

$$\sqrt{3} \cdots < \frac{1351}{780} < \frac{362}{209} < \frac{97}{56} < \frac{26}{15} < \frac{7}{4} < \frac{2}{1},$$

und $1351^2 = 3.780^2 + 1$. Wir haben also, auf Grund der von Theon von Smyrna und Proklos überlieferten Methode, die Beziehungen die Archimedes, ohne Erklärung, als bekannt, angibt.

ΓΕΩΜΕΤΡΙΚΗ ΑΛΓΕΒΡΑ.— Συμβολή εις τὴν ἔρευναν τῆς γεωμετρικῆς ἀλγέβρας τῶν Πυθαγορείων, ἐπὶ *Εὐαγγ. Σταμάτη** Ἀνεκοινώθη ὑπὸ τοῦ κ. Μιχαὴλ Στεφανίδου.

1. ΕΙΣΑΓΩΓΗ

1. Εἰς τὸν ἴδιον τὸν Πυθαγόραν ἀποδίδεται κατὰ τὴν παράδοσιν ἡ ἀπόδειξις τοῦ περιφήμου ὁμωνύμου θεωρήματος (Εὐκλείδου I, 47), ὡρισμένοι ἀκέραιαι λύσεις τῆς ἐξισώσεως $z^2 = x^2 + y^2$ καὶ ἡ ἀνακάλυψις τῶν ἀσυμμέτρων¹. Εἰς τοὺς Πυθαγορείους ἐν γένει ἀποδίδεται μετὰξὺ ἄλλων τὸ II Βιβλίον καὶ τὸ πλεῖστον τῶν ἀριθμητικῶν τῶν Στοιχείων τοῦ Εὐκλείδου, καὶ ἡ εὕρεσις τῶν ἀκεραίων λύσεων τῆς ἐξισώσεως $y^2 = 2x^2 \mp 1$, (1), αἵτινες χρησιμεύουσι διὰ τὸν ὑπολογισμόν τῆς κατὰ προσέγγισιν ἀριθμητικῆς τιμῆς τῆς $\sqrt{2}$. Αἱ ἀκέραιαι λύσεις τῆς ἐξισώσεως (1) ὀνομάζονται ὡς γνωστόν, πλευρικοὶ καὶ διαμετρικοὶ ἀριθμοί, διότι οἱ μὲν ἐκ τούτων ἀντιστοιχοῦσιν εἰς τὰς πλευράς, οἱ δὲ εἰς τὰς διαγωνίους τετραγώνων².

Ὁ νόμος σχηματισμοῦ τῶν πλευρικῶν καὶ διαμετρικῶν ἀριθμῶν διεσώθη ὑπὸ τοῦ Θεώωνος τοῦ Σμυρναίου καὶ ἔχει ὡς κάτωθι:

	Πλευρικοὶ ἀριθμοί	Διαμετρικοὶ ἀριθμοί
1.	1	1
2.	$1 + 1 = 2$	$2 \cdot 2 + 1 = 3$
3.	$2 + 3 = 5$	$2 \cdot 2 + 3 = 7$

* EVANGFLOS STAMATIS. A contribution to the investigation of the geometrical algebra of the Pythagoreans.

¹ ΜΙΧΑΗΛ ΣΤΕΦΑΝΙΔΟΥ, Εἰσαγωγή εἰς τὴν ἱστορίαν τῶν Φυσικῶν Ἐπιστημῶν, σ. 55-65 καὶ 102-3, Ἀθήναι 1938.— Προκλὸς εἰς σχόλια Εὐκλείδου I, σ. 65 καὶ 438. Ἐκδ. Friedlein, Teubner.— Ίμβρίχου V. P. 246, ἔκδ. L. Deubner, Teubner.

² Theonis Smyrnaei, Philosophi Platonici, ἔκδ. E. Hiller, σ. 43, Teubner. E. ΣΤΑΜΑΤΗΣ, Εὐκλείδου Γεωμετρία-Θεωρία ἀριθμῶν, τόμ. II, σ. 8. (Ἔργ. Ἐκδ. Σχολικῶν Βιβλίων, 1953 Ἀθήναι).— PAUL-HENRI MICHEL, De Pythagore à Euclide. p. 438, Paris 1950.— (Soc. d'ed. Les Belles Lettres). M. CANTOR Vorlesungen über Geschichte der Mathematik u. Theon von Smyrna.— T. HEATH A history of Greek mathematics I. p. 91, Oxford 1921 at the Clarendon Press.— R. MORIS COHEN and J. E. DRABKIN, A source book in Greek science, p. 43, McGraw-Hill book company inc. New York, Toronto, London, 1984.

Stamatis, Evangelos

SYMBOLIS EIS TEN EREUNAN TES GEOMETRIKES ALGEBRAS TON PYTHAGOREION

In: Praktika tes Akademias Athenon, 30 - 1955-p.262-282

4.	$5 + 7 = 12$	$2 \cdot 5 + 7 = 17$
5.	$12 + 17 = 29$	$2 \cdot 12 + 17 = 41$
.	.	.
.	.	.
.	.	.

$$\text{Είναι δε } \frac{1}{1} < \frac{7}{5} < \frac{41}{29} < \dots \sqrt{2} \dots < \frac{17}{12} < \frac{3}{2},$$

και

$$1^2 = 2 \cdot 1^2 - 1$$

$$3^2 = 2 \cdot 2^2 + 1$$

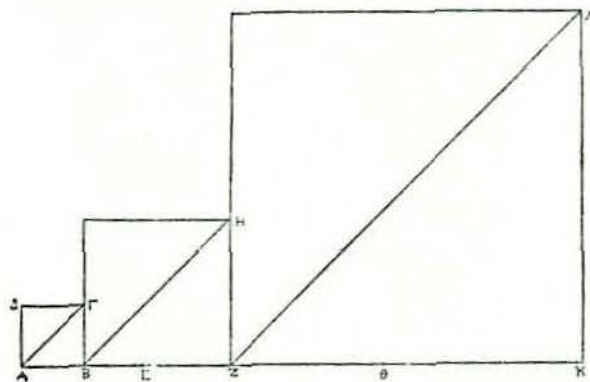
$$7^2 = 2 \cdot 5^2 - 1$$

$$17^2 = 2 \cdot 12^2 + 1$$

$$41^2 = 2 \cdot 29^2 - 1 \quad \text{κλπ.}$$

Ἡ γεωμετρικὴ ἀπόδειξις τοῦ νόμου σχηματισμοῦ τῶν πλευρικῶν καὶ διαμετρικῶν ἀριθμῶν μνημονεύεται ὑπὸ τοῦ Πρόκλου¹ καὶ ἔχει ὡς ἐξῆς:

Ἐστω τετραγώνον πλευρᾶς $AB=a_1$ καὶ διαγωνίου $AG=\delta_1$, (σχ. 1), ὅτε εἶναι



Σχ. 1.

$\delta_1^2 = 2a_1^2$. Ἐπὶ τῆς προεκτάσεως τῆς AB λαμβάνομεν τμήμα $BE=AB=a_1$ καὶ ἐν συνεχεῖα τμήμα $EZ=AG=\delta_1$. Κατὰ τὸν Εὐκλείδην II, 10 θὰ εἶναι $(2a_1 + \delta_1)^2 + \delta_1^2 = 2a_1^2 + 2(a_1 + \delta_1)^2$.

Καὶ ἐπειδὴ $\delta_1^2 = 2a_1^2$, θὰ ἔχωμεν δι' ἀφαίρέσεως τούτου κατὰ μέλη ἐκ τῆς προηγουμένης ἐξισώσεως, $(2a_1 + \delta_1)^2 = 2(a_1 + \delta_1)^2$. Ἡ σχέσις ὅμως αὕτη σημαίνει ὅτι ἢ μὲν $(a_1 + \delta_1) = BE + EZ$ εἶναι πλευρὰ, ἢ δὲ $(2a_1 + \delta_1) = AB + BE + EZ$ εἶναι διαγωνίος τετραγώνου, ἢ BH . Ἐὰν ἐπὶ τῆς προεκτάσεως τῆς BZ λάβωμεν τμήμα

¹ Σχόλια εἰς Πολιτεῖαν Πλάτωνος, τόμ. II σ. 24 κ.ε. 393 κ.ε. ὑπὸ F. HULTSCH, ἐκδ. Kroll, Teubner.

$Z\Theta = BZ$ καὶ ἐν συνεχείᾳ τμήμα $\Theta K = BH$ καὶ ἐφαρμόσωμεν τὸ εὐκλείδειον θεώρημα II, 10, θὰ λάβωμεν νέον τετράγωνον, τοῦ ὁποῖου ἡ μὲν πλευρὰ θὰ εἶναι ἢ $Z\Theta + \Theta K = 3\alpha_1 + 2\delta_1$, ἡ δὲ διαγώνιος ἢ $Z\Lambda = BK = 2BZ + \Theta K = 4\alpha_1 + 3\delta_1$.

