive multiplication by fifths ($3/2$). He uses an arithmetic mean and a harmonic mean for generating number for a succession of musical octaves, fourths, and fifths. In fact, in music it is the insertion of the arithmetic and harmonic means between the two extremes in double ratios, representing the octave double, that gives the progression known as the *musical proportion*, i.e. 1, $4/3$, $3/2$ and 2, respectively, representing the frequencies of a fundamental, fourth, fifth and octave.

For example, for the first interval

$$\text{Arithmetic mean} = (1 + 2)/2 = 3/2.$$

The harmonic mean of two numbers is the reciprocal of the arithmetic mean of their reciprocals. For 1 and 2, the reciprocals are 1 and $1/2$, whose arithmetic mean is $(1 + 1/2)/2$ or $3/4$. Thus,

$$\text{Harmonic mean} = 4/3.$$

Taking the interval between the fourth ($4/3$) and the fifth ($3/2$) as a full tone

$$\frac{3/4}{3/2} = \frac{3}{2} \times \frac{2}{3} = \frac{9}{8}.$$

Plato then fills up the scale with intervals of $9/8$, the tone; hence, intervals of $256/243$ as remainders, the half tone.

Plato generated the musical scale with *arithmetic calculations*, and not by the division of the vibrating string in different proportions as did the Pythagoreans.

The Pythagoreans supposedly constructed the musical scale by experimenting with a stretched string with a moveable bridge. For a string divided in half the harmonic mean between $1/2$ and 1 is $2/3$, the musical fifth, and the arithmetic mean between $1/2$ and 1 is $3/4$, the musical fourth. This gives the progression 1, $3/4$, $2/3$, $1/2$. They are

8/16 or $1/2$	Octave	Diapason
4/6 or $2/3$	Fifth	Diapente
9/12 or $3/4$	Fourth	Diatessaron

In comparing these two progressions, an inversion of ratios and a crossing of functional positions between the arithmetic and harmonic means is evident (Table 1) [4].

To the Pythagoreans the solar system consisted of 10 planets revolving in circles around a central fire. The planets produced harmonious sounds according to their distances from the centre. The distances between the planets were similar to the subdivisions of a stretched string. This was called *Musica Mundana*, which is usually translated as Music of the Spheres. The produced sounds are so exquisite that ordinary ears are unable to hear it. This music is present in all cycles and rhythms of nature such as biological cycles, seasons and the movements of the planets. In the *Republic* in the *Myth of Er*, Plato with homage to Pythagoras writes about the cosmos, "...upon each of its circles stood a siren who was carried round with its movements, uttering the concords of a single scale", and he, in his *Timaeus*, describes the circles of heaven subdivided according to the musical ratios. There he describes the forming of the circular paths for the stars by the Creator, "He cut the whole fabric into two strips, which He placed crosswise at their middle points to form a shape like the letter X; He then bent the ends round in a circle and fastened them to each other ...to make two circles, one inner and one outer".

Kepler (1571–1630 AD) writes in his *Harmonice Munde* [5] that he wishes "to erect the magnificent edifice of the harmonic system of the musical scale ...as God, the Creator Himself, has expressed it in harmonising the heavenly motions". And later, "I grant you that no sounds are given forth, but I affirm ...that the

movements of the planets are modulated according to harmonic proportions".

2.5. Systems of proportions based on the musical ratios

The renaissance architect Leone Battista Alberti (1404–1472 AD) in his *Ten Books of Architecture* [6] writes, "[I am] convinced of the truth of Pythagoras' saying, that Nature is sure to act consistently.... I conclude that the same numbers by means of which the agreement of sounds affect our ears with delight are the very same which please our eyes and our minds". This is in agreement with the idea of Plato that those ratios pleasing to the ear would also be pleasing to the eye. Musical ratios have therefore a close relationship with art or architecture, and they can be regarded as basis for artistic designs.

2.6. The Platonic Solids

In the *Timaeus*, Plato describes the way the Divine Creator made up the visible world. The five elements were attributed to the basic solids, called the Platonic Solids. These are the only possible *regular* polyhedra whose faces are identical *regular* polygons: *tetrahedron* with four equilaterally triangular faces, *cube* with six square faces, *octahedron* with eight equilaterally triangular faces, *dodecahedron* with 12 regularly pentagonal faces, and *icosahedron* with 20 equilaterally triangular faces (Fig. 1).

2.7. The elements linked to the Platonic Solids

Plato, in his *Timaeus*, shows that the four basic elements of the world are earth, air, fire and water. He associates four of the *Platonic Solids* with the four elements; the cube with earth, the icosahedron with water, the tetrahedron with fire and the octahedron with air, "we must proceed to distribute the figures [the solids] we have just described between fire, earth, water, and air.... Let us assign the cube to earth, for it is the most immobile of the four bodies and most retentive of shape, the least mobile of the remaining figures

Table 1
Comparison between musical proportions from the viewpoint of vibration and string length

	Note	HM (Fourth)	AM (Fifth)	Octave
Vibration	1	4	3	2
	6	$\frac{3}{8}$	$\frac{2}{9}$	12
String length	12	9	8	6
	1	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$

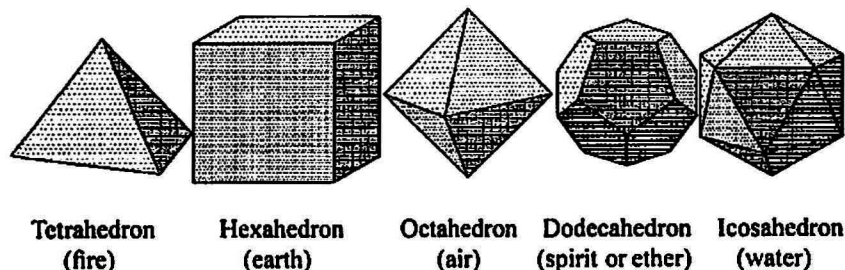


Fig. 1. The Platonic Solids.

(icosahedron) to water, the most mobile (tetrahedron) to fire, the intermediate (octahedron) to air”.

Plato writes about a *certain fifth composition* used by the Creator in creating the universe, “there still remained a fifth construction, which the god used for embroidering the constellations on the whole heaven”. By *embroidering the constellations* he means the container or *whole* that is *dodecahedron* with 12 faces that are regular *pentagons*. The 12 faces are related to the zodiac and the *whole cosmos*. The *Golden Ratio* governs the shape of a pentagon and for Pythagoreans it symbolises the generation of the cosmos, spirit or ether. Thus, the dodecahedron was associated with the fifth element *ether* or heaven or the cosmos (Fig. 1).

