

ΕΥΑΓΓΕΛΟΥ ΣΤΑΝΑΤΗ*

STAMATIS E., Συμβολή εις την έρευναν της γεωμετρικής αλγέβρας των Πυθαγορείων
[avec un résumé en angl.] : Platon VIII 1956 144-157. La formule d'Archimède relative à l'approximation arithmétique de π se laisse intercepter par

ΣΥΜΒΟΛΗ ΕΙΣ ΤΗΝ ΕΡΕΥΝΑΝ ΤΗΣ ΓΕΩΜΕΤΡΙΚΗΣ ΑΛΓΕΒΡΑΣ ΤΩΝ ΠΥΘΑΓΟΡΕΙΩΝ(*)

I. ΕΙΣΑΓΩΓΗ

1. Εις τὸν ἴδιον τὸν Πυθαγόραν ἀποδίδεται κατὰ τὴν παράδοσιν ἡ ἀπόδει-
ξις τοῦ περιφύτου ὁμωνόμου θεωρήματος (Εὐκλείδου I, 47), ὡρισμένη ἀκέραιαι
λύσεις τῆς ἐξίσωσης $z^2 = x^2 + y^2$ καὶ ἡ ἀνακάλυψις τῶν ἀσυμμέτρων (*). Εἰς τοὺς
Πυθαγορείους ἐν γένει ἀποδίδεται μεταξύ ἄλλων τὸ II Βιβλίον καὶ τὸ πλεῖστον
τῶν ἀριθμητικῶν τῶν Στοιχείων τοῦ Εὐκλείδου καὶ ἡ εὕρεσις τῶν ἀκεραίων λύ-
σεων τῆς ἐξίσωσης $y^2 = x^2 \mp 1$, (1), αἵτινες χρησιμεύουσι διὰ τὸν ὑπολογισμὸν
τῆς κατὰ προσέγγισιν ἀριθμητικῆς τιμῆς τῆς $\sqrt{2}$. Αἱ ἀκέραιαι λύσεις τῆς ἐξίσω-
σεως (1) ὀνομάζονται, ὡς γνωστόν, πλευρικοὶ καὶ διαμετρικοὶ ἀριθμοί, διότι οἱ
μὲν ἐκ τούτων ἀντιστοιχοῦσιν εἰς τὰς πλευράς, οἱ δὲ εἰς τὰς διαγωνίους τε-
τραγώνων (**).

Ὁ νόμος σχηματισμοῦ τῶν πλευρικῶν καὶ διαμετρικῶν ἀριθμῶν διεσώθη
ὑπὸ τοῦ Θεωνοῦ τοῦ Σμυρναίου καὶ ἔχει ὡς κάτωθι :

	Πλευρικοὶ ἀριθμοί	Διαμετρικοὶ ἀριθμοί
1.	1	1
2.	1 + 1 = 2	2 . 1 + 1 = 3
3.	2 + 3 = 5	2 . 2 + 3 = 7
4.	5 + 7 = 12	2 . 5 + 7 = 17
5.	12 + 17 = 29	2 . 12 + 17 = 41
.	.	.

*) Evangelos Stamatīs, A contribution to the investigation of the geometrical alge-
bra of the Pythagoreans.

1) Ἀνεκρινώθη ἐν τῇ Ἀκαδημίᾳ Ἀθηνῶν κατὰ τὴν συνεδρίαν τῆς 2ας Ἰουνίου 1955 διὰ
τοῦ ἀκαδημαϊκοῦ κ. Μιχαὴλ Στεφανίδου.

2) Μιχαὴλ Στεφανίδου, Εἰσαγωγή εἰς τὴν ἱστορίαν τῶν Φυσικῶν Ἐπιστημῶν, σ. 55-65
καὶ 102-3, Ἀθήναι 1938. — Προκίος, Σχόλια εἰς Εὐκλείδην I, σ. 65 καὶ 138, ἐκδ. Friedlein,
Teubner. — Iamblicus V. P. 216, ἐκδ. L. Deubner, Teubner.

3) Theonīs Smyrnaei, Philosophi Platonici, ἐκδ. E. Hiller, σ. 43, Teubner. — E. Στα-
ματίου, Εὐκλείδου Γεωμετρία - Θεωρία ἀριθμῶν, τόμ. II, σ. 8 (Ὅργαν. Ἐκδ. Σχολικῶν Βιβλίων,
Ἀθήναι 1953). — Paul - Henri Michel, De Pythagore à Euclide, p. 133, Paris 1950 (Soc. d'
éd. Les Belles Lettres). — M. Cantor, Vorlesungen über Geschichte der Mathematik unter
Theon von Smyrna. — T. Heath, A history of Greek mathematics I, p. 91, Oxford 1921,
At the Clarendon Press. — R. Morris Cohen - J. E. Drachin, A source book in Greek science,
p. 13, McGraw - Hill book company inc. New York, Toronto, London, 1913.

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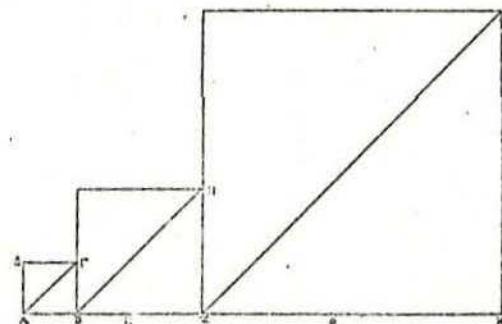
$$\text{Εἶναι δὲ } \frac{1}{1} < \frac{7}{5} < \frac{41}{29} < \dots < \frac{17}{12} < \frac{3}{2},$$

καὶ

$$\begin{aligned} 1^2 &= 2, \quad 1^2 - 1 \\ 3^2 &= 2, \quad 2^2 + 1 \\ 7^2 &= 2, \quad 5^2 - 1 \\ 17^2 &= 2, \quad 12^2 + 1 \\ 41^2 &= 2, \quad 29^2 - 1 \quad \text{κλπ.} \end{aligned}$$

Ἡ γεωμετρικὴ ἀπόδειξις τοῦ νόμου σχηματισμοῦ τῶν πλευρικῶν καὶ διαμε-
τρικῶν ἀριθμῶν μνημονεύεται ὑπὸ τοῦ Πρόκλου (*) καὶ ἔχει ὡς ἑξῆς :

* Ἐστὼ τετραγώνων πλευρᾶς $AB = \alpha$, καὶ διαγωνίου $AG = \delta_1$, (σχ. 1), ὅτε εἶναι



Σχ. 1

$\delta_1^2 = 2\alpha_1^2$. Ἐπὶ τῆς προεκτάσεως τῆς AB λαμβάνομεν τμήμα $BE = AB = \alpha$, καὶ ἐν
συνεχείᾳ τμήμα $EZ = AG = \delta_1$. Κατὰ τὸν Εὐκλείδην II, 10 θὰ εἶναι :

$$(2\alpha_1 + \delta_1)^2 + \delta_1^2 = 2\alpha_1^2 + 2(\alpha_1 + \delta_1)^2.$$

Καὶ ἐπειδὴ $\delta_1^2 = 2\alpha_1^2$, θὰ ἔχωμεν δι' ἀφαιρέσεως τούτου κατὰ μέλη ἐκ τῆς
προηγούμενης ἐξίσωσεως, $(2\alpha_1 + \delta_1)^2 = 2(\alpha_1 + \delta_1)^2$. Ἡ σχέσηις δὲ αὕτη σημαίνει ὅτι
ἢ μὲν $(\alpha_1 + \delta_1) = BE + EZ$ εἶναι πλευρά, ἢ δὲ $(2\alpha_1 + \delta_1) = AB + BE + EZ$ εἶναι διαγώνιος
τετραγώνου, ἢ BH . Ἐὰν ἐπὶ τῆς προεκτάσεως τῆς BZ λάβωμεν τμήμα $Z\Theta = BZ$ καὶ
ἐν συνεχείᾳ τμήμα $\Theta K = BH$ καὶ ἐφαρμόσωμεν τὸ εὐκλείδειον θεώρημα II, 10 θὰ
λάβωμεν νέον τετράγωνον, τοῦ ὁποῦ ἢ μὲν πλευρὰ θὰ εἶναι ἡ $Z\Theta + \Theta K = 3\alpha_1 + 2\delta_1$,
ἢ δὲ διαγώνιος ἡ $Z\Lambda = BK = 2BZ + \Theta K = 4\alpha_1 + 3\delta_1$.