Ὅθεν λαμβάνομεν τὸ ἐξῆς σχῆμα

Πλευρικοὶ ἀριθμοὶ			Διαμετρικοὶ ἀριθμοὶ		
α_1			δ_1		
α_2	$= \alpha_1$	$+ \delta_1$	δ_2	$= 2\alpha_1$	$+ \delta_1$
α_3	$= \alpha_2$	$+ \delta_2$	δ_3	$= 2\alpha_2$	$+ \delta_2$
α_4	$= \alpha_3$	$+ \delta_3$	δ_4	$= 2\alpha_3$	$+ \delta_3$
.			.		
.			.		
.			.		
α_n	$= \alpha_{n-1}$	$+ \delta_{n-1}$	δ_n	$= 2\alpha_{n-1}$	$+ \delta_{n-1}$

Ἐὰν θέσωμεν $\alpha_1 = 1$ καὶ $\delta_1 = 1$, λαμβάνομεν τοὺς κατὰ τὸν Θέωνα τὸν Σμυρναῖον πλευρικοὺς καὶ διαμετρικοὺς ἀριθμούς, ἥτοι τὰς ἀκεραίας λύσεις τῆς ἐξισώσεως $y^2 = 2x^2 \mp 1$, ἢ $\delta_n^2 = 2\alpha_n^2 + (-1)^n$, ($n = 1, 2, 3, \dots$).

Ἐκ τῆς Πολιτείας τοῦ Πλάτωνος πληροφορούμεθα ὅτι οἱ πλευρικοὶ καὶ διαμετρικοὶ ἀριθμοὶ ἦσαν γνωστοὶ εἰς αὐτόν. Ἐκεῖ ἀναγιγνώσκομεν «ἐκατὸν μὲν ἀριθμῶν ἀπὸ διαμέτρων ρητῶν πεμπάδος, δεομένων ἐνὸς ἐκάστων, ἀρρήτων δὲ δυοῖν» (546 c). Ἐνταῦθα ὁ Πλάτων ὑπαινίσσεται τοὺς πλευρικοὺς καὶ διαμετρικοὺς ἀριθμούς καὶ δὴ καὶ μίαν ἀκεραίαν λύσιν τῆς ἀνωτέρω ἐξισώσεως, τὴν $7^2 = 2 \cdot 5^2 - 1$. Τοῦτο συνάγεται ἐκ τοῦ Πρόκλου, ὅστις γράφει «ὅπου δὲ τὸ σύνεγγυς ἀγαπῶμεν, οἷον εὐρόντες ἐν γεωμετρικῇ τετράγωνον τετραγώνου διπλάσιον, ἐν ἀριθμοῖς δὲ οὐκ ἔχοντες ἐνὸς δέοντος φάμεν ἄλλον ἄλλου διπλάσιον ὑπάρχειν, ὥσπερ τοῦ ἀπὸ τῆς πεντάδος ὁ ἀπὸ τῆς ἐπτάδος διπλάσιον ἐνὸς δέοντος». Καὶ ἀλλαχοῦ «οὐ γὰρ ἐστὶ τετράγωνος ἀριθμὸς τετραγώνου διπλάσιος εἰ μὴ λέγει τις τὸν σύνεγγυς, ὁ γὰρ ἀπὸ τοῦ ζ' τοῦ ἀπὸ τοῦ ε' διπλάσιός ἐστιν ἐνὸς δέοντος¹. (Εἶναι δηλ. $7^2 = 2 \cdot 5^2 - 1$).

2. Ὁ Ἀρχιμήδης εἰς τὴν πραγματείαν αὐτοῦ «Κύκλου Μέτρησις» χρησιμοποιεῖ ἄνευ ἀποδείξεως τὰς σχέσεις

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}, \text{ καὶ } 265^2 = 3 \cdot 153^2 - 2, 1351^2 = 3 \cdot 780^2 + 1.$$

Γεωμετρικὴν ἀπόδειξιν τῶν σχέσεων τούτων ὑπεβάλομεν εἰς τὴν Ἀκαδημίαν Ἀθηνῶν².

Αὕτη συνδέεται πρὸς τοὺς πλευρικοὺς καὶ διαμετρικοὺς ἀριθμούς. Θεωροῦμεν

¹ Πρόκλος εἰς Εὐκλείδην I, σ. 61 καὶ 427, ἔκδ. G. FRIEDLEIN, Teubner.

² Βλ. Πρακτικά, ἀνωτ., σ. 255 κ. εἰς.

ισοσκελές αμβλυγώνιον τρίγωνον, τὸ $AB\Gamma$ (σχ. 2), τοῦ ὁποῖου ἡ μεγαλύτερα γωνία, ἡ $AB\Gamma$, νὰ εἶναι ἴση πρὸς τὴν ἐξωτερικὴν γωνίαν ἱσοπλεύρου τριγώνου. Κατὰ τὸν Εὐκλείδην II, 12, ἐὰν καλέσωμεν τὴν πλευρὰν $AB = \alpha_1$ καὶ τὴν $AG = \delta_1$, ἥτις βεβαίως εἶναι ἡ μεγαλύτερα διαγώνιος τοῦ ρόμβου $AB\Gamma\Delta$, θὰ εἶναι $\delta_1^2 = 3\alpha_1^2$. καὶ συνεπῶς $\delta_1 : \alpha_1 = \sqrt{3}$. Ἐφαρμόζομεν τὴν ἀκριβῶς τὴν ὑπὸ τοῦ Πρόκλου ὑποδεικνυμένην μέθοδον διὰ τὴν ἀπόδειξιν τῶν ἐκ τετραγώνων σχημάτων προκυπτόντων πλευρῶν καὶ διαμετρικῶν ἀριθμῶν. Ἐπὶ τῆς προεκτάσεως τῆς AB λαμβάνομεν τμήμα $BE = AB = \alpha_1$ καὶ ἐν συνεχείᾳ τμήμα $EZ = AG = \delta_1$. Κατὰ τὸν Εὐκλείδην II, 10 θὰ ἔχωμεν.

$$(2\alpha_1 + \delta_1)^2 + \delta_1^2 = 2\alpha_1^2 + 2(\alpha_1 + \delta_1)^2,$$

καὶ ἐκ ταύτης

$$(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2, \quad (1).$$

Εἶναι ὅρα καὶ $3(2\alpha_1 + \delta_1)^2 = 12\alpha_1^2 + 12\alpha_1\delta_1 + 3\delta_1^2$. Ἀλλὰ $\delta_1^2 = 3\alpha_1^2$. Ἐπομένως

$$3(2\alpha_1 + \delta_1)^2 = 9\alpha_1^2 + 12\alpha_1\delta_1 + 4\delta_1^2 = (3\alpha_1 + 2\delta_1)^2.$$

Ἡ σχέσηις ὅμως αὕτη σημαίνει ὅτι ἡ μὲν $(2\alpha_1 + \delta_1)$ εἶναι πλευρά, ἡ δὲ $(3\alpha_1 + 2\delta_1)$ εἶναι ἡ μεγαλύτερα διαγώνιος ὁμοίου ρόμβου πρὸς τὸν $AB\Gamma\Delta$ τοῦ $AZH\Theta$. Ἐὰν ἐπὶ τῆς προεκτάσεως τῆς AZ λάβωμεν τμήμα ἴσον πρὸς AZ καὶ ἐν συνεχείᾳ τμήμα ἴσον πρὸς AH , τότε ἔχομεν κατὰ τὸν αὐτὸν νόμον τὴν πλευρὰν καὶ τὴν μεγαλύτεραν διαγώνιον νέου ὁμοίου ρόμβου πρὸς τὸν ἀρχικόν, ἥτοι πλευρὰ μὲν εἶναι ἡ $2AZ + AH$, διαγώνιος δὲ μεγαλύτερα, ἡ $3AZ + 2AH$, ἢ $7\alpha_1 + 4\delta_1$ καὶ $12\alpha_1 + 7\delta_1$ ἀντιστοίχως. Καλοῦντες τὰς τιμὰς τῶν πλευρῶν πλευρικοὺς ἀριθμοὺς καὶ τὰς τιμὰς τῶν μεγαλύτερων διαγωνίων, τῶν συνεχῶν κατὰ τὸν ἀνωτέρω νόμον κατασκευαζομένων ρόμβων, διαμετρικοὺς ἀριθμοὺς, θὰ ἔχωμεν

Πλευρικοὶ ἀριθμοί.	Διαμετρικοὶ ἀριθμοί.
α_1	δ_1
$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = 3\alpha_1 + 2\delta_1$
$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = 3\alpha_2 + 2\delta_2$
$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = 3\alpha_3 + 2\delta_3$
$\alpha_5 = 2\alpha_4 + \delta_4$	$\delta_5 = 3\alpha_4 + 2\delta_4$
$\vdots$	$\vdots$
$\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$	$\delta_n = 3\alpha_{n-1} + 2\delta_{n-1}$

Ἐάν θέσωμεν $\alpha=1$, $\delta=1$ λαμβάνομεν

(Α) Πλευρικοί ἀριθμοί.	Διαμετρικοί ἀριθμοί.
$\alpha_1=1$	$\delta_1=1$
$\alpha_2=3$	$\delta_2=5$
$\alpha_3=11$	$\delta_3=19$
$\alpha_4=41$	$\delta_4=71$
$\alpha_5=153$	$\delta_5=265$
.	.
.	.
.	.