3. Sacred geometry in nature

In nature, systems of patterns as geometric structures of form and proportion can be found from the minutest particles to the greater cosmos. Life is interwoven with geometric forms, such as the angles of atomic bonds in the molecules, the spherical shape of the cell that itself develops with a geometric progression from one to two to four to eight cells and beyond, the helical spirals of DNA, and the lattice patterns of crystals.

Geometry is the practice of forms through the measure and relationships, by which means each form can be unfolded out of a preceding one, i.e. *geometrical archetypes*. Reality, as Plato stated, consisted of *Arche-typal Ideas*, or pure essences, of which the visible world is only a reflection. The senses cannot perceive this metaphysical realm. Geometry makes use of the visible forms to describe these Ideas. Sacred Geometry opens out the oneness underlying all geometric forms and the inseparable relationship of the part to the whole, and continuously reminds the Unity and sacred origin of all things created.

Alberti, in his *Ten Books of Architecture*, defines the natural beauty of forms and proposes that beauty is an agreement of constituent parts with the *Law of Nature*. “The ancients...did in their works propose to themselves chiefly the imitation of Nature, as the greatest Artist at all manner of compositions”. For Alberti, *Law of Nature* is the rule of proportions and mutual correspondence of parts with and within the whole Nature, “... the rule of these proportions is best gathered from those things in which we find Nature herself to be most complete and admirable.... Nature is sure to act consistently, and with a constant analogy in all her operations”.

3.1. The Golden Ratio

The Golden Ratio (also called as the Golden Proportion, Golden Mean, Divine Ratio, Divine Pro-

portion, sacred cut, or simply ϕ (ratio)) is a supra rational or transcendent ratio found in fundamental forms: plants, flowers, viruses, DNA, shells, planets and galaxies. It is often designated by the Greek letter ϕ , for Phideas (c. 490–430 BC), Athenian sculptor and artistic director of the construction of the Parthenon, who supposedly used the Golden Ratio in his work. Although the Golden Ratio is first and foremost a proportion, not a number, as a numerical quantity it is $\phi = (1 + \sqrt{5})/2$ about 1.618. The Golden Ratio is the unique ratio of two terms when the ratio of the larger term to the smaller term is in the same way as the smaller plus larger to the larger (Fig. 2). It symbolises the regeneration and progression and extension from the Unity as each generation is linked to its ancestors. It is the perfect division of the Unity.

The Golden Ratio has some unique properties:

- $\phi = 1 + 1/(1 + 1/(1 + 1/(1 + 1/(1 + 1/(\dots)))))) \dots$
- $(1/\phi) + 1 = \phi; 1 + \phi = \phi^2; \phi + \phi^2 = \phi^3; \phi^2 + \phi^3 = \phi^4; \text{ad infinitum.}$

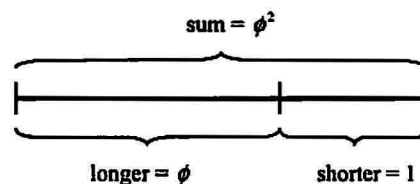
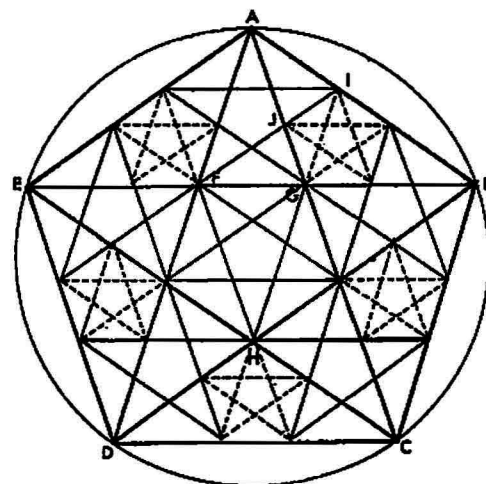


Fig. 2. The Golden Ratio.



$AB = 1, EG = FB = 1, EB = \phi (1.618), GB = \phi - 1 = 1/\phi (0.618), GI = FG = 1 - 1/\phi (0.382), FG = 1/\phi^2 (0.382), JG = 1/\phi^3 (0.236)$

Fig. 3. The Golden Ratio in the pentagon and pentagram.

- Ratio of segments in pentagon and pentagram (five-pointed star) sacred to Plato and Pythagoras (Fig. 3). The 12 faces of the Platonic Solid the dodecahedron (Fig. 1) are pentagons containing the Golden Ratio, thus equating this figure with the cosmos by Plato. In a pentagram each larger (or smaller) section is related by the ϕ ratio, so that a power series of the Golden Ratio raised to successively higher (or lower) powers is automatically generated: $\phi, \phi^2, \phi^3, \phi^4, \phi^5$, etc.
- Human canon is composed of the Golden Ratio.
- ϕ = ratio of adjacent terms of the Fibonacci Series evaluated at infinity. The Fibonacci Series is a set of numbers that begins with 1 and 1 and each term thereafter is the sum of the prior two terms, i.e. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, The Fibonacci Series can be found in the ratio of the number of spiral arms in daisies, in the chronology of rabbit populations (Fig. 4), in the sequence of leaf patterns twisted around a branch (Fig. 5), and many places in nature where self-generating patterns are in effect. The relationship between two successive numbers of this series tends to approach ϕ .

The Golden Ratio ϕ is the most pleasing aesthetic proportion. Throughout the history of art and architecture traditional artists adopted the Divine Ratio as sacred measure and aesthetic proportion in order to embody the spirit in the matter. The Great Pyramid in Egypt contains this ratio (Fig. 6). Greeks sought divinity at the Oracle of Del-Phi. Pythagoras was initiated in Egypt's Mystery School temples in the ancient city of Phi-la-del-Phi-a. The outline of the Parthenon at the Acropolis near Athens is enclosed by a Golden Rectangle by design and it is composed of many ϕ -rectangles (Fig. 7).

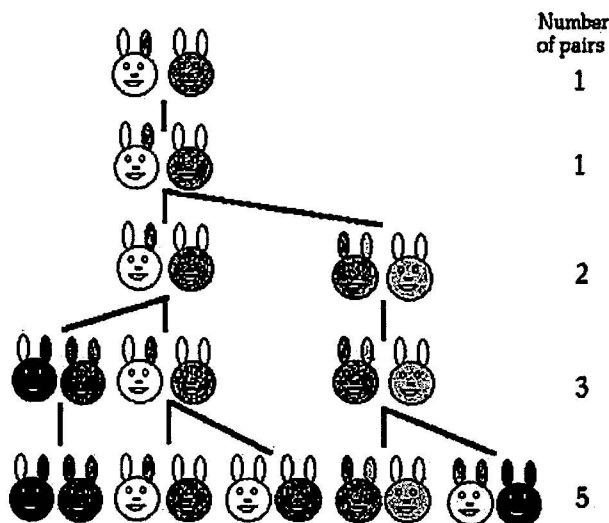


Fig. 4. The Fibonacci Series in the chronology of rabbit populations.