Ὅθεν λαμβάνομεν τὸ ἑξῆς σχῆμα :

Πλευρικοὶ ἀριθμοί	Διαμετρικοὶ ἀριθμοί
α_1	δ_1
$\alpha_2 = \alpha_1 + \delta_1$	$\delta_2 = 2\alpha_1 + \delta_1$
$\alpha_3 = \alpha_2 + \delta_2$	$\delta_3 = 2\alpha_2 + \delta_2$
$\alpha_4 = \alpha_3 + \delta_3$	$\delta_4 = 2\alpha_3 + \delta_3$
$\vdots$	$\vdots$
$\alpha_n = \alpha_{n-1} + \delta_{n-1}$	$\delta_n = 2\alpha_{n-1} + \delta_{n-1}$

Ἐὰν θέσωμεν $\alpha_1 = 1$ καὶ $\delta_1 = 1$, λαμβάνομεν τοὺς κατὰ τὸν Θεώνα τὸν Σμυρ-

1) Σχόλια εἰς Πολύβιον Πλάτωνος, τόμ. II, σ. 21 κ. ε., 303 κ. ε. ὑπὸ F. Hultsch, ἐκδ.
Kroll, Teubner.

ναίον πλευρικούς και διαμετρικούς αριθμούς, ήτοι τὰς ἀκεραίας λύσεις τῆς ἐξίσωσης $y^2 = 2x^2 + 1$, ἢ $\delta v^2 = 2\alpha^2 + (-1)^n$, ($n=1, 2, 3, \dots$).

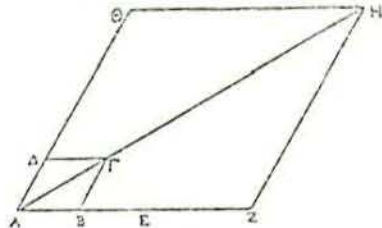
Ἐκ τῆς Πολιτείας τοῦ Πλάτωνος πληροφοροῦμεθα, ὅτι οἱ πλευρικοὶ καὶ διαμετρικοὶ ἀριθμοὶ ἦσαν γνωστοὶ εἰς αὐτόν. Ἐκεῖ ἀναγιγνώσκουμεν «ἐκείνων μὲν ἀριθμῶν ἀπὸ διαμέτρων ἡγίων περιόδου, διαιρέσεων ἐνὸς ἐκάστων, ἀρρήτων δὲ δυοῖν» (545 α). Ἐνταῦθα δὲ Πλάτων ὑπαινίσσεται τοὺς πλευρικοὺς καὶ διαμετρικοὺς ἀριθμοὺς καὶ δὴ καὶ μίαν ἀκεραίαν λύσιν τῆς ἀνωτέρω ἐξισώσεως, τὴν $7^2 = 2 \cdot 5^2 + 1$. Τοῦτο συνάγεται ἐκ τοῦ Πρόκλου, ὅστις γράφει «ὅπου δὲ τὸ σύνγεγνε ἀρᾶσθαι, οἷον εἰρόντες ἐν γεωμετρίᾳ τετραγώνου διπλασίον, ἐν ἀριθμοῖς δὲ οὐκ ἔχοντες ἐνὸς δέοντος φερῖν ἄλλον ἄλλον διπλασίον ἐπὶ ἑαυτῶν, ὥστε τοῦ ἀπὸ τῆς περιόδου δὲ ἀπὸ τῆς ἐπὶ τῆς διπλασίον ἐνὸς δέοντος». Καὶ ἀλλαχοῦ «οὗ γὰρ ἐστὶ τετραγώνος ἀριθμὸς τετραγώνου διπλασίονος εἰ μὴ λίγαι εἰς τὸν σύνγεγνε. δὲ γὰρ ἀπὸ τοῦ ζ' τοῦ ἀπὸ τοῦ ε' διπλασίονος ἐστὶν ἐνὸς δέοντος» (1). (Εἶναι δηλ. $7^2 = 2 \cdot 5^2 + 1$).

2. Ὁ Ἀρχιμήδης εἰς τὴν πραγματείαν αὐτοῦ «Κύκλου Μέτρησις» χρησιμοποιοῖ ἀνευ ἀποδείξεως τὰς σχέσεις

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}, \text{ καὶ } 265^2 = 3 \cdot 153^2 - 2, 1351^2 = 3 \cdot 780^2 + 1$$

Γεωμετρικὴν ἀπόδειξιν τῶν σχέσεων τούτων ὑπεβάλομεν εἰς τὴν Ἀκαδημίαν Ἀθηνῶν (1).

Αὕτη συνδέεται πρὸς τοὺς πλευρικοὺς καὶ διαμετρικοὺς ἀριθμοὺς. Θεωροῦμεν ἰσοσκελεῖς ἀμβλυγώνιον τρίγωνον,



Σχ. 2

τὸ ΑΒΓ (σχ. 2), τοῦ ὁποῦ ἡ μεγαλύτερα γωνία, ἢ ΑΒΓ, νὰ εἶναι ἴση πρὸς τὴν ἐξωτερικὴν γωνίαν ἰσοπλευροῦ τριγώνου. Κατὰ τὸν Εὐκλείδην II, 12, ἐὰν καλέσωμεν τὴν πλευρὰν ΑΒ=α, καὶ τὴν ΑΓ=δ, ἡττις βεβαίως εἶναι ἡ μεγαλύτερα διαγώνιος τοῦ ῥόμβου ΑΒΓΔ, θὰ εἶναι $\delta_1^2 = 3\alpha_1^2$, καὶ συνεπῶς $\delta_1 : \alpha_1 = \sqrt{3}$. Ἐφαρμόζομεν τώρα ἀκριβῶς τὴν ὑπὸ τοῦ Πρόκλου ὑποδεικνυμένην μέθοδον διὰ τὴν ἀπόδειξιν τῶν ἐκ τετραγώνων σχημάτων

προκύπτόντων πλευρικῶν καὶ διαμετρικῶν ἀριθμῶν. Ἐπὶ τῆς προεκτάσεως τῆς ΑΒ λαμβάνομεν τμήμα ΒΕ=ΑΒ=α, καὶ ἐν συνεχείᾳ τμήμα ΕΖ=ΑΓ=δ. Κατὰ τὸν Εὐκλείδην II, 10 θὰ ἔχωμεν: $(2\alpha_1 + \delta_1)^2 + \delta_1^2 = 2\alpha_1^2 + 2(\alpha_1 + \delta_1)^2$, καὶ ἐκ ταύτης: $(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2$ (1).

Εἶναι ἄρα καὶ $3(2\alpha_1 + \delta_1)^2 = 12\alpha_1^2 + 12\alpha_1\delta_1 + 3\delta_1^2$. Ἀλλὰ $\delta_1^2 = 3\alpha_1^2$. Ἐπομένως $3(2\alpha_1 + \delta_1)^2 = 9\alpha_1^2 + 12\alpha_1\delta_1 + 4\delta_1^2 = (3\alpha_1 + 2\delta_1)^2$.