Οἱ ἀριθμοὶ οὗτοι παρέχουσι τὰς ἀκεραίας λύσεις τῆς ἐξισώσεως $y^2=3x^2-2$, ἥτοι εἶναι:

$$1^2=3 \cdot 1^2-2$$

$$5^2=3 \cdot 3^2-2$$

$$19^2=3 \cdot 11^2-2, \text{ κλπ.}$$

Ἐάν θέσωμεν $\alpha_1=1$, $\delta_1=2$ λαμβάνομεν:

(Β) Πλευρικοί ἀριθμοί.	Διαμετρικοί ἀριθμοί.
$\alpha_1=1$	$\delta_1=2$
$\alpha_2=4$	$\delta_2=7$
$\alpha_3=15$	$\delta_3=26$
$\alpha_4=56$	$\delta_4=97$
$\alpha_5=209$	$\delta_5=362$
$\alpha_6=780$	$\delta_6=1351$
.	.
.	.
.	.

Οἱ ἀριθμοὶ οὗτοι παρέχουσι τὰς ἀκεραίας λύσεις τῆς ἐξισώσεως $y^2=3x^2+1$, ἥτοι εἶναι

$$2^2=3 \cdot 1^2+1$$

$$7^2=3 \cdot 4^2+1$$

$$26^2=3 \cdot 15^2+1, \text{ κλπ.}$$

Οἱ λόγοι $\frac{\delta_n}{\alpha_n}$ τῶν (Α) καὶ (Β) ἀποτελοῦσι δύο ἀκολουθίας ἐκ τῶν ὁποίων ἡ ἢ μὲν τῶν (Α) εἶναι αὐξουσα, ἡ δὲ τῶν (Β) φθίνουσα. Τὸ κοινὸν φράγμα τούτων εἶναι ἡ $\sqrt{3}$, ἥτοι εἶναι

$$1) \frac{1}{1} < \frac{5}{3} < \frac{19}{11} < \frac{41}{21} < \frac{265}{153} < \dots \sqrt{3} \dots < \frac{1351}{780} < \frac{362}{209} < \frac{97}{56} < \frac{26}{15} < \frac{7}{4} < \frac{2}{1},$$

καὶ 2) $265^2=3 \cdot 153^2-2$, $1351^2=3 \cdot 780^2+1$, ὡς χρησιμοποιεῖται ταῦτα ἄνευ ἀποδείξεως, ὡς γνωστὰ, ὁ Ἀρχιμήδης.

II.

Ἐκ τῶν ἀνωτέρω ἐκτεθέντων συνάγομεν τὸ συμπέρασμα ὅτι οἱ Πυθαγόρειοι ἐγνώριζον καὶ τὰς ἀκεραίας λύσεις τῆς ἐξισώσεως

$$\delta_v^2 = \lambda a_v^2 + (\lambda - 4)^v (-1)^v, \quad (v=1, 2, 3, \dots \text{ καὶ } \lambda \geq 5,$$

ἀκεραῖος μὴ τετράγωνος), καὶ ὅτι ἡ $\sqrt{\lambda}$ εἶναι τὸ κοινὸν φράγμα δύο ἀκολουθιῶν, μιᾶς αὐξοῦσης καὶ μιᾶς φθινούσης, διότι ἡ ἀπόδειξις τούτων εἶναι ἀκριβῶς ἡ αὐτὴ πρὸς τὰς ἀνωτέρω ἐκτεθείσας.

Παρέχομεν τὴν ἀπόδειξιν διὰ τὰς ἐξισώσεις

$$\delta_v^2 = 5a_v^2 + (5-4)^v (-1)^v$$

$$\delta_v^2 = 6a_v^2 + (6-4)^v (-1)^v$$

$$\delta_v^2 = 7a_v^2 + (7-4)^v (-1)^v$$

$$\delta_v^2 = 8a_v^2 + (8-4)^v (-1)^v$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\delta_v^2 = 17a_v^2 + (17-4)^v (-1)^v,$$

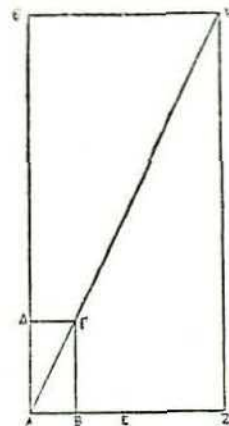
καὶ τὴν $\sqrt{5}, \sqrt{6}, \sqrt{7}, \dots, \sqrt{17}$. Εἶναι δὲ γνωστὸν ἐκ τοῦ Θεαιτήτου τοῦ Πλάτωνος ὅτι ὁ Θεόδωρος¹ (ὁ Κυρηναῖος, ὅστις θεωρεῖται Πυθαγόρειος) ἀπέδειξε τὸ ἀσύμμετρον τῆς $\sqrt{3}, \sqrt{5}, \dots, \sqrt{17}$, (Θεαιτήτου 147 D-148 B).

II. 1. $\delta_v^2 = 5a_v^2 + (5-4)^v (-1)^v$ καὶ $\sqrt{5}$.

Θεωροῦμεν ὀρθογώνιον παραλληλόγραμμον, τὸ ABΓΔ (σχ. 3), ἐνθα ἔστω $AB=a_1$, $BΓ=2a_1$ καὶ ἡ διαγώνιος $AΓ=\delta_1$. Εἶναι ἄρα $\delta_1^2=5a_1^2$, καὶ $\frac{\delta_1}{a_1}=\sqrt{5}$.

Ἐπὶ τῆς προεκτάσεως τῆς AB λαμβάνομεν τμήμα $BE=AB=a_1$ καὶ ἐν συνεχείᾳ τμήμα $EΖ=AΓ=\delta_1$. Κατὰ τὸν Εὐκλείδην II, 10 θὰ ἔχωμεν.

$$(2a_1 + \delta_1)^2 + \delta_1^2 = 2a_1^2 + 2(a_1 + \delta_1)^2, \text{ καὶ ἐκ ταύτης}$$



Σχ. 3.

¹ 1. PAULY - WISSOWA, Realenzyklopädie unter Theodoros. Dort Literaturangabe: M. CANTOR, E. FRANK, F. HULTSCH, G. JUNGE, H. VOGT, H. G. ZEUTHEN, EVA SACHS, T. BONNESEN, H. HASSE-H. SCHOLZ, T. HEATH. — 2. Und W. L. VAN DER WAERDEN, Die Arithmetik der Pythagoreer II. Die Theorie des Irrationalen, *Mathem. Annalen*, 120, 5./6. Heft, 1940, Springer Verlag, Berlin, Göttingen, Heidelberg. — 3. K. REIDEMEISTER, Die Arithmetik der Griechen, 1940, Leipzig. — 4. J. E. HOFMANN, Geschichte der Mathematik I, S. 26-27, Berlin, 1953. (*Sammlung Götschen*, 226. Walter de Gruyter und Co.) 5. ROBERT S. BRUMBAUGH, Plato's Mathematical Imagination, p. 146, Indiana University Press, Bloomington, 1954.

$$(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2$$

Είναι ἔρα καὶ $5(2\alpha_1 + \delta_1)^2 = 20\alpha_1^2 + 20\alpha_1\delta_1 + 5\delta_1^2$.

Ἀλλὰ $\delta_1^2 = 5\alpha_1^2$. Ἐπομένως

$$5(2\alpha_1 + \delta_1)^2 = 20\alpha_1^2 + 5\alpha_1^2 + 20\alpha_1\delta_1 + 4\delta_1^2 = (5\alpha_1 + 2\delta_1)^2.$$

Ἡ σχέσις ὅμως αὕτη σημαίνει ὅτι ἡ μὲν $(2\alpha_1 + \delta_1) = AZ$ εἶναι πλευρά, ἡ δὲ $(5\alpha_1 + 2\delta_1) = AH$ εἶναι διαγώνιος ὁμοίου πρὸς τὸ ἀρχικὸν ὀρθογωνίου παραλληλογράμμου τοῦ $AZH\Theta$. Κατὰ τὸν προφανῆ νόμον τῆς κατασκευῆς ἐν συνεχείᾳ ὁμοίων ὀρθογωνίων παραλληλογράμμων θὰ ἔχωμεν:

Πλευρικοὶ ἀριθμοί.

Διαμετρικοὶ ἀριθμοί.