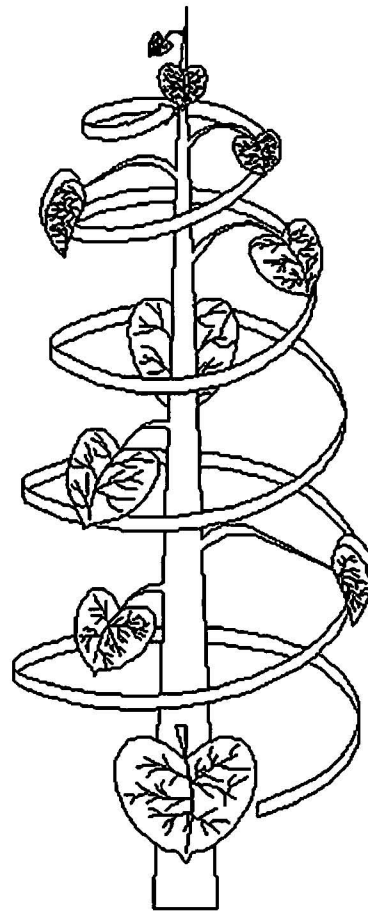


Fig. 5. The Fibonacci Series in the distribution of leaves around a central stem: 3 leaves in 5 turns, 5 leaves in 8 turns.

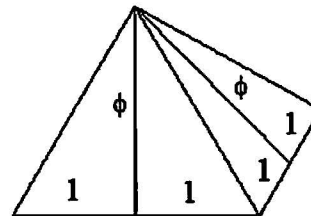


Fig. 6. The Great Pyramid and the Golden Ratio.

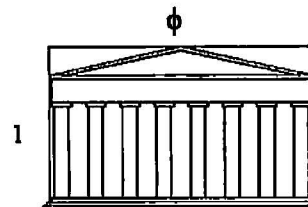


Fig. 7. Parthenon, the Golden Rectangle.

3.2. Spirals

As a mathematical fact, all figures that grow by gnomonic expansion generate intersections upon which

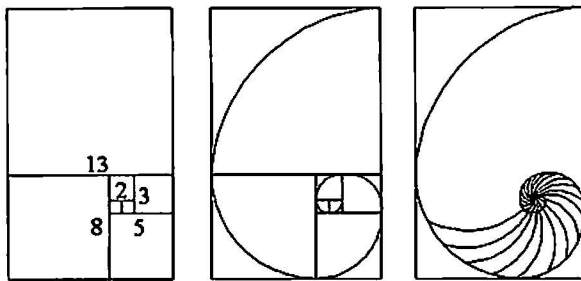


Fig. 8. *Nautilus pompilius*, the Golden Mean spiral.

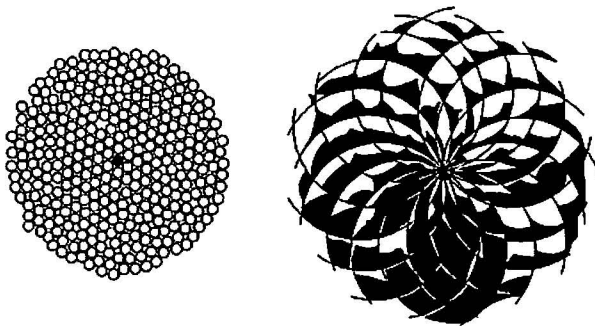


Fig. 9. Seed distribution in sunflower and pinecone based on the Golden Spiral.

spirals can be constructed. The spiral created by a recursive nest of Golden Rectangles (rectangles with relative side lengths of 1 and ϕ , or successive terms in the Fibonacci Series) can be found in a myriad of places in nature; in a snake coil, in an elephant trunk, in the cochlea of the inner ear, and in the shape of the *Nautilus pompilius* shell, for which the same proportions for each expanded chamber that is added is used (Fig. 8).

This spiral is present in the growth patterns of many plants. For example, in the sunflower the distribution of seeds is based on the Golden Mean spirals. There are 55 clockwise spirals overlaid onto either 34 or 89 counter-clockwise spirals that are parts of the Fibonacci Series (Fig. 9).

For the Pythagoreans this form symbolises the dynamics of the rhythmic generation of the cosmos, and represents universal love. In Persian architecture, arabesque patterns are based on ascending spirals with succession of form elements indicating the idea of infinity and multiplicity, as the creation of the universe. The harmonious and rhythmic movements of repeated arabesque patterns express return to the Unity (Fig. 10).

4. Sacred geometry in Persian architecture

Traditional architecture represents the Cosmos in earthly dimensions. In an architectural monument all



Fig. 10. Arabesque as the cosmic spiral, the journey towards the Unity through multiplicity, Chahar-Bagh madrasa, Isfahan, 1706–14 AD.

dimensions, both in its integrity (height, length and width) and in its components (including geometrical surface patterns), are interrelated and never divorced from geometry. As man shares with nature a commonality of proportion, traditional architect utilises geometry for a further exploration into the processes of Nature in order to lead the contemplative mind from the sensible to the intelligible world.

Geometry plays a fundamental role in design of Persian architectural monuments. From the viewpoint of *exterior* functioning, the use of geometry as *art* for creation of shapes, patterns and proportions reminds the *Great Architecture of the World* and recalls the Archetypes. The art of geometry is thus the key element to make a correspondence between the building and the Ideas that the builder has in his mind. From the viewpoint of *interior* functioning, geometry as *science* for selection of structural dimensions such as height, length and width of the building and its structural elements governs the *structural behaviour* of the building, the *behaviour* that follows the geometry. The right geometry makes the building behave correctly.

There are some comprehensive research works dedicated to the field of metaphysical [7–11] and mathematical [12–18] aspects of Persian architecture that have made it possible to uncover a part of the profound knowledge used in Persian traditional architecture.

4.1. Sacred geometry in patterns

For traditional architect geometric patterns are as aspects of the multiplicity of the Unity. The repeated patterns symbolise the idea of infinity and timelessness. The beauty and harmony observed in geometric patterns reveal the geometrical order that reflects a higher and more profound order viewed as Cosmic Laws. Spiritual man seeks geometric patterns as means of understanding the Creator.

4.2. The mathematics of two-dimensional geometric patterns

In Persian architecture geometric patterns as spatial concepts are used to fill surfaces; patterns or motifs grow side by side to cover a surface. If one wished to cover a planar surface with *regular* shapes or polygons, leaving no spaces between the meeting-points of the vertices, what are such *regular polygons*? It could be shown that as a mathematical fact there are only three regular polygons, known as the regular equipartitions of the plane surface, that may be used to fill a surface area exactly where the vertices sum up to 360 degrees: the triangle, the square, and the hexagon (Fig. 11).