Ἡ σχέση αὕτη σημαίνει ὅτι ἡ μὲν $(2\alpha_1 + \delta_1)$ εἶναι πλευρά, ἡ δὲ $(3\alpha_1 + 2\delta_1)$ εἶναι ἡ μεγαλύτερα διαγώνιος ὁμοίου ῥόμβου πρὸς τὸν ΑΒΓΔ τοῦ ΑΖΗΘ. Ἐάν ἐπὶ τῆς προεκτάσεως τῆς ΑΖ λάβωμεν τμήμα ἴσον πρὸς ΑΖ καὶ ἐν συνεχείᾳ τμήμα ἴσον πρὸς ΑΗ, τότε ἔχομεν κατὰ τὸν αὐτὸν νόμον τὴν πλευρὰν καὶ τὴν μεγαλύτεραν διαγώνιον νέου ὁμοίου ῥόμβου πρὸς τὸν ἀρχικόν, ἡτοι πλευρὰ μὲν εἶναι ἡ $2\alpha_2 + \delta_2$, διαγώνιος δὲ μεγαλύτερα ἡ $3\alpha_2 + 2\delta_2$ ἢ $7\alpha_1 + 4\delta_1$ καὶ $12\alpha_1 + 7\delta_1$ ἀντιστοίχως. Καλοῦντες τὰς τιμὰς τῶν πλευρῶν πλευρικοὺς ἀριθμοὺς

1) Πρόκλος εἰς Εὐκλείδην I, σ. 61 καὶ 427, 122. G. Friedlein, Teubner.

2) Βλ. Πρακτικὰ Ἀκαδημίας Ἀθηνῶν, 2.6.1935, σ. 255 κ. ἐξ. καὶ προηγούμενον τεύχος «Ἠλλάτων».

ἀριθμοὺς καὶ τὰς τιμὰς τῶν μεγαλύτερων διαγώνων, τῶν συνεχῶν κατὰ τὸν ἀνωτέρω νόμον κατασκευαζομένων ῥόμβων, διαμετρικοὺς ἀριθμοὺς, θὰ ἔχωμεν:

Πλευρικοὶ ἀριθμοὶ	Διαμετρικοὶ ἀριθμοὶ
α_1	δ_1
$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = 3\alpha_1 + 2\delta_1$
$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = 3\alpha_2 + 2\delta_2$
$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = 3\alpha_3 + 2\delta_3$
$\alpha_5 = 2\alpha_4 + \delta_4$	$\delta_5 = 3\alpha_4 + 2\delta_4$
$\vdots$	$\vdots$
$\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$	$\delta_n = 3\alpha_{n-1} + 2\delta_{n-1}$

Ἐάν θέσωμεν $\alpha=1$, $\delta=1$, λαμβάνομεν:

(Α) Πλευρικοὶ ἀριθμοὶ	Διαμετρικοὶ ἀριθμοὶ
$\alpha_1 = 1$	$\delta_1 = 1$
$\alpha_2 = 3$	$\delta_2 = 5$
$\alpha_3 = 11$	$\delta_3 = 19$
$\alpha_4 = 41$	$\delta_4 = 71$
$\alpha_5 = 153$	$\delta_5 = 265$
$\vdots$	$\vdots$

Οἱ ἀριθμοὶ οὗτοι παρέχουσι τὰς ἀκεραίας λύσεις τῆς ἐξισώσεως $y^2 = 3x^2 - 2$, ἡτοι εἶναι: $1^2 = 3 \cdot 1^2 - 2$, $5^2 = 3 \cdot 3^2 - 2$, $19^2 = 3 \cdot 11^2 - 2$, κλπ.

Ἐάν θέσωμεν $\alpha_1=1$, $\delta_1=2$, λαμβάνομεν:

(Β) Πλευρικοὶ ἀριθμοὶ	Διαμετρικοὶ ἀριθμοὶ
$\alpha_1 = 1$	$\delta_1 = 2$
$\alpha_2 = 4$	$\delta_2 = 7$
$\alpha_3 = 15$	$\delta_3 = 26$
$\alpha_4 = 56$	$\delta_4 = 97$
$\alpha_5 = 209$	$\delta_5 = 362$
$\alpha_6 = 780$	$\delta_6 = 1351$
$\vdots$	$\vdots$

Οἱ ἀριθμοὶ οὗτοι παρέχουσι τὰς ἀκεραίας λύσεις τῆς ἐξισώσεως $y^2 = 3x^2 + 1$, ἡτοι εἶναι: $2^2 = 3 \cdot 1^2 + 1$, $7^2 = 3 \cdot 4^2 + 1$, $26^2 = 3 \cdot 15^2 + 1$, κλπ.

Οἱ λόγοι $\frac{\delta_n}{\alpha_n}$ τῶν (Α) καὶ (Β) ἀποτελοῦσι δύο ἀκολουθίας, ἐκ τῶν ὁποίων ἡ μὲν τῶν (Α) εἶναι ἀόξανομένη, ἡ δὲ τῶν (Β) φθίνουσα. Τὸ κοινὸν φράγμα τούτων εἶναι ἡ $\sqrt{3}$, ἡτοι εἶναι

$$1) \frac{1}{1} < \frac{5}{3} < \frac{19}{11} < \frac{41}{21} < \frac{265}{153} < \dots < \sqrt{3} < \dots < \frac{1351}{780} < \frac{362}{209} < \frac{97}{56} < \frac{26}{15} < \frac{7}{4} < \frac{2}{1},$$

καὶ 2) $265^2 = 3 \cdot 153^2 - 2$, $1351^2 = 3 \cdot 780^2 + 1$, ὡς χρησιμοποιεῖ ταῦτα ἀνευ ἀποδείξεως, ὡς γνωστὰ, ὁ Ἀρχιμήδης.

II.

Ἐκ τῶν ἀνωτέρω ἐκτεθέντων συνάγομεν τὸ συμπέρασμα ὅτι οἱ Πυθαγόρειοι ἐγνώριζον καὶ τὰς ἀκεραίας λύσεις τῆς ἐξισώσεως

$$\delta v^2 = \lambda \alpha^2 + (\lambda - 4)^n (-1)^n$$

($n=1, 2, 3, \dots$ καὶ $\lambda \geq 5$, ἀκεραῖος μὴ τετράγωνος), καὶ ὅτι ἡ $\sqrt{\lambda}$ εἶναι τὸ κοινὸν

οράγμα δύο ακολουθιών, μιας αύξανουσας και μιας φθινούσας, διότι η απόδειξις τούτων είναι ακριβώς η αὐτή πρὸς τὰς ἀνωτέρω ἐκτεθείσας.

Παρέχοντες τὴν ἀπόδειξιν διὰ τὰς ἐξισώσεις:

$$\begin{aligned} \delta_1 &= 5\alpha_1 + (5-4)^n (-1)^n \\ \delta_2 &= 6\alpha_1 + (5-4)^n (-1)^n \\ \delta_3 &= 7\alpha_1 + (7-4)^n (-1)^n \\ \delta_4 &= 8\alpha_1 + (8-4)^n (-1)^n \\ &\vdots \\ \delta_n &= 17\alpha_1 + (17-4)^n (-1)^n \end{aligned}$$

καὶ τὴν $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, ..., $\sqrt{17}$. Εἶναι δὲ γνωστὸν ἐκ τοῦ Θεωρήματος τοῦ Πλάτωνος, ὅτι ὁ Θεόδωρος (1) (ὁ Κυρηναῖος, ὅστις θεωρεῖται Πυθαγόρειος) ἀπέδειξε τὸ ἀσύμμετρον τῆς $\sqrt{3}$, $\sqrt{5}$, ..., $\sqrt{17}$ (Θεώρητος 147 D—148 B).