α_1	δ_1
$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = 5\alpha_1 + 2\delta_1$
$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = 5\alpha_2 + 2\delta_2$
$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = 5\alpha_3 + 2\delta_3$
$\vdots$	$\vdots$
$\vdots$	$\vdots$
$\vdots$	$\vdots$
$\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$	$\delta_n = 5\alpha_{n-1} + 2\delta_{n-1}$

Ἐὰν θέσωμεν $\alpha_1 = 1$, $\delta_1 = 2$ λαμβάνομεν

$$\alpha_2 = 4, \quad \delta_2 = 9$$

$$\alpha_3 = 17, \quad \delta_3 = 38$$

$$\alpha_4 = 72, \quad \delta_4 = 161$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

Εἶναι δὲ $\frac{2}{1} < \frac{38}{17} < \dots < \sqrt{5} \dots < \frac{161}{72} < \frac{9}{4}$, καὶ

$$2^2 = 5 \cdot 1^2 - 1$$

$$9^2 = 5 \cdot 4^2 + 1$$

$$38^2 = 5 \cdot 17^2 - 1$$

$$\vdots$$

$$\vdots$$

$$\delta_n^2 = 5 \cdot \alpha_n^2 + (5-4)^n (-1)^n.$$

II. 2. $\delta_n^2 = 10\alpha_n^2 + (10-4)^n (-1)^n$, καὶ $\sqrt{10}$.

Εἰς τὸ προηγούμενον σχῆμα 3 λαμβάνομεν $AB = \alpha_n$, $B\Gamma = 3\alpha_n$, ὁπότε $\delta_n^2 = 10\alpha_n^2$, καὶ $\frac{\delta_n}{\alpha_n} = \sqrt{10}$.

Ἐφαρμόζοντες τὴν προηγουμένην κατασκευὴν (II. 1) λαμβάνομεν

$$(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2$$

$$\text{Είναι άρα και } 10(2\alpha_1 + \delta_1)^2 = 40\alpha_1^2 + 40\alpha_1\delta_1 + 10\delta_1^2$$

$$\text{Άλλά } \delta_1^2 = 10\alpha_1^2. \text{ Έπομένως}$$

$$10(2\alpha_1 + \delta_1)^2 = 40\alpha_1^2 + 60\alpha_1^2 + 40\alpha_1\delta_1 + 4\delta_1^2 = (10\alpha_1 + 2\delta_1)^2.$$

Η σχέσις όμως αύτη σημαίνει ότι ή μὲν $(2\alpha_1 + \delta_1)$ εἶναι πλευρά, ή δὲ $(10\alpha_1 + 2\delta_1)$ διαγώνιος ὁμοίου ὀρθογωνίου παραλληλογράμμου πρὸς τὸ ἀρχικόν. Ὁ νόμος τῆς κατασκευῆς τῶν ὁμοίων ἐν συνεχείᾳ παραλληλογράμμων εἶναι προφανής. Ὅθεν θὰ εἶναι

Πλευρικοὶ ἀριθμοί.

$$\begin{aligned} \alpha_1 & \\ \alpha_2 &= 2\alpha_1 + \delta_1 \\ \alpha_3 &= 2\alpha_2 + \delta_2 \\ \alpha_4 &= 2\alpha_3 + \delta_3 \\ &\vdots \\ \alpha_n &= 2\alpha_{n-1} + \delta_{n-1} \end{aligned}$$

Διαμετρικοὶ ἀριθμοί.

$$\begin{aligned} \delta_1 & \\ \delta_2 &= 10\alpha_1 + 2\delta_1 \\ \delta_3 &= 10\alpha_2 + 2\delta_2 \\ \delta_4 &= 10\alpha_3 + 2\delta_3 \\ &\vdots \\ \delta_n &= 10\alpha_{n-1} + 2\delta_{n-1} \end{aligned}$$

$$\text{Ἐὰν θέσωμεν } \alpha_1 = 1 \text{ και } \delta_1 = 2 \text{ λαμβάνομεν}$$

$$\alpha_2 = 4 \quad \delta_2 = 14$$

$$\alpha_3 = 22 \quad \delta_3 = 68$$

$$\alpha_4 = 112 \quad \delta_4 = 356$$

$$\vdots$$

$$\text{Εἶναι δὲ } \frac{2}{1} < \frac{68}{22} < \dots \sqrt{10} \dots < \frac{356}{112} < \frac{14}{4}$$

και

$$2^2 = 10 \cdot 1^2 - 6$$

$$14^2 = 10 \cdot 4^2 + 6^2$$

$$68^2 = 10 \cdot 22^2 - 6^3$$

$$356^2 = 10 \cdot 112^2 + 6^4$$

$$\vdots$$

$$\delta_n^2 = 10 \cdot \alpha_n^2 + (10 - 4)^n (-1)^n$$

$$\text{II. 3} \quad \delta_n^2 = 17\alpha_n^2 + (17-4)^n (-1)^n, \text{ και } \sqrt{17}.$$

Εἰς τὸ αὐτὸ σχῆμα 3 λαμβάνομεν $AB = \alpha_1$, $B\Gamma = 4\alpha_1$, $A\Gamma = \delta_1$, ὁπότε εἶναι

$$\delta_1^2 = 17\alpha_1^2 \text{ και } \frac{\delta_1}{\alpha_1} = \sqrt{17}. \text{ Ἐφαρμόζομεν πάλιν τὴν κατασκευὴν (II. 1) ὁπότε ἔχομεν}$$

$$(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2.$$

Είναι ἄρα καὶ $17(2\alpha_1 + \delta_1)^2 = 68\alpha_1^2 + 68\alpha_1\delta_1 + 17\delta_1^2$. Ἀλλὰ $\delta_1^2 = 17\alpha_1^2$.
 Ἐπομένως $17(2\alpha_1 + \delta_1)^2 = 68\alpha_1^2 + 221\alpha_1^2 + 68\alpha_1\delta_1 + 4\delta_1^2 = (17\alpha_1 + 2\delta_1)^2$.

Ἡ σχέσις ὁμῶς αὕτη σημαίνει ὅτι ἡ μὲν $(2\alpha_1 + \delta_1)$ εἶναι πλευρά, ἡ δὲ $(17\alpha_1 + 2\delta_1)$ διαγώνιος ὁμοίου ὀρθογωνίου παραλληλογράμμου πρὸς τὸ ἀρχικόν.

Ὁ νόμος σχηματισμοῦ τῶν ὁμοίων ὀρθογ. παραλληλογράμμων εἶναι προφανής.

Ὅθεν θὰ ἔχωμεν

Πλευρικοὶ ἀριθμοί.

$$\begin{aligned} \alpha_1 & \\ \alpha_2 &= 2\alpha_1 + \delta_1 \\ \alpha_3 &= 2\alpha_2 + \delta_2 \\ \alpha_4 &= 2\alpha_3 + \delta_3 \\ &\vdots \\ \alpha_n &= 2\alpha_{n-1} + \delta_{n-1} \end{aligned}$$

Διαμετρικοὶ ἀριθμοί.

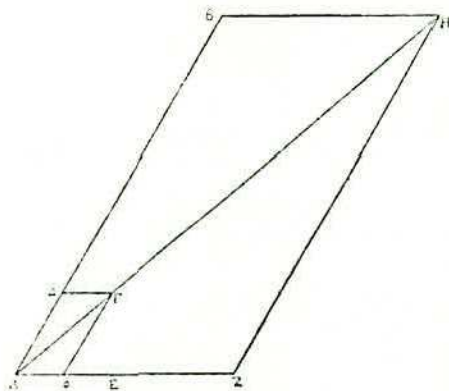
$$\begin{aligned} \delta_1 & \\ \delta_2 &= 17\alpha_1 + 2\delta_1 \\ \delta_3 &= 17\alpha_2 + 2\delta_2 \\ \delta_4 &= 17\alpha_3 + 2\delta_3 \\ &\vdots \\ \delta_n &= 17\alpha_{n-1} + 2\delta_{n-1} \end{aligned}$$

Ἐὰν θέσωμεν $\alpha_1 = 1$ καὶ $\delta_1 = 2$ λαμβάνομεν

$$\begin{aligned} \alpha_2 &= 4 & \delta_2 &= 21 \\ \alpha_3 &= 29 & \delta_3 &= 110 \\ \alpha_4 &= 168 & \delta_4 &= 713 \end{aligned}$$

$$\text{Εἶναι δὲ } \frac{2}{1} < \frac{110}{29} < \dots < \sqrt{17} \dots < \frac{713}{168} < \frac{21}{4}, \quad \text{καὶ}$$

$$\begin{aligned} 2^2 &= 17 \cdot 1^2 - 13 \\ 21^2 &= 17 \cdot 4^2 + 13^2 \\ 110^2 &= 17 \cdot 29^2 - 13^3 \\ 713^2 &= 17 \cdot 168^2 + 13^4 \\ &\vdots \\ \delta_n^2 &= 17 \cdot \alpha_n^2 + (17-4)^n (-1)^n \end{aligned}$$



Σχ. 4.

Π. 4.