The combinations of these three regular polygons form eight *semi-regular* equipartitions in which the vertices are similar on all occasions (Fig. 12) and fourteen *demi-regular* equipartitions in which the vertices vary. These are the basic space-filling patterns, also known as mosaics, grids, lattices, or tessellations [12,13].

Let

n = number of sides of each regular polygon,

$$\frac{n-2}{n} 180^\circ = \text{interior angle of each vertex of each polygon,}$$

$$\frac{360}{((n-2)/n)180} = 2 + \frac{4}{n-2} = \text{number of such polygons at each vertex.}$$

To have a whole number for n greater than 2 there must be:

$n = 3, 4, 6$ for triangle, square and hexagon, respectively.

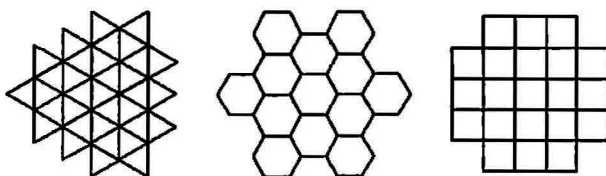


Fig. 11. The three regular patterns that exactly fill a two-dimensional surface.

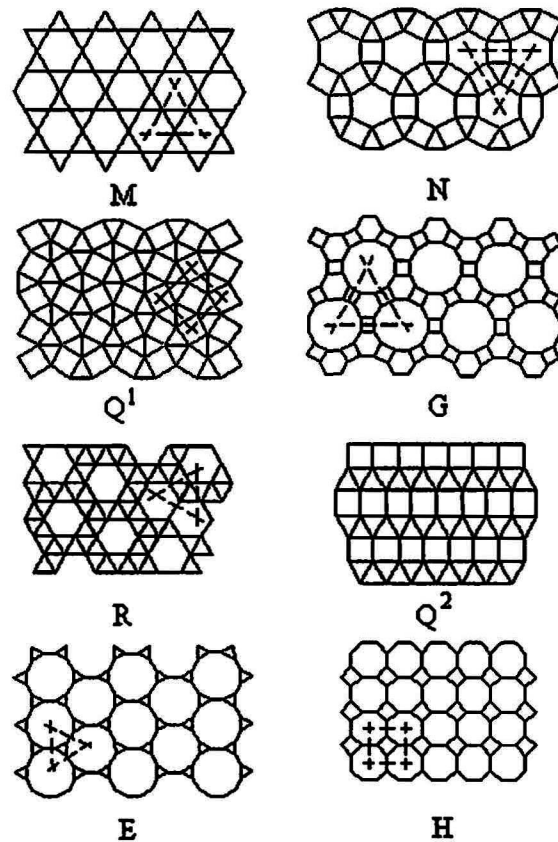


Fig. 12. The eight semi-regular patterns.

It can be shown that there cannot be less than three polygons nor more than six around a vertex. The following equation holds for polygons surrounding a vertex:

$$3 \leq \text{possible number of polygons around a vertex} \leq 6, \quad (1)$$

therefore,

$$\left(\frac{n_1-2}{n_1} + \frac{n_2-2}{n_2} + \frac{n_3-2}{n_3} \right) 180^\circ = 360^\circ \quad \text{for 3 polygons,} \quad (2)$$

or

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{1}{2} \quad \text{for 3 polygons,} \quad (3)$$

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} = 1 \quad \text{for 4 polygons,} \quad (4)$$

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \frac{1}{n_5} = \frac{3}{2} \quad \text{for 5 polygons,} \quad (5)$$

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \frac{1}{n_5} + \frac{1}{n_6} = 2 \quad \text{for 6 polygons.} \quad (6)$$

It can be seen that there are *seventeen* possible solutions in whole numbers, as shown in Table 2. The three marked with an asterisk, K, P and S, can be removed as they are the first and only totally regular solutions; three hexagons, four squares and six triangles per vertex. A, B, C, D, E and J can occur at only one point and will not generate a continuous

Table 2
Seventeen possible patterns of triangle, square or hexagon

Code letter	Faces						Code letter ^a	Faces					
	n_1	n_2	n_3	n_4	n_5	n_6		n_1	n_2	n_3	n_4	n_5	n_6
A	3	7	42				*K	6	6	6			
B	3	8	24				L	3	3	4	12		
C	3	9	18				M	3	3	6	6		
D	3	10	15				N	3	4	4	6		
E	3	12	12				*P	4	4	4	4		
F	4	5	20				Q ²	3	3	3	4	4	
G	4	6	12				R	3	3	3	3	6	
H	4	8	8				*S	3	3	3	3	3	3
J	5	5	10										

^aThe only totally regular solutions.

Table 3
Eight semi-regular patterns

New number	Code letter	Faces				
		n_1	n_2	n_3	n_4	n_5
1	M	3	6	3	6	
2	N	3	4	6	4	
3	Q ¹	3	3	4	3	4
4	G	4	6	12		
5	R	3	3	3	3	6
6	E	3	12	12		
7	H	4	8	8		
8	L	3	3	4	12	

Table 4
Fourteen demi-regular patterns

No	Code letters ^a	Faces					Faces					Faces					
		n_1	n_2	n_3	n_4	n_5	n_1	n_2	n_3	n_4	n_5	n_1	n_2	n_3	n_4	n_5	n_6
1	E+L	3	12	12			3	4	3	12							
2	L+(1)	3	3	4	12							3	3	3	3	3	3
3	L+Q ¹	3	4	3	12		3	3	4	12		3	4	3	3	4	
4	N+G	6	4	3	4		12	6	4								
5	L+Q ¹ +(1)	3	3	4	12		3	4	3	3	4	3	3	3	3	3	3
6	M+M ¹	3	6	3	6		6	6	3	3							
7	N+Q ¹	4	3	4	6		3	4	3	3	4						
8	N+Q ² +Q ¹	4	3	6	4		3	3	3	4	4	3	3	4	3	4	
9	Q ¹ +(1)						4	3	4	3	3	3	3	3	3	3	3
10	Q ¹ +(1)						3	3	4	3	4	3	3	3	3	3	3
11	Q ² +Q ¹ +(1)	3	3	3	4	4	3	3	4	3	4	3	3	3	3	3	3
12	Q ² +Q ¹ +(1)	3	3	3	4	4	3	3	4	3	4	3	3	3	3	3	3
13	N ¹ +N	4	4	3	6		4	3	4	6							
14	N+Q ²	4	3	4	6		3	3	3	4	4						

^a(1) = regular equipartition.

pattern covering a whole surface, and therefore can be discounted. Thus there remain only eight cases. From symmetry viewpoint these can be divided into eight semi-regular equipartitions whose vertices are similar on each occasion (Fig. 12, Table 3), and into 14 semi-regular equipartitions whose vertices vary (Table 4).