II. 1. $\delta_n = 5\alpha_n + (5-4)^n (-1)^n$ καὶ $\sqrt{5}$. Θεωροῦμεν ὀρθογώνιον παραλληλόγραμμον, τὸ ABΓΔ (σχ. 3), ἐνθα ἔστω AB=α, ΒΓ=2α, καὶ ἡ διαγώνιος ΑΓ=δ. Εἶναι ἄρα

$$\delta^2 = 5\alpha^2 \quad \text{καὶ} \quad \frac{\delta}{\alpha} = \sqrt{5}.$$

Ἐπὶ τῆς προεκτάσεως τῆς AB λαμβάνομεν τμήμα BE=AB=α, καὶ ἐν συνεχείᾳ τμήμα EZ=ΑΓ=δ. Κατὰ τὸν Εὐκλείδην II, 10, θά ἔχωμεν:

$$(2\alpha + \delta)^2 + \delta^2 = 2\alpha^2 + 2(2\alpha + \delta)\delta$$

καὶ ἐκ ταύτης $(2\alpha + \delta)^2 = 4\alpha^2 + 4\alpha\delta + \delta^2$

Εἶναι ἄρα καὶ

$$5(2\alpha + \delta)^2 = 20\alpha^2 + 20\alpha\delta + 5\delta^2.$$

Ἀλλὰ $\delta^2 = 5\alpha^2$. Ἐπομένως

$$5(2\alpha + \delta)^2 = 20\alpha^2 + 5\alpha^2 + 20\alpha\delta + 4\delta^2 = (5\alpha + 2\delta)^2.$$

Ἡ σχέσηις ὅμως αὕτη σημαίνει, ὅτι ἡ μὲν $(2\alpha + \delta) = AZ$ εἶναι πλευρά, ἡ δὲ $(5\alpha + 2\delta) = AH$ εἶναι διαγώνιος ὁμοίου πρὸς τὸ ἀρχικὸν ὀρθογώνιον παραλληλογράμμου τοῦ AZHΘ. Κατὰ τὸν προφανῆ νόμον τῆς κατασκευῆς ἐν συνεχείᾳ ὁμοίων ὀρθογωνίων παραλληλογράμμων θά ἔχωμεν:

Πλευρικοὶ ἀριθμοὶ	Διαμετρικοὶ ἀριθμοὶ
α_1	δ_1
$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = 5\alpha_1 + 2\delta_1$
$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = 5\alpha_2 + 2\delta_2$
$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = 5\alpha_3 + 2\delta_3$
$\vdots$	$\vdots$
$\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$	$\delta_n = 5\alpha_{n-1} + 2\delta_{n-1}$

1) *Panitz - Wissen, Realenzyklopädie unter Theodoros Dort Literaturangabe: M. Cantor, E. Frank, F. Hultsch, G. Junge, H. Vogt, H. G. Zeuthen, Eva Sachs, T. Bonnesen, H. Hasse-H. Scholz, T. Heath.* — 2. Und W. L. van der Waerden, *Die Arithmetik der Pythagoreer II. Die Theorie des Irrationalen*, Mathem. Annalen, 120, 5/6 Heft, 1949, Springer Verlag, Berlin, Göttingen, Heidelberg — 3. K. Reidemeister, *Die Arithmetik der Griechen*, Leipzig, 1910. — 4. J. E. Hofmann, *Geschichte der Mathematik I*, S. 26-27, Berlin, 1953 (Sammlung Göschen, 226, Walter de Gruyter und Co.). — 5. Robert S. Brown, *Plato's Mathematical Imagination*, p. 110, Indiana Univ. Press, Bloomington, 1954.

Ἐάν θέσωμεν $\alpha_1 = 1$, καὶ $\delta_1 = 2$ λαμβάνομεν

$$\begin{aligned} \alpha_2 &= 4, & \delta_2 &= 9 \\ \alpha_3 &= 17, & \delta_3 &= 38 \\ \alpha_4 &= 72, & \delta_4 &= 161 \\ &\vdots & &\vdots \end{aligned}$$

Εἶναι δὲ $\frac{2}{1} < \frac{38}{17} < \dots < \frac{161}{72} < \frac{9}{4}$, καὶ

$$\begin{aligned} 2^2 &= 5 \cdot 1^2 - 1 \\ 9^2 &= 5 \cdot 4^2 + 1 \\ 38^2 &= 5 \cdot 17^2 - 1 \\ &\vdots \\ \delta_n^2 &= 5 \cdot \alpha_n^2 + (5-4)^n (-1)^n. \end{aligned}$$

II. 2. $\delta_n = 10\alpha_n + (10-4)^n (-1)^n$, καὶ $\sqrt{10}$.

Εἰς τὸ προηγούμενον σχῆμα 3 λαμβάνομεν AB=α, ΒΓ=3α, ὁπότε

$$\delta^2 = 10\alpha^2 \quad \text{καὶ} \quad \frac{\delta}{\alpha} = \sqrt{10}.$$

Ἐφαρμόζοντες τὴν προηγουμένην κατασκευὴν (II. 1) λαμβάνομεν

$$(2\alpha + \delta)^2 = 4\alpha^2 + 4\alpha\delta + \delta^2$$

Εἶναι ἄρα καὶ $10(2\alpha + \delta)^2 = 40\alpha^2 + 40\alpha\delta + 10\delta^2$

Ἀλλὰ $\delta^2 = 10\alpha^2$. Ἐπομένως

$$10(2\alpha + \delta)^2 = 40\alpha^2 + 60\alpha^2 + 40\alpha\delta + 4\delta^2 = (10\alpha + 2\delta)^2.$$

Ἡ σχέσηις ὅμως αὕτη σημαίνει, ὅτι ἡ μὲν $(2\alpha + \delta)$ εἶναι πλευρά, ἡ δὲ $(10\alpha + 2\delta)$ διαγώνιος ὁμοίου ὀρθογωνίου παραλληλογράμμου πρὸς τὸ ἀρχικόν. Ὁ νόμος τῆς κατασκευῆς τῶν ὁμοίων ἐν συνεχείᾳ παραλληλογράμμων εἶναι προφανής.

Ὅθεν θά εἶναι

Πλευρικοὶ ἀριθμοὶ Διαμετρικοὶ ἀριθμοὶ

α_1	δ_1
$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = 10\alpha_1 + 2\delta_1$
$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = 10\alpha_2 + 2\delta_2$
$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = 10\alpha_3 + 2\delta_3$
$\vdots$	$\vdots$
$\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$	$\delta_n = 10\alpha_{n-1} + 2\delta_{n-1}$

Ἐάν θέσωμεν $\alpha_1 = 1$ καὶ $\delta_1 = 2$ λαμβάνομεν

$$\begin{aligned} \alpha_2 &= 4, & \delta_2 &= 14 \\ \alpha_3 &= 22, & \delta_3 &= 68 \\ \alpha_4 &= 112, & \delta_4 &= 356 \\ &\vdots & &\vdots \end{aligned}$$

Εἶναι δὲ $\frac{2}{1} < \frac{68}{22} < \dots < \frac{356}{112} < \frac{14}{4}$

καὶ

$$\begin{aligned} 2^2 &= 10 \cdot 1^2 - 6 \\ 14^2 &= 10 \cdot 4^2 + 6 \\ 68^2 &= 10 \cdot 22^2 - 6 \\ 356^2 &= 10 \cdot 112^2 + 6 \\ &\vdots \\ \delta_n^2 &= 10 \cdot \alpha_n^2 + (10-4)^n (-1)^n \end{aligned}$$

II. 3. $\delta_n = 17\alpha_n + (17-4)^n (-1)^n$, καὶ $\sqrt{17}$.

Εἰς τὸ αὐτὸ σχῆμα 3 λαμβάνομεν AB=α, ΒΓ=4α, ὁπότε εἶναι

$\delta_1 = 17\alpha_1$ και $\frac{\delta_1}{\alpha_1} = \sqrt{17}$. Εφαρμόζομεν πάλιν τὴν κατασκευὴν (II. 1) ὁπότε ἔχομεν

$$(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2.$$

Εἶναι ἄρα καὶ $17(2\alpha_1 + \delta_1)^2 = 68\alpha_1^2 + 68\alpha_1\delta_1 + 17\delta_1^2$. Ἀλλὰ $\delta_1 = 17\alpha_1$.