Θεωροῦμεν τὸ ρομβοειδὲς παραλληλόγραμμον ΑΒΓΔ (σχ. 4) ἐνθα γωνία

$$\text{ΑΒΓ} = 120^\circ, \text{ΑΒ} = \alpha_1, \text{ΒΓ} = 2\alpha_1$$

καὶ ἡ μεγαλυτέρα διαγώνιος ΑΓ = δ_1 . Κατὰ τὸν Εὐκλείδην Π, 12 θὰ εἶναι $\delta_1^2 = 7\alpha_1^2$

ὁπότε $\frac{\delta_1}{\alpha_1} = \sqrt{7}$. Πάλιν ἐφαρμόζομεν τὴν

αὐτὴν κατασκευὴν (Π. 1), ὁπότε λαμβάνοντες $\text{ΒΕ} = \alpha_1, \text{ΕΖ} = \delta_1$, θὰ ἔχωμεν κατὰ τὸν Εὐκλείδην Π, 10.

$$(2a_1 + \delta_1)^2 + \delta_1^2 = 2a_1^2 + 2(a_1 + \delta_1)^2, \quad \text{ἐξ ἧς}$$

$$(2a_1 + \delta_1)^2 = 4a_1^2 + 4a_1\delta_1 + \delta_1^2.$$

$$\text{Εἶναι ἄρα καὶ} \quad 7(2a_1 + \delta_1)^2 = 28a_1^2 + 28a_1\delta_1 + 7\delta_1^2.$$

$$\text{Ἀλλὰ} \quad \delta_1^2 = 7a_1^2. \quad \text{Ἐπομένως}$$

$$7(2a_1 + \delta_1)^2 = 28a_1^2 + 21a_1^2 + 28a_1\delta_1 + 4\delta_1^2 = (7a_1 + 2\delta_1)^2.$$

Ἡ σχέσις ὅμως αὕτη σημαίνει ὅτι ἡ μὲν $(2a_1 + \delta_1)$ εἶναι πλευρά, ἡ δὲ $(7a_1 + 2\delta_1)$ διαγώνιος μεγαλύτερα ὁμοίου πρὸς τὸ ἀρχικὸν ρομβοειδοῦς παραλληλογράμμου. Κατὰ τὸν προφανῆ νόμον κατασκευῆς ὁμοίων ρομβοειδῶν παραλληλογράμμων θὰ ἔχωμεν:

Πλευρικοὶ ἀριθμοί.

$$\begin{aligned} a_1 & \\ a_2 &= 2a_1 + \delta_1 \\ a_3 &= 2a_2 + \delta_2 \\ a_4 &= 2a_3 + \delta_3 \\ &\vdots \\ a_n &= 2a_{n-1} + \delta_{n-1} \end{aligned}$$

Διαμετρικοὶ ἀριθμοί.

$$\begin{aligned} \delta_1 & \\ \delta_2 &= 7a_1 + 2\delta_1 \\ \delta_3 &= 7a_2 + 2\delta_2 \\ \delta_4 &= 7a_3 + 2\delta_3 \\ &\vdots \\ \delta_n &= 7a_{n-1} + 2\delta_{n-1} \end{aligned}$$

$$\text{Ἐὰν θέσωμεν} \quad a_1 = 1 \quad \text{καὶ} \quad \delta_1 = 2, \text{ λαμβάνομεν}$$

$$\begin{aligned} a_2 &= 4 & \delta_2 &= 11 \\ a_3 &= 19 & \delta_3 &= 50 \\ a_4 &= 88 & \delta_4 &= 233 \\ &\vdots & & \end{aligned}$$

$$\text{Εἶναι δὲ} \quad \frac{2}{1} < \frac{50}{19} < \dots < \sqrt{7} \dots < \frac{233}{88} < \frac{11}{4},$$

καὶ

$$\begin{aligned} 2^2 &= 7 \cdot 1^2 - 3 \\ 11^2 &= 7 \cdot 4^2 + 3^2 \\ 50^2 &= 7 \cdot 19^2 - 3^3 \\ 233^2 &= 7 \cdot 88^2 + 3^4 \\ &\vdots \\ \delta_n^2 &= 7 \cdot a_n^2 + (7-4)^n (-1)^n. \end{aligned}$$

II. 5. Εἰς τὸ προηγούμενον σχῆμα 4 λαμβάνομεν $AB = a_1$, $BF = 3a_1$, $AF = \delta_1$. Κατὰ τὸν Εὐκλείδην II, 12 εἶναι $\delta_1^2 = 13a_1^2$, καὶ $\frac{\delta_1}{a_1} = \sqrt{13}$. Πάλιν ἐφαρμόζομεν τὴν

Πλευρικοί αριθμοί.		Διαμετρικοί αριθμοί.	
α_1		δ_1	
$\alpha_2 = 2\alpha_1 + \delta_1$		$\delta_2 = 13\alpha_1 + 2\delta_1$	
$\alpha_3 = 2\alpha_2 + \delta_2$		$\delta_3 = 13\alpha_2 + 2\delta_2$	
$\alpha_4 = 2\alpha_3 + \delta_3$		$\delta_4 = 13\alpha_3 + 2\delta_3$	
$\vdots$		$\vdots$	
$\vdots$		$\vdots$	
$\vdots$		$\vdots$	
$\alpha_v = 2\alpha_{v-1} + \delta_{v-1}$		$\delta_v = 13\alpha_{v-1} + 2\delta_{v-1}$	
Ἐάν θέσωμεν	$\alpha_1 = 1$	$\delta_1 = 2$	λαμβάνομεν
	$\alpha_2 = 4$	$\delta_2 = 17$	
	$\alpha_3 = 25$	$\delta_3 = 86$	
	$\alpha_4 = 136$	$\delta_4 = 497$	
	$\vdots$	$\vdots$	
	$\vdots$	$\vdots$	
	$\vdots$	$\vdots$	
Εἶναι δε	$\frac{2}{1} < \frac{86}{25} < \cdots \sqrt{13} < \cdots \frac{497}{136} < \frac{17}{4}$,		καὶ
	$2^2 = 13 \cdot 1^2 - 9$		
	$17^2 = 13 \cdot 4^2 + 9^2$		
	$86^2 = 13 \cdot 25^2 - 9^2$		
	$497^2 = 13 \cdot 136^2 + 9^4$		
	$\vdots$		
	$\vdots$		
	$\vdots$		
	$\delta_v^2 = 13\alpha_v^2 + (13-4)^v (-1)^v$.		

Ἐκ τῶν ἀνωτέρω ἐκτεθεισῶν κατασκευῶν καὶ ἀποδείξεων καθίσταται αὐτονόητος ὁ σχηματισμὸς ἀντιστοιχῶν πλευρικῶν καὶ διαμετρικῶν ἀριθμῶν δι' ἀλγεβρικοῦ καθαρῶς ὑπολογισμοῦ καὶ οὐχὶ γεωμετρικοῦ, διὰ τὴν $\sqrt{6} \cdot \sqrt{8} \cdot \sqrt{11} \cdot \sqrt{12} \cdot \sqrt{14} \cdot \sqrt{15}$, ... καὶ τὰς συναφεῖς ἐξισώσεις.

Οὕτω θὰ εἶναι

$$\text{II. 6. Διὰ } \delta_v^2 = 6a_v^2 + (6-4)^v (-1)^v \text{ καὶ } 1/\sqrt{6}.$$

a_1	δ_1
$a_2 = 2a_1 + \delta_1$	$\delta_2 = 6a_1 + 2\delta_1$
$a_3 = 2a_2 + \delta_2$	$\delta_3 = 6a_2 + 2\delta_2$
$a_4 = 2a_3 + \delta_3$	$\delta_4 = 6a_3 + 2\delta_3$
⋮	⋮
⋮	⋮
$a_v = 2a_{v-1} + \delta_{v-1}$	$\delta_v = 6a_{v-1} + 2\delta_{v-1}$

Ἐὰν θέσωμεν $a_1 = 1$	καὶ $\delta_1 = 2$	λαμβάνομεν
$a_2 = 4$	$\delta_2 = 10$	
$a_3 = 18$	$\delta_3 = 44$	
$a_4 = 80$	$\delta_4 = 196$	

$$\text{Εἶναι δὲ } \frac{2}{1} < \frac{44}{18} < \dots < \sqrt{6} \dots < \frac{196}{80} < \frac{10}{4}, \text{ καὶ}$$

$$\begin{aligned} 2^2 &= 6 \cdot 1^2 - 2 \\ 10^2 &= 6 \cdot 4^2 + 2^2 \\ 44^2 &= 6 \cdot 18^2 - 2^3 \\ 196^2 &= 6 \cdot 80^2 + 2^4 \end{aligned}$$

⋮

$$\delta_v^2 = 6 \cdot a_v^2 + (6-4)^v (-1)^v.$$

$$\text{II. 7. Διὰ } \delta_v^2 = 8a_v^2 + (8-4)^v (-1)^v \text{ καὶ } \sqrt{8}.$$

a_1	δ_1
$a_2 = 2a_1 + \delta_1$	$\delta_2 = 8a_1 + 2\delta_1$
$a_3 = 2a_2 + \delta_2$	$\delta_3 = 8a_2 + 2\delta_2$
$a_4 = 2a_3 + \delta_3$	$\delta_4 = 8a_3 + 2\delta_3$
⋮	⋮
⋮	⋮
$a_v = 2a_{v-1} + \delta_{v-1}$	$\delta_v = 8a_{v-1} + 2\delta_{v-1}$