Geometric patterns have been used widely in Persian architecture. Fig. 13(a) shows semi-regular pattern No. 1 (M), a combination of triangles and hexagons. In Fig. 13(b) the same pattern is used in a tile lattice pattern in the Jami mosque in Yazd (fourteenth century AD), Central Iran.

4.3. Mechanical features of Persian architectural patterns

In Fig. 14 tile (in the Jami mosque in Isfahan, 15th century AD), door (in the Imam-Zadeh Ismail Shrine in Isfahan, 15th and 17th centuries AD) and window (in the Chahar-Bagh madrassa in Isfahan, 1706–14 AD) decorations based on mathematical patterns are shown. In design of wooden doors and windows, geometrical

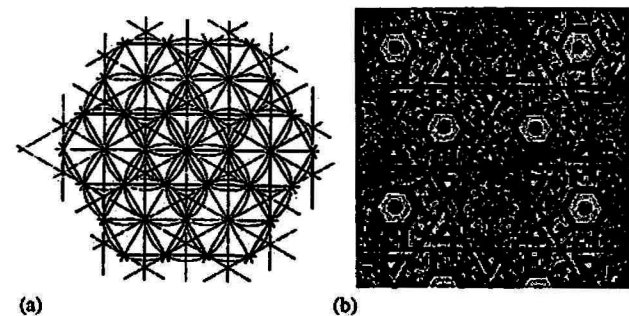


Fig. 13. Semi-regular tile lattice pattern, Jami mosque, Yazd, 14th century AD.

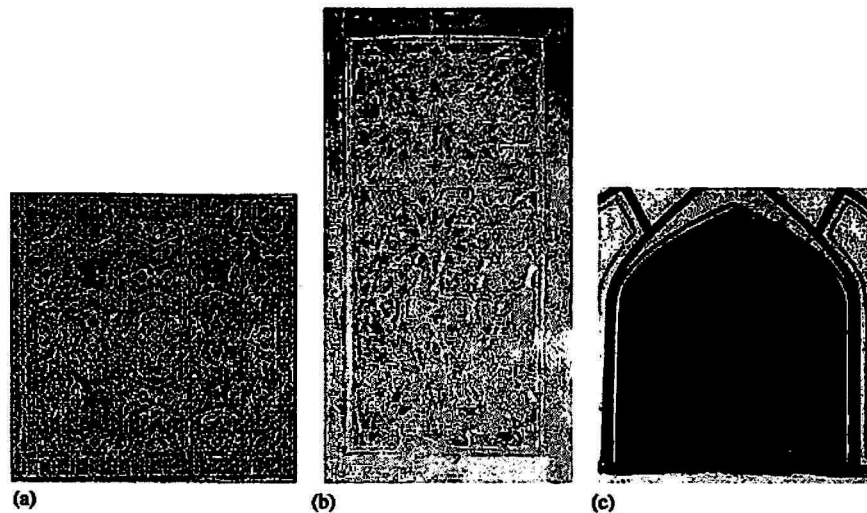


Fig. 14. Decoration based on mathematical patterns: (a) tile, Jami mosque, Isfahan, 15th century AD; (b) door, Imam-Zadeh Ismail Shrine, Isfahan, 15th and 17th centuries AD; (c) window, Chahar-Bagh madrassa, Isfahan, 1706–14 AD.

patterns enable small pieces of wood to be used economically and allow the whole combination to conform to climatic changes of temperature and humidity. There is no need for string, glue or screws to hold the wooden pieces together.

4.4. Platonic Ideas in Persian architectural patterns

Recall that the pentagon, containing the Golden Ratio, is the shape for the faces of the Platonic Solid the dodecahedron (Fig. 1), symbolising the cosmos or ether. The pentagon is in mutual relation with the pentagram (Fig. 3) and the spiral (Fig. 8) all representing the generation of the cosmos, universal love and *rebirth* (Fig. 15). Such concepts can be explained through Persian Islamic architectural patterns.

Fig. 16 demonstrates a combination of geometric patterns and calligraphy using tile in a wall in the Jami mosque in Yazd. As Kritchlow [13] states about a similar geometric pattern found in the Jami mosque in Isfahan, it shows the arrangement of ten peripheral pentagons, with a pentagram (five-pointed star) inside, arranged symmetrically around a star decagon, with arms related in a Golden Mean proportion to the side of the pentagon. The sacred name Muhammad, the Cosmic or Divine Man, is rotated around a five-pointed star.

It can be added to Kritchlow's statements that the number five, represented by the five-pointed star (pentagram) standing on two *legs*, symbolises the Perfect Man. The name Muhammad calls for the pentagram (man) to be *reborn* as a *Whole Person*. The number 10, represented by the decagon outside and the decagon (10-pointed) star inside, symbolises the return to the Unity. The whole pattern indicates that creation is a continual inverting exchange between eternal Divine

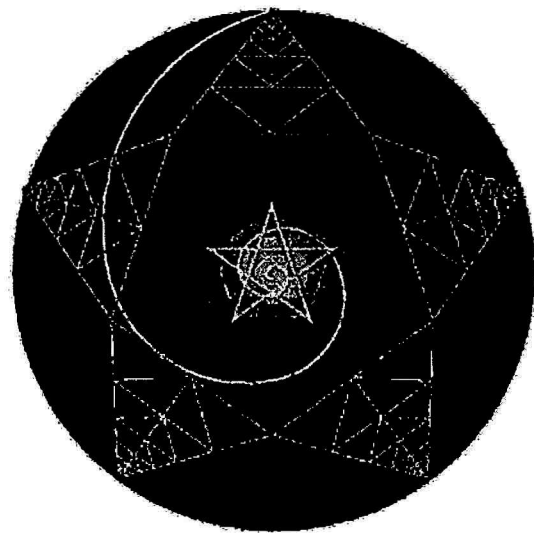


Fig. 15. Pentagon, pentagram and spiral all contain the Golden Ratio and represent the Cosmic Rebirth.

Man and humanity. The Divine Man incarnates continuously, and the Divinity reflects Itself in matter so as to become perceptible. Man is not a mere constituent part of the universe, but the original goal and the final stage of creation.

4.5. Geometrical analysis of historical buildings

Geometrical analysis of many Persian historical buildings has proven that a complete knowledge of proportions, in particular the Golden Ratio, was widely used in Persian architecture and it was the basis of Persian aesthetics.