Ἐπομένως $17(2\alpha_1 + \delta_1)^2 = 68\alpha_1^2 + 221\alpha_1\delta_1 + 4\delta_1^2 = (17\alpha_1 + 2\delta_1)^2$.

Ἡ σχέσηις ὁμῶς αὕτη σημαίνει, ὅτι ἡ μὲν $(2\alpha_1 + \delta_1)$ εἶναι πλευρά, ἡ δὲ $(17\alpha_1 + 2\delta_1)$ διαγώνιος ὁμοίου ὀρθογωνίου παραλληλογράμμου πρὸς τὸ ἀρχικόν. Ὁ νόμος σχηματισμοῦ τῶν ὁμοίων ὀρθογ. παραλληλογράμμων εἶναι προφανής.

Ὅθεν θὰ ἔχωμεν

Πλευρικοὶ ἀριθμοὶ

Διαμετρικοὶ ἀριθμοὶ

$$\begin{aligned} \alpha_1 &= 2\alpha_1 + \delta_1 \\ \alpha_2 &= 2\alpha_2 + \delta_2 \\ \alpha_3 &= 2\alpha_3 + \delta_3 \\ \alpha_4 &= 2\alpha_4 + \delta_4 \\ &\vdots \\ \alpha_n &= 2\alpha_{n-1} + \delta_{n-1} \end{aligned}$$

$$\begin{aligned} \delta_1 &= 17\alpha_1 + 2\delta_1 \\ \delta_2 &= 17\alpha_2 + 2\delta_2 \\ \delta_3 &= 17\alpha_3 + 2\delta_3 \\ &\vdots \\ \delta_n &= 17\alpha_{n-1} + 2\delta_{n-1} \end{aligned}$$

Ἐάν θέσωμεν

$$\begin{aligned} \alpha_1 &= 1 \\ \alpha_2 &= 4 \\ \alpha_3 &= 29 \\ \alpha_4 &= 168 \end{aligned}$$

καὶ

$$\begin{aligned} \delta_1 &= 2 \\ \delta_2 &= 21 \\ \delta_3 &= 110 \\ \delta_4 &= 713 \end{aligned}$$

λαμβάνομεν

$$\text{Εἶναι δὲ } \frac{2}{1} < \frac{110}{29} < \sqrt{17} < \frac{713}{168} < \frac{21}{4}, \text{ καὶ}$$

$$\begin{aligned} 2^2 &= 17 \cdot 1^2 - 13 \\ 21^2 &= 17 \cdot 4^2 + 13 \\ 110^2 &= 17 \cdot 29^2 - 13 \\ 713^2 &= 17 \cdot 168^2 + 13 \\ &\vdots \\ \delta_n^2 &= 17 \cdot \alpha_n^2 + (17-4)^n (-1)^n \end{aligned}$$

II. 4. Θεωροῦμεν τὸ ῥομβοειδὲς παραλληλόγραμμον ΑΒΓΔ (σχ. 4) ἐνθα γωνία ΑΒΓ=120°, ΑΒ=α₁, ΒΓ=2α₁ καὶ ἡ μεγαλύτερα διαγώνιος ΑΓ=δ₁.

Κατὰ τὸν Εὐκλείδην II, 12 θὰ εἶναι $\delta_1^2 = 7\alpha_1^2$ ὁπότε $\frac{\delta_1}{\alpha_1} = \sqrt{7}$.

Πάλιν ἐφαρμόζομεν τὴν αὐτὴν κατασκευὴν (II. 1), ὁπότε λαμβάνοντες ΒΕ=α₁, ΕΖ=δ₁,

θὰ ἔχωμεν κατὰ τὸν Εὐκλείδην II, 10:

$$(2\alpha_1 + \delta_1)^2 + \delta_1^2 = 2\alpha_1^2 + 2(2\alpha_1 + \delta_1)\delta_1.$$

Ἐξ ἧς $(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2$.

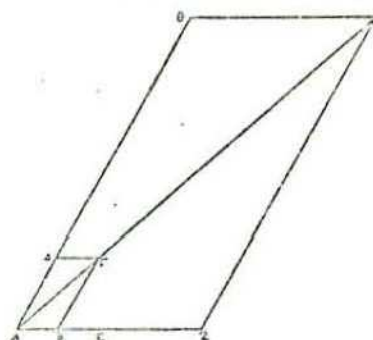
Εἶναι ἄρα καὶ

$$7(2\alpha_1 + \delta_1)^2 = 28\alpha_1^2 + 28\alpha_1\delta_1 + 7\delta_1^2.$$

Ἀλλὰ $\delta_1^2 = 7\alpha_1^2$. Ἐπομένως

$$7(2\alpha_1 + \delta_1)^2 = 28\alpha_1^2 + 21\alpha_1\delta_1 + 20\alpha_1\delta_1 + 4\delta_1^2 = (7\alpha_1 + 2\delta_1)^2.$$

Ἡ σχέσηις ὁμῶς αὕτη σημαίνει, ὅτι ἡ μὲν $(2\alpha_1 + \delta_1)$ εἶναι πλευρά, ἡ δὲ $(7\alpha_1 + 2\delta_1)$ διαγώνιος μεγαλύτερα ὁμοίου πρὸς τὸ ἀρχικόν ῥομβοειδοῦς παραλληλογράμμου. Κατὰ τὸν προφανή νόμον κατασκευῆς ὁμοίων ῥομβοειδῶν παραλληλογράμμων θὰ ἔχωμεν:



Σχ. 4

Πλευρικοὶ ἀριθμοὶ

Διαμετρικοὶ ἀριθμοὶ

$$\begin{aligned} \alpha_1 &= 2\alpha_1 + \delta_1 \\ \alpha_2 &= 2\alpha_2 + \delta_2 \\ \alpha_3 &= 2\alpha_3 + \delta_3 \\ &\vdots \\ \alpha_n &= 2\alpha_{n-1} + \delta_{n-1} \end{aligned}$$

$$\begin{aligned} \delta_1 &= 7\alpha_1 + 2\delta_1 \\ \delta_2 &= 7\alpha_2 + 2\delta_2 \\ \delta_3 &= 7\alpha_3 + 2\delta_3 \\ &\vdots \\ \delta_n &= 7\alpha_{n-1} + 2\delta_{n-1} \end{aligned}$$

Ἐάν θέσωμεν

$$\begin{aligned} \alpha_1 &= 1 \\ \alpha_2 &= 4 \\ \alpha_3 &= 19 \\ \alpha_4 &= 88 \end{aligned}$$

καὶ

$$\begin{aligned} \delta_1 &= 2, \text{ λαμβάνομεν} \\ \delta_2 &= 11 \\ \delta_3 &= 50 \\ \delta_4 &= 233 \end{aligned}$$

Εἶναι δὲ

$$\frac{2}{1} < \frac{50}{19} < \sqrt{7} < \frac{233}{88} < \frac{11}{4},$$

καὶ

$$\begin{aligned} 2^2 &= 7 \cdot 1^2 - 3 \\ 11^2 &= 7 \cdot 4^2 + 3 \\ 50^2 &= 7 \cdot 19^2 - 3 \\ 233^2 &= 7 \cdot 88^2 + 3 \\ &\vdots \\ \delta_n^2 &= 7 \cdot \alpha_n^2 + (7-4)^n (-1)^n. \end{aligned}$$