Ἐὰν θέσωμεν $a_1 = 1$	καὶ $\delta_1 = 2$	λαμβάνομεν
$a_2 = 4$	$\delta_2 = 12$	

$$\begin{array}{ll} \alpha_3 = 20 & \delta_3 = 56 \\ \alpha_4 = 96 & \delta_4 = 272 \\ \vdots & \vdots \\ \vdots & \vdots \end{array}$$

$$\text{Εἶναι } \delta_1 = \frac{2}{1} < \frac{56}{20} < \cdots \sqrt{8} \cdots < \frac{272}{96} < \frac{12}{4}, \quad \text{καὶ}$$

$$2^2 = 8 \cdot 1^2 - 4$$

$$12^2 = 8 \cdot 4^2 + 4^2$$

$$59^2 = 8 \cdot 20^2 - 4^3$$

$$272^2 = 8 \cdot 96^2 + 4^4$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\delta_v^2 = 8 \cdot \alpha_v^2 + (8 - 4)^v (-1)^v.$$

$$\text{II. 8. Διὰ } \delta_v^2 = 11\alpha_v^2 + (11 - 4)^v (-1)^v, \text{ καὶ } \sqrt{11}.$$

$$\begin{array}{ll} \alpha_1 & \delta_1 \\ \alpha_2 = 2\alpha_1 + \delta_1 & \delta_2 = 11\alpha_1 + 2\delta_1 \\ \alpha_3 = 2\alpha_2 + \delta_2 & \delta_3 = 11\alpha_2 + 2\delta_2 \\ \alpha_4 = 2\alpha_3 + \delta_3 & \delta_4 = 11\alpha_3 + 2\delta_3 \\ \vdots & \vdots \\ \vdots & \vdots \\ \alpha_v = 2\alpha_{v-1} + \delta_{v-1} & \delta_v = 11\alpha_{v-1} + 2\delta_{v-1} \end{array}$$

$$\begin{array}{ll} \text{Ἐάν θέσωμεν } \alpha_1 = 1 & \text{καὶ } \delta_1 = 2 \quad \text{λαμβάνομεν} \\ \alpha_2 = 4 & \delta_2 = 15 \\ \alpha_3 = 23 & \delta_3 = 74 \\ \alpha_4 = 120 & \delta_4 = 401, \end{array}$$

$$\text{Εἶναι } \delta_1 = \frac{2}{1} < \frac{74}{23} < \cdots \sqrt{11} \cdots < \frac{401}{120} < \frac{15}{4}, \quad \text{καὶ}$$

$$2^2 = 11 \cdot 1^2 - 7$$

$$15^2 = 11 \cdot 4^2 + 7^2$$

$$74^2 = 11 \cdot 23^2 - 7^3$$

$$401^2 = 11 \cdot 120^2 + 7^4$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\delta_v^2 = 11\alpha_v^2 + (11 - 4)^v (-1)^v.$$

II. 9. Διὰ $\delta_v^2 = 12a_v^2 + (12-4)^v(-1)^v$, καὶ $\sqrt{12}$.

$$\begin{array}{ll} a_1 & \delta_1 \\ a_2 = 2a_1 + \delta_1 & \delta_2 = 12a_1 + 2\delta_1 \\ a_3 = 2a_2 + \delta_2 & \delta_3 = 12a_2 + 2\delta_2 \\ a_4 = 2a_3 + \delta_3 & \delta_4 = 12a_3 + 2\delta_3 \\ \vdots & \vdots \\ a_v = 2a_{v-1} + \delta_{v-1} & \delta_v = 12a_{v-1} + 2\delta_{v-1} \end{array}$$

Ἐὰν θέσωμεν $a_1 = 1$ λαμβάνομεν
 $a_2 = 4$ $\delta_1 = 2$
 $a_3 = 24$ $\delta_2 = 16$
 $a_4 = 128$ $\delta_3 = 80$
 $\delta_4 = 448$

Εἶναι δὲ $\frac{2}{1} < \frac{80}{24} < \dots \sqrt{12} \dots < \frac{448}{128} < \frac{16}{4}$, καὶ

$$\begin{array}{l} 2^2 = 12 \cdot 1^2 - 8 \\ 16^2 = 12 \cdot 4^2 + 8^2 \\ 80^2 = 12 \cdot 24^2 - 8^3 \\ 448^2 = 12 \cdot 128^2 + 8^4 \\ \vdots \end{array}$$

$$\delta_v^2 = 12 \cdot a_v^2 + (12-4)^v(-1)^v.$$

II. 10. Διὰ $\delta_v^2 = 14a_v^2 + (14-4)^v(-1)^v$, καὶ $\sqrt{14}$.

$$\begin{array}{ll} a_1 & \delta_1 \\ a_2 = 2a_1 + \delta_1 & \delta_2 = 14a_1 + 2\delta_1 \\ a_3 = 2a_2 + \delta_2 & \delta_3 = 14a_2 + 2\delta_2 \\ a_4 = 2a_3 + \delta_3 & \delta_4 = 14a_3 + 2\delta_3 \\ \vdots & \vdots \\ a_v = 2a_{v-1} + \delta_{v-1} & \delta_v = 14a_{v-1} + 2\delta_{v-1} \end{array}$$

Ἐὰν θέσωμεν $a_1 = 1$ λαμβάνομεν
 $a_2 = 4$ $\delta_1 = 2$
 $a_3 = 26$ $\delta_2 = 18$
 $a_4 = 144$ $\delta_3 = 92$
 $\delta_4 = 548$

Εἶναι δὲ $\frac{2}{1} < \frac{92}{26} < \dots \sqrt{14} \dots < \frac{548}{144} < \frac{18}{4}$, καὶ

$$\begin{aligned}
 2^2 &= 14 \cdot 1^2 - 10 \\
 18^2 &= 14 \cdot 4^2 + 10^2 \\
 92^2 &= 14 \cdot 26^2 - 10^3 \\
 548^2 &= 14 \cdot 144^2 + 10^4 \\
 &\vdots \\
 \delta_v^2 &= 14 \cdot \alpha_v^2 + (14-4)^v (-1)^v.
 \end{aligned}$$

II. 11. Διὰ $\delta_v^2 = 15\alpha_v^2 + (15-4)(-1)^v$, καὶ $\sqrt{15}$.

$$\begin{array}{ll}
 \alpha_1 & \delta_1 \\
 \alpha_2 = 2\alpha_1 + \delta_1 & \delta_2 = 15\alpha_1 + 2\delta_1 \\
 \alpha_3 = 2\alpha_2 + \delta_2 & \delta_3 = 15\alpha_2 + 2\delta_2 \\
 \alpha_4 = 2\alpha_3 + \delta_3 & \delta_4 = 15\alpha_3 + 2\delta_3 \\
 \vdots & \vdots \\
 \alpha^v = 2\alpha_{v-1} + \delta_{v-1} & \delta_v = 15\alpha_{v-1} + 2\delta_{v-1}
 \end{array}$$

$$\begin{array}{lll}
 \text{Ἐὰν θέσωμεν} & \alpha_1 = 1 & \delta_1 = 2 \quad \text{λαμβάνομεν} \\
 & \alpha_2 = 4 & \delta_2 = 19 \\
 & \alpha_3 = 27 & \delta_3 = 98 \\
 & \alpha_4 = 152 & \delta_4 = 601 \\
 & \vdots & \vdots
 \end{array}$$

$$\text{Εἶναι δὲ} \quad \frac{2}{1} < \frac{98}{27} < \dots \sqrt{15} \dots < \frac{601}{152} < \frac{19}{4}, \quad \text{καὶ}$$

$$\begin{aligned}
 2^2 &= 15 \cdot 1^2 - 11 \\
 19^2 &= 15 \cdot 4^2 + 11^2 \\
 98^2 &= 15 \cdot 27^2 - 11^3 \\
 601^2 &= 15 \cdot 152^2 + 11^4 \\
 &\vdots \\
 \delta_v^2 &= 15 \cdot \alpha_v^2 + (15-4)^v (-1)^v.
 \end{aligned}$$

III. 1. Ἐκ τῶν προηγουμένων παρατηροῦμεν ὅτι ἡ κατὰ τὸ εὐκλείδειον θεώρημα II, 10 γεωμετρικὴ κατασκευὴ, τὴν ὁποίαν μνημονεύει ὁ Πρόκλος, ἄγει εἰς τὴν ταυτότητα (1). $(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2$, τὴν ἀποδεικνυσμένην κατὰ τὸ εὐκλείδειον θεώρημα II, 4. Ἐὰν $\delta_1^2 = \lambda\alpha_1^2$, ἐνθα

$\lambda \geq 5$, ἀκέραιος μὴ τετράγωνος, θὰ εἶναι καὶ $(\lambda-4)\delta_1^2 = (\lambda-4)\lambda\alpha_1^2$, ὁπότε ἐκ τῆς (1) ἔχομεν

$$\lambda(2\alpha_1 + \delta_1)^2 = 4\lambda\alpha_1^2 + 4\lambda\alpha_1\delta_1 + (\lambda-4)\lambda\alpha_1^2 + 4\delta_1^2 = (\lambda\alpha_1 + 2\delta_1)^2.$$