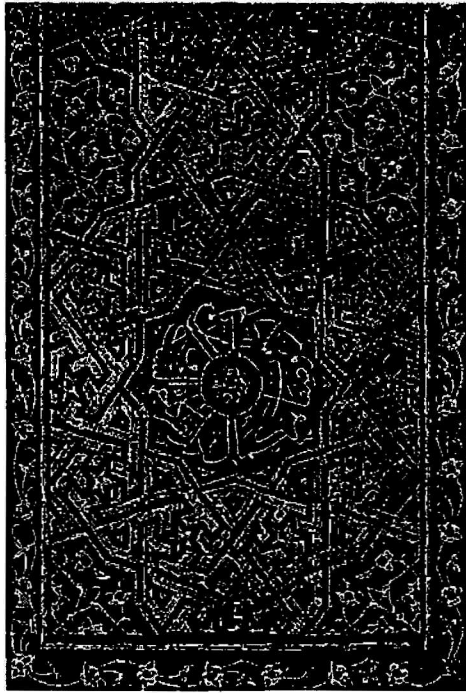


Fig. 16. Combination of geometric patterns and calligraphy, indicating Sacred Geometry and the Cosmic Man, Jami mosque, Yazd, 14th century AD.

In many Persian buildings the plan and elevation were set out in a framework of squares and equilateral triangles, whose intersections gave all the important fixed points, such as the width and height of doors, the width, length and height of galleries, the position of inscriptions, etc. Thus, the size of every part was related to every other part in some defined proportion. A building was not, therefore, a collection of odd components, but a harmonious configuration of proportionally related elements, which gave movement to space and satisfied the eye.

For example, geometrical analysis shows that a complete knowledge of the Golden Ratio is applied in the plan of Persepolis (518–330 BC) as shown in Fig. 17.

The Golden Ratio has been masterly used in the design of the Taj-al-Mulk dome dated 1088 AD, in Jami mosque in Isfahan. The outer diameter of the dome is 11.7 m. Its height from the ground level is 20 m. The thickness of the dome shell diminishes from the base to the apex. Schroeder [14] gives a wonderful description of the building, "[it] is the most beautiful structure in Persia.... In each of these aspects [aesthetics, geometry and mechanics] the building is remarkable". He thoroughly explains the aesthetic and geometrical features of the monument. He shows the sophisticated application of the Golden Ratio, in such a way that the lesser part is below, in the dimensions of the dome and the chamber below. His geometrical analysis proves that

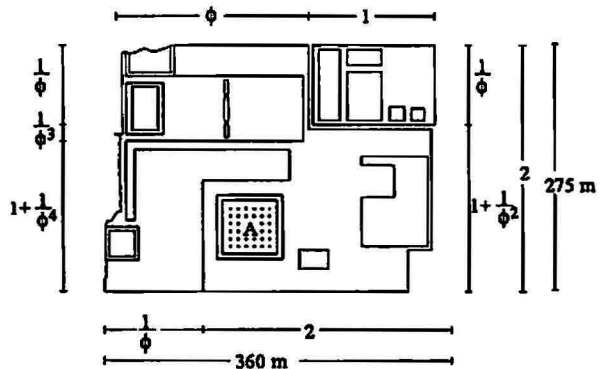


Fig. 17. Persepolis, 518–330 BC, the use of the Golden Ratio in the plan.

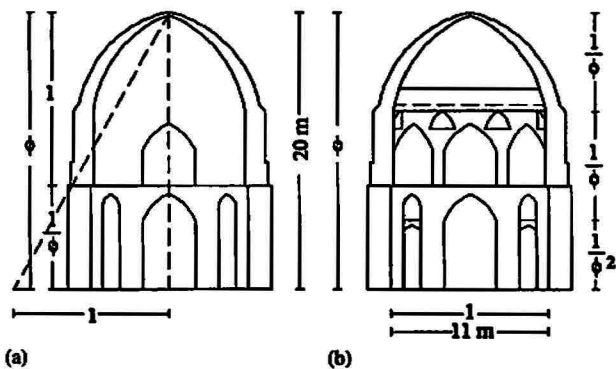


Fig. 18. The Taj-al-Mulk dome and the Golden Ratio, Isfahan, 1088 AD: (a) Schroeder's [14] geometrical analysis; (b) Hejazi's suggestion.

the architect of the building has taken a pentagon, which is generated between the sides of a grand equilateral triangle the apex of which is the peak of the dome, as a symbol for the ratio; and the proportions of the Golden Section determine the dimensions of each single element in the structure, such as the peak of the dome, the height of the whole zone of transition from the base to the dome, the peak of the octagonal arch, the peak of the lower main arch, the height of the lower side-arches and the height of the window (Fig. 18(a)). It is significant that one of the main motifs in the brick decoration of the dome interior is a triangle and rhombus figure, which plays a key role to the whole system of geometrical adjustments.

It may alternatively be shown that the rule of the Golden Ratio, in such a way that the lesser part is above, can be applied to the structure. It is suggested that the dimension of the Golden Rectangle in which the vertical cross-section of the dome is lying could have been alternatively used as a module by the architect. This specific Golden Rectangle can be traced repetitively from the dome towards the base and then to other

sections of the building; hence, generating the Golden Mean spiral (Fig. 18(b)).

Aesthetically, the Ali Qapu building (1597–1668 AD), in Isfahan, shows the application of the Golden Ratio in architecture. If the width of the building is considered as

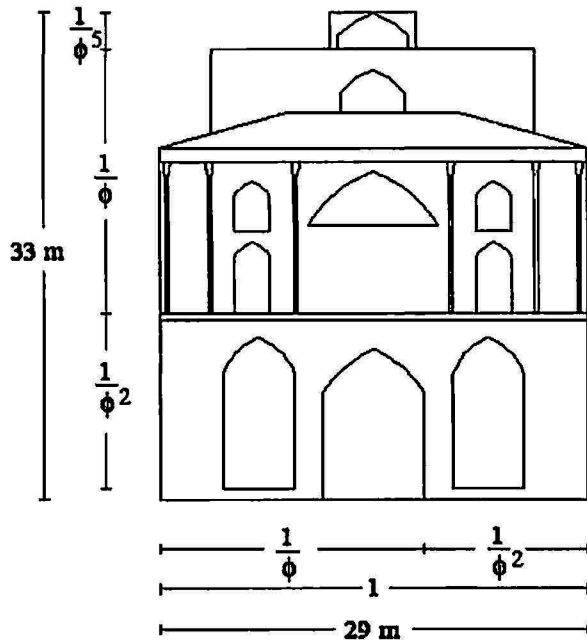


Fig. 19. Ali Qapu, Isfahan, 1597–1668 AD, the use of the Golden Ratio.

unity, important points such as the corners of the entrance to the building and the heights of different levels produce ratios of the Golden Ratio (Fig. 19).