II. 5. Εἰς τὸ προηγούμενον σχῆμα 4 λαμβάνομεν ΑΒ=α₁, ΒΓ=3α₁, ΑΓ=δ₁.

Κατὰ τὸν Εὐκλείδην II, 12 εἶναι $\delta_1^2 = 13\alpha_1^2$ καὶ $\frac{\delta_1}{\alpha_1} = \sqrt{13}$. Πάλιν ἐφαρμόζομεν τὴν αὐτὴν κατασκευὴν ὡς καὶ προηγουμένως, ἥτοι λαμβάνομεν ΒΕ=α₁, ΕΖ=δ₁, ὁπότε κατὰ τὸ II, 10 τοῦ Εὐκλείδου θὰ εἶναι

$$(2\alpha_1 + \delta_1)^2 + \delta_1^2 = 2\alpha_1^2 + 2(2\alpha_1 + \delta_1)\delta_1, \text{ ἔξ ἧς}$$

$$(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2.$$

Εἶναι ἄρα καὶ $13(2\alpha_1 + \delta_1)^2 = 52\alpha_1^2 + 52\alpha_1\delta_1 + 13\delta_1^2$.

Ἀλλὰ $\delta_1^2 = 13\alpha_1^2$. Ἐπομένως

$$13(2\alpha_1 + \delta_1)^2 = 52\alpha_1^2 + 117\alpha_1\delta_1 + 52\alpha_1\delta_1 + 4\delta_1^2 = (13\alpha_1 + 2\delta_1)^2.$$

Ἡ σχέσηις ὁμῶς αὕτη σημαίνει, ὅτι ἡ μὲν $(2\alpha_1 + \delta_1)$ εἶναι πλευρά, ἡ δὲ $(13\alpha_1 + 2\delta_1)$ διαγώνιος μεγαλύτερα ὁμοίου ῥομβοειδοῦς παραλληλογράμμου πρὸς τὸ ἀρχικόν. Κατὰ τὸν προφανή νόμον κατασκευῆς τῶν ὁμοίων ῥομβοειδῶν παραλληλογράμμων θὰ ἔχωμεν

Πλευρικοὶ ἀριθμοὶ

Διαμετρικοὶ ἀριθμοὶ

$$\begin{aligned} \alpha_1 &= 2\alpha_1 + \delta_1 \\ \alpha_2 &= 2\alpha_2 + \delta_2 \\ \alpha_3 &= 2\alpha_3 + \delta_3 \\ &\vdots \\ \alpha_n &= 2\alpha_{n-1} + \delta_{n-1} \end{aligned}$$

$$\begin{aligned} \delta_1 &= 13\alpha_1 + 2\delta_1 \\ \delta_2 &= 13\alpha_2 + 2\delta_2 \\ \delta_3 &= 13\alpha_3 + 2\delta_3 \\ &\vdots \\ \delta_n &= 13\alpha_{n-1} + 2\delta_{n-1} \end{aligned}$$

Ἐάν θέσωμεν

$$\begin{aligned} \alpha_1 &= 1 \\ \alpha_2 &= 4 \\ \alpha_3 &= 25 \\ \alpha_4 &= 136 \end{aligned}$$

καὶ

$$\begin{aligned} \delta_1 &= 2, \text{ λαμβάνομεν} \\ \delta_2 &= 17 \\ \delta_3 &= 85 \\ \delta_4 &= 497 \end{aligned}$$

$$\text{Εἶναι δὲ } \frac{2}{1} < \frac{85}{25} < \sqrt{13} < \frac{497}{136} < \frac{17}{4}, \text{ καὶ}$$

$$\begin{aligned} 2^2 &= 13 \cdot 1^2 - 9 \\ 17^2 &= 13 \cdot 4^2 + 9^2 \\ 85^2 &= 13 \cdot 25^2 - 9^2 \\ 497^2 &= 13 \cdot 136^2 + 9^2 \end{aligned}$$

$$b_v^2 = 13a_v^2 + (13-4)v(-1)^v.$$

Εκ τῶν ἀνωτέρω ἐκτεθειμένων κατασκευῶν καὶ ἀποδείξεων καθίσταται αὐτονόητος ὁ σχηματισμὸς ἀντιστοιχῶν πλυστικών καὶ διαμετρικῶν ἀριθμῶν διὰ διὰ γινόμενου καθαρῶς ὑπολογισμοῦ καὶ οὐχὶ γεωμετρικοῦ, διὰ τὴν $\sqrt{5}$, $\sqrt{8}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{14}$, $\sqrt{15}$, ... καὶ τὰς συναφεῖς ἐξισώσεις.

Οὕτω θὰ εἶναι

$$\text{II. 6. Διὰ } b_v^2 = 6a_v^2 + (6-4)v(-1)^v, \text{ καὶ } \sqrt{6}.$$

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2a_1 + b_1 \\ a_3 &= 2a_2 + b_2 \\ a_4 &= 2a_3 + b_3 \\ &\vdots \\ a_v &= 2a_{v-1} + b_{v-1} \end{aligned}$$

$$b_1 = 6a_1 + 25, \quad b_2 = 6a_2 + 25, \quad b_3 = 6a_3 + 25, \quad b_v = 6a_{v-1} + 25_{v-1}$$

Ἐὰν θέλωμεν $a_1 = 1$ καὶ λαμβάνομεν

$$\begin{aligned} a_2 &= 4 \\ a_3 &= 18 \\ a_4 &= 80 \end{aligned}$$

$$\text{Εἶναι δὲ } \frac{2}{1} < \frac{44}{18} < \dots \sqrt{6} \dots < \frac{196}{80} < \frac{10}{4}, \text{ καὶ}$$

$$\begin{aligned} 2^2 &= 6 \cdot 1^2 - 2 \\ 10^2 &= 6 \cdot 4^2 + 2^2 \\ 44^2 &= 6 \cdot 18^2 - 2^2 \\ 196^2 &= 6 \cdot 80^2 + 2^2 \end{aligned}$$

$$b_v^2 = 6a_v^2 + (6-4)v(-1)^v$$

$$\text{II. 7. Διὰ } b_v^2 = 8a_v^2 + (8-4)v(-1)^v \text{ καὶ } \sqrt{8}.$$

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2a_1 + b_1 \\ a_3 &= 2a_2 + b_2 \\ a_4 &= 2a_3 + b_3 \\ &\vdots \\ a_v &= 2a_{v-1} + b_{v-1} \end{aligned}$$

$$b_1 = 8a_1 + 25, \quad b_2 = 8a_2 + 25, \quad b_3 = 8a_3 + 25, \quad b_v = 8a_{v-1} + 25_{v-1}$$

Ἐὰν θέλωμεν $a_1 = 1$ καὶ λαμβάνομεν

$$\begin{aligned} a_2 &= 4 \\ a_3 &= 20 \\ a_4 &= 96 \end{aligned}$$

$$\text{Εἶναι δὲ } \frac{2}{1} < \frac{55}{20} < \dots \sqrt{8} \dots < \frac{272}{96} < \frac{12}{4}, \text{ καὶ}$$

$$\begin{aligned} 2^2 &= 8 \cdot 1^2 - 4 \\ 12^2 &= 8 \cdot 4^2 + 4^2 \\ 56^2 &= 8 \cdot 20^2 - 4^2 \\ 272^2 &= 8 \cdot 96^2 + 4^2 \end{aligned}$$

$$b_v^2 = 8a_v^2 + (8-4)v(-1)^v.$$

$$\text{II. 8. Διὰ } b_v^2 = 11a_v^2 + (11-4)v(-1)^v, \text{ καὶ } \sqrt{11}.$$

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2a_1 + b_1 \\ a_3 &= 2a_2 + b_2 \\ a_4 &= 2a_3 + b_3 \\ &\vdots \\ a_v &= 2a_{v-1} + b_{v-1} \end{aligned}$$

$$b_1 = 11a_1 + 25, \quad b_2 = 11a_2 + 25, \quad b_3 = 11a_3 + 25, \quad b_v = 11a_{v-1} + 25_{v-1}$$