Ἐπομένως θὰ εἶναι

	Πλευρικοὶ ἀριθμοί.	Διαμετρικοὶ ἀριθμοί.
1)	α_1	δ_1
	$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = \lambda\alpha_1 + 2\delta_1$
	$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = \lambda\alpha_2 + 2\delta_2$
	$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = \lambda\alpha_3 + 2\delta_3$
	$\vdots$	$\vdots$
	$\vdots$	$\vdots$
	$\alpha_v = 2\alpha_{v-1} + \delta_{v-1}$	$\delta_v = \lambda\alpha_{v-1} + 2\delta_{v-1}$
2)	$\frac{\delta_1}{\alpha_1} < \frac{\delta_2}{\alpha_2} < \frac{\delta_3}{\alpha_3} < \dots < \sqrt{\lambda} \dots < \frac{\delta_v}{\alpha_v} < \frac{\delta_1}{\alpha_1} < \frac{\delta_2}{\alpha_2}, (\alpha_1=1, \delta_1=2).$	
3)	$\delta_v^2 = \lambda\alpha_v^2 + (\lambda-4)^v (-1)^v$	

III. 2.

Εἶναι δυνατόν ἢ $\sqrt{2}$ νὰ ὑπολογισθῇ καὶ ἐκ τῶν πλευρικῶν καὶ διαμετρικῶν ἀριθμῶν τῆς μορφῆς $\alpha_v = 2\alpha_{v-1} + \delta_{v-1}$, $\delta_v = \lambda\alpha_{v-1} + 2\delta_{v-1}$, ὅταν $\lambda = 2$. Τὴν μέθοδον ταύτην καλοῦμεν γενικὴν πρὸς διάκρισιν ἀπὸ τῆς μεθόδου τῆς διασωθείσης ὑπὸ τοῦ Θέωνος τοῦ Σμυρναίου καὶ τοῦ Πρόκλου, τὴν ὁποίαν καλοῦμεν εἰδικήν.

Πρὸς σύγκρισιν παραθέτομεν τὰ ἐξαγόμενα καὶ τῶν δύο μεθόδων.

Α'. Μέθοδος εἰδική, $\alpha_v = \alpha_{v-1} + \delta_{v-1}$, $\delta_v = 2\alpha_{v-1} + \delta_{v-1}$.

$\alpha_1 = 1$	$\delta_1 = 1$
$\alpha_2 = 2$	$\delta_2 = 3$
$\alpha_3 = 5$	$\delta_3 = 7$
$\alpha_4 = 12$	$\delta_4 = 17$
$\vdots$	
$\vdots$	

$$\frac{\delta_1}{\alpha_1} < \frac{\delta_2}{\alpha_2} < \frac{\delta_3}{\alpha_3} < \dots < \sqrt{2} \dots < \frac{\delta_v}{\alpha_v} < \frac{\delta_1}{\alpha_1} < \frac{\delta_2}{\alpha_2}$$

$$1^2 = 2 \cdot 1^2 - 1, \quad 3^2 = 2 \cdot 2^2 + 1, \quad 7^2 = 2 \cdot 5^2 - 1, \quad \dots, \quad \delta_v^2 = 2\alpha_v^2 + (-1)^v.$$

[Διὰ $\alpha_1 = 1$, $\delta_1 = 2$.

$\alpha_1 = 1$	$\delta_1 = 2$
$\alpha_2 = 3$	$\delta_2 = 4$
$\alpha_3 = 7$	$\delta_3 = 10$

$$\alpha_1 = 17$$

$$\delta_1 = 24$$

$$\frac{\delta_1}{\alpha_1} > \frac{\delta_2}{\alpha_2} > \frac{\delta_3}{\alpha_3} > \dots \sqrt{2} \dots > \frac{\delta_6}{\alpha_6} > \frac{\delta_4}{\alpha_4} > \frac{\delta_2}{\alpha_2}$$

$$2^2 = 2 \cdot 1^2 + 2$$

$$4^2 = 2 \cdot 3^2 - 2$$

$$10^2 = 2 \cdot 7^2 + 2$$

$$24^2 = 2 \cdot 17^2 - 2$$

$$\delta_v^2 = 2\alpha_v^2 + (2-4)(-1)^v.$$

B'. Μεθοδος γενικη, $\alpha_v = 2\alpha_{v-1} + \delta_{v-1}$, $\delta_v = 2\alpha_{v-1} + 2\delta_{v-1}$.

Διὰ $\alpha_1 = 1$, $\delta_1 = 1$.

$$\alpha_1 = 1$$

$$\delta_1 = 1$$

$$\alpha_2 = 3$$

$$\delta_2 = 4$$

$$\alpha_3 = 10$$

$$\delta_3 = 14$$

$$\alpha_4 = 34$$

$$\delta_4 = 48$$

$$\frac{\delta_1}{\alpha_1} < \frac{\delta_2}{\alpha_2} < \frac{\delta_3}{\alpha_3} < \dots \sqrt{2} \dots$$

$$1^2 = 2 \cdot 1^2 - 1$$

$$4^2 = 2 \cdot 3^2 - 2$$

$$14^2 = 2 \cdot 10^2 - 2^2$$

$$48^2 = 2 \cdot 34^2 - 2^3$$

$$\delta_v^2 = 2 \cdot \alpha_v^2 + (2-4)^{v-1} \cdot (-1)^v.$$

Διὰ $\alpha_1 = 1$, $\delta_1 = 2$,

$$\alpha_1 = 1$$

$$\delta_1 = 2$$

$$\alpha_2 = 4$$

$$\delta_2 = 6$$

$$\alpha_3 = 14$$

$$\delta_3 = 20$$

$$\alpha_4 = 48$$

$$\delta_4 = 68$$

$$\sqrt{2} \dots < \frac{\delta_3}{\alpha_3} < \frac{\delta_2}{\alpha_2} < \frac{\delta_1}{\alpha_1}.$$

$$\begin{aligned}
 2^2 &= 2 \cdot 1^2 + 2 \\
 6^2 &= 2 \cdot 4^2 + 2^2 \\
 20^2 &= 2 \cdot 14^2 + 2^3 \\
 68^2 &= 2 \cdot 48^2 + 2^4 \\
 &\vdots \\
 \delta_v^2 &= 2 \cdot \alpha_v^2 + (2-4)^v \cdot (-1)^v.
 \end{aligned}$$

Κατ' ἀντιστοιχίαν πρὸς τὴν ὑπὸ τοῦ Ἀρχιμήδους παρεχομένην τιμὴν τῆς $\sqrt{3}$ θὰ εἴχομεν, διὰ τὴν $\sqrt{2}$,

$$\frac{1}{1} < \frac{4}{3} < \frac{14}{10} < \frac{48}{34} < \frac{164}{116} < \dots \sqrt{2} \dots < \frac{792}{560} < \frac{232}{164} < \frac{68}{48} < \frac{20}{14} < \frac{6}{4} < \frac{2}{1}.$$

Εἶναι φανερόν, ὅτι εἶναι προτιμότερα ἢ ὑπὸ τοῦ Θεώωνος τοῦ Σμυρναίου καὶ τοῦ Πρόκλου διασωθεῖσα μέθοδος διὰ τὸν ὑπολογισμόν τῆς $\sqrt{2}$.

SUMMARY

I 1. The law of formation of the side- and diameter- (diagonal-) numbers is explained by Theon of Smyrna. According to Proclus the related identity is proved by Euclid book II, proposition 10.

Side numbers $\alpha_v = \alpha_{v-1} + \delta_{v-1}$, diagonal numbers $\delta_v = 2\alpha_{v-1} + \delta_{v-1}$, $\delta_v^2 = 2\alpha_v^2 \mp 1$. For $\alpha_1 = 1$, $\delta_1 = 1$ we have $\frac{1}{1} < \frac{7}{5} < \dots \sqrt{2} \dots < \frac{17}{12} < \frac{3}{2}$.

2. Archimedes for the arithmetical approximation to π starts from a greater and a lesser limit to the value of $\sqrt{3}$, which without remark as known,

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}, \quad 265^2 = 3 \cdot 153^2 - 2, \quad 1351^2 = 3 \cdot 780^2 + 1.$$

We give the following interpretation on the archimedean formula, with pythagorean method of the side- and diagonal- numbers. In the figure 2 is $AB = \alpha_1$ the side, $AF = \delta_1$ the greater diagonal of the rhomb $ABFA$, and the greater angle $ABF = 120^\circ$. Then $\delta_1^2 = 3\alpha_1^2$, $\delta_1 : \alpha_1 = \sqrt{3}$. According to Euclid II Prop. 10 we have

$$\begin{aligned}
 (2\alpha_1 + \delta_1)^2 + \delta_1^2 &= 2\alpha_1^2 + 2(\alpha_1 + \delta_1)^2 \\
 (2\alpha_1 + \delta_1)^2 &= 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2,
 \end{aligned} \tag{1}$$

It is also $3(2\alpha_1 + \delta_1)^2 = 12\alpha_1^2 + 12\alpha_1\delta_1 + 3\delta_1^2$, and because $\delta_1^2 = 3\alpha_1^2$, $3(2\alpha_1 + \delta_1)^2 = 9\alpha_1^2 + 12\alpha_1\delta_1 + 4\delta_1^2 = (3\alpha_1 + 2\delta_1)^2$. The law

of formation of the corresponding side- and diagonal- numbers is evidently. Side numbers $\alpha_v = 2\alpha_{v-1} + \delta_{v-1}$, diagonal numbers $\delta_v = 3\alpha_{v-1} + 2\delta_{v-1}$.