Hajiqasemi's [19] geometrical analysis of the Shaykh-Lutf-Allah mosque (1601–1628 AD), in Isfahan, demonstrates the fascinating use of geometry in the design of the façade (Fig. 20). Again the Golden Mean (ratio of segments in pentagon) can be traced in the building. Two intersected pentagons, making a star decagon, determine all important points, such as the height and width of the entrance, the position and dimensions of the main and lateral arches, the boundaries of dadoes and staircase, the dimensions of the window at the top of the entrance, the tiling frames, and the ground level.

4.6. Relation between geometry and structural features

In the field of Persian historical buildings, it is meaningless to consider structural phenomena such as strength, stiffness and stability as the main and determinant design criteria. From the viewpoint of a traditional architect, although being fully aware of forces, resulting stresses and structural failures, the calculation of stress is of secondary importance. It is the function of structural elements that follows the overall form of building, as form itself has no meaning without the right function. Any structural (stress) analysis and verification of historical buildings is justifiable, only if it is a part of, and in the course of, the entire analysis of buildings which should include geometrical, natural and

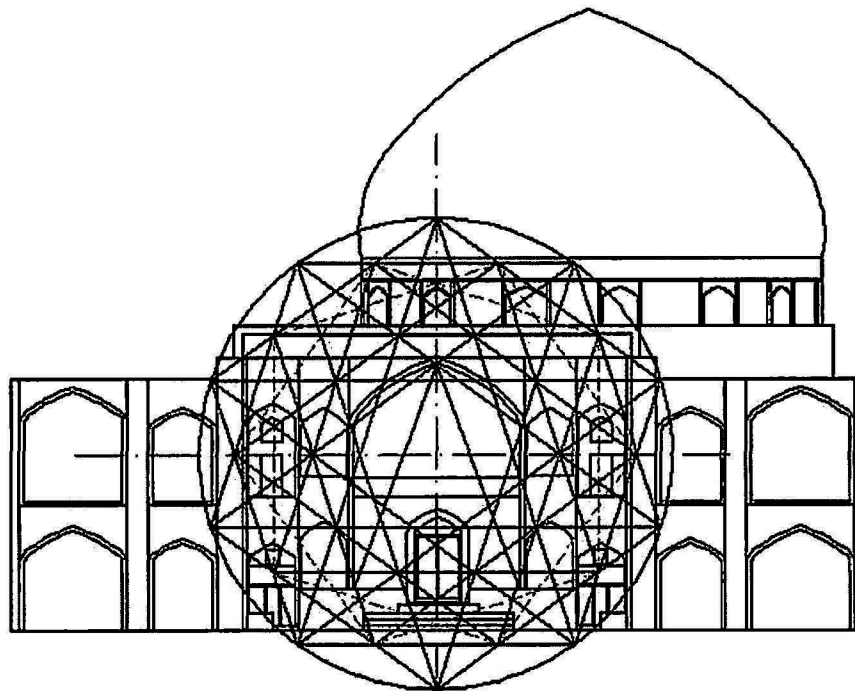


Fig. 20. Shaykh-Lutf-Allah mosque, Isfahan, 1601–1628 AD, geometrical analysis.

supernatural characteristics. It is unlikely to find a historical building whose construction rules and structural design are incorrect according to the modern structural engineering codes. If any part of a building is found to be defective, it should be attributed to the misconception of the structural function of the building or to the insufficiency of modern rules used to assess the behaviour of the building. Modern structural analysis of historical buildings is an alternative tool to uncover the profound knowledge of traditional master-builders in their works, and to improve the modern theories of structures and set new design criteria.

As a fundamental principle in traditional art of building, the functioning and stability of a building follow its geometry; a perfect geometry guarantees the stability. This principle can be traced in many historical Persian buildings.

4.6.1. Optimum design of wooden structures

It has been shown that structural design of the wooden structure of the Ali Qapu building has been relatively optimum and structurally a masterpiece according to the modern codes [15]. The optimum design is very much related to the Golden Ratio used in the dimensions of the whole building (Fig. 19).

4.6.2. The shape of momentless tensionless masonry domes

Recalling the Taj-al-Mulk dome with its striking geometry, containing the Golden Ratio and Golden Mean spiral, a number of researchers have investigated its structural behaviour, anticipating exceptional structural features. Farshad [20] shows that for weight loading the dimensions of the Taj-al-Mulk dome exactly match the mathematical formulae for the shape of the meridional curve and thickness variation of masonry domes without tensile stresses and bending forces. The equilibrium equations for symmetrically loaded shells of revolution are:

$$\frac{d}{d\varphi}(rN_\varphi) - rN_\theta \cos \varphi = -p_\varphi r r_1, \quad (7)$$

$$\frac{N_\varphi}{r_1} + \frac{N_\theta}{r_2} = p_r, \quad (8)$$

where φ and θ are meridional and circumferential angles, N_φ and N_θ are meridional and hoop forces, r is the radius of the circle normal to the axis of revolution, r_1 and r_2 are meridional and circumferential radii, and p_φ and p_r are the components of external load per unit area in the φ and r directions, respectively (Fig. 21). For weight load, that is p (per unit area), N_φ is always compressive, then by eliminating N_θ , that is $N_\theta = 0$, in regions where it may become tensile, the formulae for the variation of meridional thickness h and

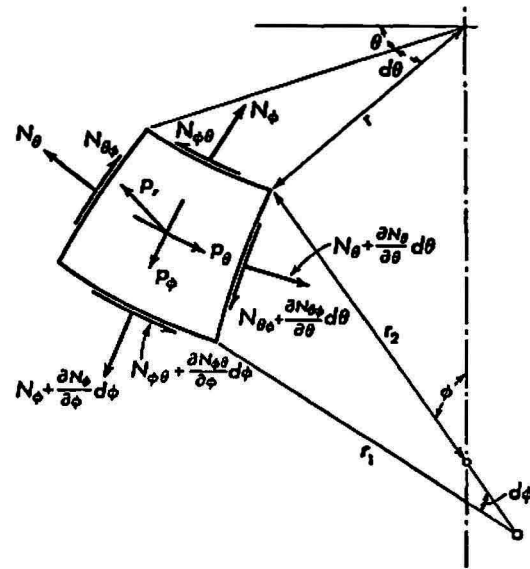


Fig. 21. Shell element.

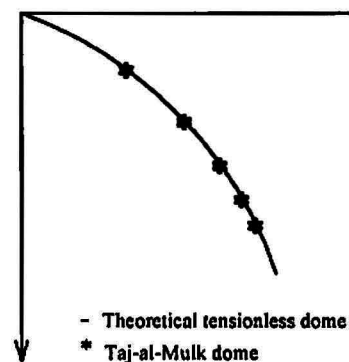


Fig. 22. Comparison of the meridional shape of theoretically perfect dome and the Taj-al-Mulk dome.