Ἐὰν θέλωμεν $a_1 = 1$ καὶ λαμβάνομεν

$$\begin{aligned} a_2 &= 4 \\ a_3 &= 23 \\ a_4 &= 120 \end{aligned}$$

$$\text{Εἶναι δὲ } \frac{2}{1} < \frac{74}{23} < \dots \sqrt{11} \dots < \frac{401}{120} < \frac{15}{4}, \text{ καὶ}$$

$$\begin{aligned} 2^2 &= 11 \cdot 1^2 - 7 \\ 15^2 &= 11 \cdot 4^2 + 7^2 \\ 74^2 &= 11 \cdot 23^2 - 7^2 \\ 401^2 &= 11 \cdot 120^2 + 7^2 \end{aligned}$$

$$b_v^2 = 11a_v^2 + (11-4)v(-1)^v.$$

$$\text{II. 9. Διὰ } b_v^2 = 12a_v^2 + (12-4)v(-1)^v, \text{ καὶ } \sqrt{12}.$$

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2a_1 + b_1 \\ a_3 &= 2a_2 + b_2 \\ a_4 &= 2a_3 + b_3 \\ &\vdots \\ a_v &= 2a_{v-1} + b_{v-1} \end{aligned}$$

$$b_1 = 12a_1 + 25, \quad b_2 = 12a_2 + 25, \quad b_3 = 12a_3 + 25, \quad b_v = 12a_{v-1} + 25_{v-1}$$

Ἐὰν θέλωμεν $a_1 = 1$ καὶ λαμβάνομεν

$$\begin{aligned} a_2 &= 4 \\ a_3 &= 24 \\ a_4 &= 123 \end{aligned}$$

$$\text{Εἶναι δὲ } \frac{2}{1} < \frac{80}{24} < \dots \sqrt{12} \dots < \frac{448}{123} < \frac{16}{4}, \text{ καὶ}$$

$$\begin{aligned} 2^2 &= 12 \cdot 1^2 - 8 \\ 16^2 &= 12 \cdot 4^2 + 8^2 \\ 80^2 &= 12 \cdot 24^2 - 8^2 \\ 448^2 &= 12 \cdot 123^2 + 8^2 \end{aligned}$$

$$b_v^2 = 12a_v^2 + (12-4)v(-1)^v$$

$$\text{II. 10. Διὰ } b_v^2 = 14a_v^2 + (14-4)v(-1)^v, \text{ καὶ } \sqrt{14}.$$

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2a_1 + b_1 \\ a_3 &= 2a_2 + b_2 \\ a_4 &= 2a_3 + b_3 \\ &\vdots \\ a_v &= 2a_{v-1} + b_{v-1} \end{aligned}$$

$$b_1 = 14a_1 + 25, \quad b_2 = 14a_2 + 25, \quad b_3 = 14a_3 + 25, \quad b_v = 14a_{v-1} + 25_{v-1}$$

Ἐὰν θέλωμεν $a_1 = 1$ καὶ λαμβάνομεν

$$\begin{aligned} a_2 &= 4 \\ a_3 &= 26 \\ a_4 &= 144 \end{aligned}$$

$$b_1 = 14, \quad b_2 = 18, \quad b_3 = 92, \quad b_4 = 548$$

Είναι δε $\frac{2}{1} < \frac{92}{26} < \dots \sqrt{14} \dots < \frac{543}{144} < \frac{18}{4}$, και

$$\begin{aligned} 2^2 &= 14, & 1^2 &= 10 \\ 18^2 &= 14, & 4^2 &= 10^2 \\ 92^2 &= 14, & 25^2 &= 10^3 \\ 543^2 &= 14, & 144^2 &= 10^4 \\ &\vdots & & \end{aligned}$$

$$\delta_n^2 = 14 \cdot \alpha_n^2 + (14-4)^n (-1)^n.$$

II. 11. Διά $\delta_n^2 = 15\alpha_n^2 + (15-4)^n (-1)^n$, και $\sqrt{15}$.

α_1	δ_1
$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = 15\alpha_1 + 2\delta_1$
$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = 15\alpha_2 + 2\delta_2$
$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = 15\alpha_3 + 2\delta_3$
$\vdots$	$\vdots$
$\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$	$\delta_n = 15\alpha_{n-1} + 2\delta_{n-1}$
Εάν θέσωμεν	
$\alpha_1 = 1$	$\delta_1 = 2$
$\alpha_2 = 4$	$\delta_2 = 19$
$\alpha_3 = 27$	$\delta_3 = 93$
$\alpha_4 = 152$	$\delta_4 = 601$
$\vdots$	$\vdots$

λαμβάνομεν

Είναι δε $\frac{2}{1} < \frac{98}{27} < \dots \sqrt{15} \dots < \frac{601}{152} < \frac{19}{4}$, και

$$\begin{aligned} 2^2 &= 15, & 1^2 &= 11 \\ 19^2 &= 15, & 4^2 &= 11^2 \\ 93^2 &= 15, & 27^2 &= 11^3 \\ 601^2 &= 15, & 152^2 &= 11^4 \\ &\vdots & & \end{aligned}$$

$$\delta_n^2 = 15 \cdot \alpha_n^2 + (15-4)^n (-1)^n.$$

III. 1. Έκ τῶν προηγουμένων παρατηρούμεν, ὅτι ἡ κατὰ τὸ εὐκλείδειον θεώρημα II. 10 γεωμετρικὴ κατασκευὴ, τὴν ὁποῖαν μνημονεύει ὁ Πρόκλος, ἀγείρει εἰς τὴν ταυτότητα, (1), $(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2$,

τὴν ἀποδεικνυομένην κατὰ τὸ εὐκλείδειον θεώρημα II. 4. Ἐάν $\delta_1^2 = \lambda\alpha_1^2$, ἐνθα $\lambda > 5$, ἀκέραιος μὴ τετράγωνος, θὰ εἶναι καὶ $(\lambda-4)\delta_1^2 = (\lambda-4)\lambda\alpha_1^2$, ὁπότε ἐκ τῆς (1) ἔχομεν $\lambda(2\alpha_1 + \delta_1)^2 = 4\lambda\alpha_1^2 + 4\lambda\alpha_1\delta_1 + (\lambda-4)\lambda\alpha_1^2 + 4\delta_1^2 = (\lambda\alpha_1 + 2\delta_1)^2$.

Ἐπομένως θὰ εἶναι,

1) Πλευρικοὶ ἀριθμοὶ	Διαμετρικοὶ ἀριθμοὶ
α_1	δ_1
$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = \lambda\alpha_1 + 2\delta_1$
$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = \lambda\alpha_2 + 2\delta_2$
$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = \lambda\alpha_3 + 2\delta_3$
$\vdots$	$\vdots$
$\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$	$\delta_n = \lambda\alpha_{n-1} + 2\delta_{n-1}$

2) $\frac{\delta_1}{\alpha_1} < \frac{\delta_2}{\alpha_2} < \frac{\delta_3}{\alpha_3} < \dots \sqrt{\lambda} \dots < \frac{\delta_n}{\alpha_n} < \frac{\delta_{n+1}}{\alpha_{n+1}} < \frac{\delta_{n+2}}{\alpha_{n+2}}$, ($\alpha_1=1$, $\delta_1=2$).