When	$a_1 = 1$	$\delta_1 = 1$	When	$a_1 = 1$	$\delta_1 = 2$
	$a_2 = 3$	$\delta_2 = 5$		$a_2 = 4$	$\delta_2 = 7$
	$a_3 = 11$	$\delta_3 = 19$		$a_3 = 15$	$\delta_3 = 26$
	$a_4 = 41$	$\delta_4 = 71$		$a_4 = 56$	$\delta_4 = 97$
	$a_5 = 153$	$\delta_5 = 265$		$a_5 = 209$	$\delta_5 = 362$
				$a_6 = 780$	$\delta_6 = 1351$

and $\delta_v^2 = 3a_v^2 - 2$

and $\delta_v^2 = 3a_v^2 + 1$.

$$\frac{1}{1} < \frac{5}{3} < \frac{19}{11} < \frac{71}{41} < \frac{265}{153} < \dots < \sqrt{3} \dots < \frac{1351}{780} < \frac{362}{209} < \frac{97}{56} < \frac{26}{15} < \frac{7}{4} < \frac{2}{1}$$

II. In the following we start from the identity (1).

1. In the figure 3 we take $AB = a_1$, $B\Gamma = 2a_1$, $A\Gamma = \delta_1$. Then $\delta_1^2 = 5a_1^2$, $\delta_1 : a_1 = \sqrt{5}$, and $5(2a_1 + \delta_1)^2 = 20a_1^2 + 20a_1\delta_1 + 5\delta_1^2$. Because $\delta_1^2 = 5a_1^2$ is $5(2a_1 + \delta_1)^2 = 20a_1^2 + 5a_1^2 + 20a_1\delta_1 + 4\delta_1^2 = (5a_1 + 2\delta_1)^2$.

Side numbers $a_v = 2a_{v-1} + \delta_{v-1}$, diagonal numbers, $\delta_v = 5a_{v-1} + 2\delta_{v-1}$, $\delta_v^2 = 5a_v^2 + (5-4)^v (-1)^v$.

$$\text{When } a_1 = 1, \delta_1 = 2, \quad \frac{2}{1} < \frac{38}{14} < \dots < \sqrt{5} \dots < \frac{161}{72} < \frac{9}{4}.$$

2. In the same figure 3 we take $AB = a_1$, $B\Gamma = 3a_1$, $A\Gamma = \delta_1$. Then $\delta_1^2 = 10a_1^2$, $\delta_1 : a_1 = \sqrt{10}$, and $10(2a_1 + \delta_1)^2 = 40a_1^2 + 40a_1\delta_1 + 10\delta_1^2$. Because $\delta_1^2 = 10a_1^2$ is $10(2a_1 + \delta_1)^2 = 100a_1^2 + 40a_1\delta_1 + 4\delta_1^2 = (10a_1 + 2\delta_1)^2$.

Side numbers $a_v = 2a_{v-1} + \delta_{v-1}$, diagonal numbers $\delta_v = 10a_{v-1} + 2\delta_{v-1}$, $\delta_v^2 = 10a_v^2 + (10-4)^v (-1)^v$. When $a_1 = 1$, $\delta_1 = 2$,

$$\frac{2}{1} < \frac{68}{22} < \dots < \sqrt{10} \dots < \frac{356}{112} < \frac{14}{4}.$$

3. In the same figure 3 we take $AB = a_1$, $B\Gamma = 4a_1$, $A\Gamma = \delta_1$. Then $\delta_1^2 = 17a_1^2$, $\delta_1 : a_1 = \sqrt{17}$, and $17(2a_1 + \delta_1)^2 = 68a_1^2 + 68a_1\delta_1 + 17\delta_1^2$. Because $\delta_1^2 = 17a_1^2$ is $17(2a_1 + \delta_1)^2 = 289a_1^2 + 68a_1\delta_1 + 4\delta_1^2 = (17a_1 + 2\delta_1)^2$.

Side numbers $a_v = 2a_{v-1} + \delta_{v-1}$, D. N. $\delta_v = 17a_{v-1} + 2\delta_{v-1}$, $\delta_v^2 = 17a_v^2 + (17-4)^v (-1)^v$.

$$\text{When } a_1 = 1, \delta_1 = 2, \quad \frac{2}{1} < \frac{110}{29} < \dots < \sqrt{17} \dots < \frac{713}{168} < \frac{21}{14}.$$

4. In the figure 4 we take $AB = a_1$, $B\Gamma = 2a_1$, $A\Gamma = \delta_1$. The angle $AB\Gamma = 120^\circ$. Then $\delta_1^2 = 7a_1^2$, $\delta_1 : a_1 = \sqrt{7}$, and $7(2a_1 + \delta_1)^2 = 28a_1^2 + 28a_1\delta_1 + 7\delta_1^2$. Because $\delta_1^2 = 7a_1^2$, $7(2a_1 + \delta_1)^2 = 49a_1^2 + 28a_1\delta_1 + 4\delta_1^2 = (7a_1 + 2\delta_1)^2$.

S.N., $a_v = 2a_{v-1} + \delta_{v-1}$, D.N., $\delta_v = 7a_{v-1} + 2\delta_{v-1}$, $\delta_v^2 = 7a_v^2 + (7-4)^v (-1)^v$.

$$\text{When } a_1 = 1, \delta_1 = 2, \quad \frac{2}{1} < \frac{50}{19} < \dots < \sqrt{7} \dots < \frac{233}{88} < \frac{11}{4}.$$

5. In the same figure 4 we take $AB = a_1$, $B\Gamma = 3a_1$, $A\Gamma = \delta_1$. Then $\delta_1^2 = 13a_1^2$, $\delta_1 : a_1 = \sqrt{13}$, $13(2a_1 + \delta_1)^2 = 52a_1^2 + 52a_1\delta_1 + 13\delta_1^2$.

Because $\delta_1^2 = 13a_1^2$ is $13(2a_1 + \delta_1)^2 = 169a_1^2 + 52a_1\delta_1 + 4\delta_1^2 = (13a_1 + 2\delta_1)^2$.
 S.N., $a_v = 2a_{v-1} + \delta_{v-1}$, D.N., $\delta_v = 13a_{v-1} + 2\delta_{v-1}$, $\delta_v^2 = 13a_v^2 + (13-4)^v(-1)^v$.

$$\frac{2}{1} < \frac{86}{25} < \dots \sqrt{13} \dots < \frac{497}{136} < \frac{17}{4}$$

In the same way we take the side—and the diagonal—numbers for the $\sqrt{6}$, $\sqrt{8}$, $\sqrt{11}$, $\sqrt{14}$, $\sqrt{15}$. (We mention Theaetetus of Plato 147 D).

III. If $\lambda \geq 5$, integer no square number and $\delta_1^2 = \lambda a_1^2$, then $(\lambda-4)\delta_1^2 = (\lambda-4)\lambda a_1^2$, and according to the identity (1)

$$\lambda(2a_1 + \delta_1)^2 = 4\lambda a_1^2 + 4\lambda a_1\delta_1 + \lambda\delta_1^2, \\ = 4\lambda a_1^2 + (\lambda-4)\lambda a_1^2 + 4\lambda a_1\delta_1 + 4\delta_1^2 = (\lambda a_1 + 2\delta_1)^2.$$

Side numbers

Diagonal numbers

a_1 $a_2 = 2a_1 + \delta_1$ $a_3 = 2a_2 + \delta_2$ $a_4 = 2a_3 + \delta_3$ $\vdots$ $a_v = 2a_{v-1} + \delta_{v-1}$	δ_1 $\delta_2 = \lambda a_1 + 2\delta_1$ $\delta_3 = \lambda a_2 + 2\delta_2$ $\delta_4 = \lambda a_3 + 2\delta_3$ $\vdots$ $\delta_v = \lambda a_{v-1} + 2\delta_{v-1}$
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$$\frac{\delta_1}{a_1} < \frac{\delta_2}{a_2} < \frac{\delta_3}{a_3} < \dots \sqrt{\lambda} \dots < \frac{\delta_v}{a_v} < \frac{\delta_4}{a_4} < \frac{\delta_2}{a_2}, \text{ and}$$

$\delta_v^2 = \lambda a_v^2 + (\lambda-4)^v(-1)^v$. We take here always $a_1 = 1$, $\delta_1 = 2$.
 $\lambda=2$ and $\lambda=3$ are special cases.

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