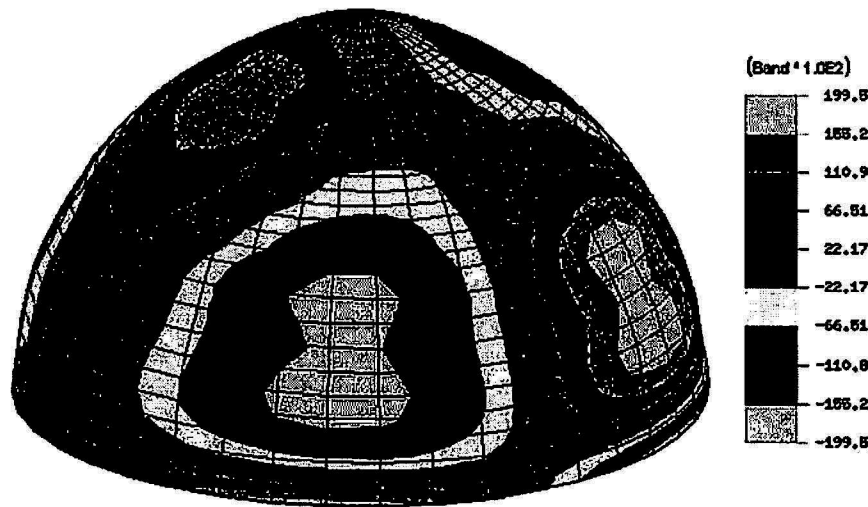
radius r of the shell will, respectively, be:

$$h = h_0 r^{1/\nu} \quad (9)$$

$$r^2 = -\frac{2A}{ph} \ln \frac{1 + \sin \varphi}{\cos \varphi} + B, \quad (10)$$

where h_0 and ν are reference thickness and Poisson's ratio, respectively, and A and B are constant values that can be obtained according to boundary conditions. The meridional shapes of theoretically perfect dome and the Taj-al-Mulk dome are plotted in Fig. 22. The agreement between the two shapes is striking.

The finite element method is a powerful tool for verification of the above mentioned statements. Structural analysis of the dome, using the finite element method, shows that the resultant stresses due to the system of bending forces are negligible compared with



Taj-al-Mulk Dome
Hoop stress (σ_θ , N/m²)

Fig. 23. Taj-al-Mulk dome, hoop stress σ_θ due to earthquake load is much less than allowable stresses.

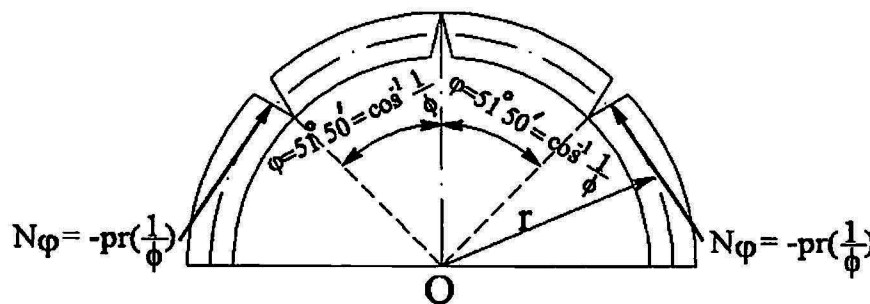


Fig. 24. Crack pattern dominated by the Golden Ratio in a hemispherical dome under weight load.

the system of membrane forces not only for weight load but also for wind and temperature and more significantly for the dynamic effects of earthquakes (Fig. 23).

Finite element analysis of the Taj-al-Mulk dome proves that if different shapes of cross-section, or different variation of meridional thickness, were used for the dome shell the magnitude of stresses and forces induced in the dome would increase and the design would not be perfect any more.

Aesthetically, geometrically, architecturally and structurally the Taj-al-Mulk dome is the ideal dome. In this building there is no distinction between geometry (shape or art) and mathematics (structural functioning or science). It is the perfect union of geometry and stability. Here art and science become architecture.

4.6.3. Relation between the Golden Ratio and crack patterns in circular shapes

Persian architects have always disliked the use of circular shape in constructing load bearing arches,

vaults and domes, because hinging cracks occurred at a certain meridional angle that caused the failure mechanism and collapse of the structure. In cases that they used this shape, they changed the radius of the shape well before the meridional angle $51^\circ 50'$ in order to avoid tensile forces. Solving the equilibrium equation for a spherical dome under weight load shows that this is the angle where the sign of hoop force N_θ changes from negative (compressive) to positive (tensile), that is $N_\theta = 0$. Masonry materials have no tensile strength and therefore cracks occur at this angle (Fig. 24). For a hemispherical dome of radius r subjected to its self weight p (per unit area), it can be shown that:

$$N_\theta = pr \frac{1 - \cos \varphi - \cos^2 \varphi}{1 + \cos \varphi}. \quad (11)$$

The sign of the hoop force changes at a value of φ given by

$$1 - \cos \varphi - \cos^2 \varphi = 0. \quad (12)$$

This yields

$$\cos \varphi = \frac{-1 + \sqrt{5}}{2} = \frac{1}{\phi}, \quad (13)$$

or

$$\varphi = 51^\circ 50'. \quad (14)$$

It is interesting to evaluate the meridional force N_φ at this angle to show that it is inversely proportional to the Golden Ratio:

$$N_\varphi = -pr \frac{1}{1 + \cos \varphi} = -pr \left(\frac{1}{\phi} \right). \quad (15)$$

This angle and meridional force could be called the *Golden Angle* and the *Golden Force*, respectively. Therefore, the meridional angle dominated by the Golden Ratio is the critical location for the stability of a spherical dome, or a circular arch or vault.

Similar to the cases of the Ali Qapu building and the Taj-al-Mulk dome it can be concluded that structural stability has direct relation with geometrical characteristics of a building. Again the Golden Ratio (geometric proportions) rules the stability (mechanics).

5. Conclusions

The triumph of Persian architecture lies in its perfect realisation of the essential importance of the utilisation of science and art. The practice of Sacred Geometry, both in scientific and artistic dimensions, is an intrinsic character of Persian architecture upon which this traditional style of architecture has been developed. Persian architecture is a mirror to reflect Divine Beauty, the Beauty that is Itself Sacred, and as a harmonious interrelation of rational proportions it could only be reflected through patterns that are exactly constructed upon *right* proportions. Sacred Geometry is the powerful tool to create the right proportions in architecture in order to make a correspondence between the heaven and the earth. Sacred Geometry and proportions found in many natural life-forms have been masterly used by the Persian traditional architect to build up a traditional style of architecture that indicates the methods of right

design and correct engineering. In many traditional structures geometry rules the stability. Persian architecture is the *Sacred Geometrisation of Divine Beauty*.

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