3) $\delta_n^2 = \lambda\alpha_n^2 + (\lambda-4)^n (-1)^n$

III. 2. Εἶναι δυνατόν ἢ $\sqrt{2}$ νὰ ὑπολογισθῇ καὶ ἐκ τῶν πλευρικῶν καὶ διαμετρικῶν ἀριθμῶν τῆς μορφῆς $\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$, $\delta_n = \lambda\alpha_{n-1} + 2\delta_{n-1}$, εἰαν $\lambda=2$. Τὴν μέθοδον ταύτην καλοῦμεν γενικὴν πρὸς διάκρισιν ἀπὸ τῆς μεθόδου τῆς διασκέδασης ὑπὸ τοῦ Θέωνος τοῦ Σμυρναίου καὶ τοῦ Πρόκλου, τὴν ὁποῖαν καλοῦμεν εἰδικήν.

Πρὸς σύγκρισιν παραθέτομεν τὰ ἐξαγόμενα καὶ τῶν δύο μεθόδων.

A'. Μέθοδος εἰδική, $\alpha_n = \alpha_{n-1} + \delta_{n-1}$, $\delta_n = 2\alpha_{n-1} + \delta_{n-1}$.

$\alpha_1 = 1$	$\delta_1 = 1$
$\alpha_2 = 2$	$\delta_2 = 3$
$\alpha_3 = 5$	$\delta_3 = 7$
$\alpha_4 = 12$	$\delta_4 = 17$
$\vdots$	$\vdots$

$$\frac{\delta_1}{\alpha_1} < \frac{\delta_2}{\alpha_2} < \frac{\delta_3}{\alpha_3} < \dots \sqrt{2} \dots < \frac{\delta_n}{\alpha_n} < \frac{\delta_{n+1}}{\alpha_{n+1}} < \frac{\delta_{n+2}}{\alpha_{n+2}}$$

$$1^2 = 2 \cdot 1^2 - 1, \quad 3^2 = 2 \cdot 2^2 + 1, \quad 7^2 = 2 \cdot 5^2 - 1, \quad \delta_n^2 = 2\alpha_n^2 + (-1)^n.$$

[Διὰ $\alpha_1=1$, $\delta_1=2$

$\alpha_1 = 1$	$\delta_1 = 2$
$\alpha_2 = 3$	$\delta_2 = 4$
$\alpha_3 = 7$	$\delta_3 = 10$
$\alpha_4 = 17$	$\delta_4 = 24$
$\vdots$	$\vdots$

$$\frac{\delta_1}{\alpha_1} > \frac{\delta_2}{\alpha_2} > \frac{\delta_3}{\alpha_3} > \dots \sqrt{2} \dots > \frac{\delta_n}{\alpha_n} > \frac{\delta_{n+1}}{\alpha_{n+1}} > \frac{\delta_{n+2}}{\alpha_{n+2}}$$

$$\begin{aligned} 2^2 &= 2 \cdot 1^2 + 2 \\ 4^2 &= 2 \cdot 3^2 - 2 \\ 10^2 &= 2 \cdot 7^2 + 2 \\ 24^2 &= 2 \cdot 17^2 - 2 \\ &\vdots \end{aligned}$$

$$\delta_n^2 = 2\alpha_n^2 + (2-4)^n (-1)^n.$$

B'. Μέθοδος γενική, $\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$, $\delta_n = 2\alpha_{n-1} + 2\delta_{n-1}$.

Διὰ $\alpha_1 = 1$, $\delta_1 = 1$.

$\alpha_1 = 1$	$\delta_1 = 1$
$\alpha_2 = 3$	$\delta_2 = 4$
$\alpha_3 = 10$	$\delta_3 = 14$
$\alpha_4 = 34$	$\delta_4 = 48$
$\vdots$	$\vdots$

$$\frac{\delta_1}{\alpha_1} < \frac{\delta_2}{\alpha_2} < \frac{\delta_3}{\alpha_3} < \dots \sqrt{2}$$

$$\begin{aligned} 1^2 &= 2 \cdot 1^2 - 1 \\ 4^2 &= 2 \cdot 3^2 - 2 \\ 14^2 &= 2 \cdot 10^2 - 2^2 \\ 48^2 &= 2 \cdot 34^2 - 2^3 \\ &\vdots \end{aligned}$$

$$\delta_n^2 = 2 \cdot \alpha_n^2 + (2-4)^n (-1)^n$$

Διὰ $\alpha_1 = 1$, $\delta_1 = 2$,

$\alpha_1 = 1$	$\delta_1 = 2$
$\alpha_2 = 4$	$\delta_2 = 6$
$\alpha_3 = 14$	$\delta_3 = 20$
$\alpha_4 = 48$	$\delta_4 = 68$
$\vdots$	$\vdots$

$$\sqrt{2} \dots < \frac{\delta_1}{\alpha_1} < \frac{\delta_2}{\alpha_2} < \frac{\delta_3}{\alpha_3}$$

$$\begin{aligned} 2^2 &= 2 \cdot 1^2 + 2 \\ 6^2 &= 2 \cdot 4^2 + 2 \\ 20^2 &= 2 \cdot 14^2 + 2 \\ 68^2 &= 2 \cdot 48^2 + 2 \\ &\vdots \\ \delta_n^2 &= 2 \cdot \alpha_n^2 + (2-4)^n \cdot (-1)^n. \end{aligned}$$

Κατ' ἀντιστοιχίαν πρὸς τὴν ὑπὸ τοῦ Ἀρχιμήδους παρεχομένην τιμὴν τῆς $\sqrt{3}$ ὅα εἰχομεν διὰ τὴν $\sqrt{2}$

$$\frac{1}{1} < \frac{4}{3} < \frac{14}{10} < \frac{48}{34} < \frac{164}{116} < \dots < \sqrt{2} < \dots < \frac{792}{560} < \frac{232}{164} < \frac{68}{48} < \frac{20}{14} < \frac{6}{4} < \frac{2}{1}.$$

Εἶναι φανερόν, ὅτι εἶναι προτιμότερα ἢ ὑπὸ τοῦ Θέωνος τοῦ Σμυρναίου καὶ τοῦ Πρόκλου διασώθεισα μέθοδος διὰ τὸν ὀπολογισμὸν τῆς $\sqrt{2}$.

SUMMARY

I. 1. The law of formation of the side- and diameter- (diagonal-) numbers is explained by Theon of Smyrna. According to Proclus the related identity is proved by Euclid book II, proposition 10.

Side numbers $\alpha_n = \alpha_{n-1} + \delta_{n-1}$, diagonal numbers $\delta_n = 2\alpha_{n-1} + \delta_{n-1}$, $\delta_n^2 = 2\alpha_n^2 + 1$. For $\alpha_1=1$, $\delta_1=1$ we have $\frac{1}{1} < \frac{4}{3} < \dots < \sqrt{2} < \dots < \frac{17}{12} < \frac{8}{5}$.

2. Archimedes for the arithmetical approximation to π starts from a greater and a lesser limit to the value of $\sqrt{3}$, which without remark as known:

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}, \quad 265^2 = 3 \cdot 153^2 - 2, \quad 1351^2 = 3 \cdot 780^2 + 1.$$

We give the following interpretation on the archimedean formula, with pythagorean method of the side- and diagonal-numbers. In the figure 2 is $AB=\alpha_1$ the side $AF=\delta_1$, the greater diagonal of the rhomb $ABFA$, and the greater angle $ABF=120^\circ$. Then, $\delta_1^2=3\alpha_1^2$, $\delta_1:\alpha_1=\sqrt{3}$. According to Euclid II Prop. 10 we have

$$(2\alpha_1 + \delta_1)^2 + \delta_1^2 = 2\alpha_1^2 + 2(\alpha_1 + \delta_1)^2, \quad (2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2. \quad (I)$$

It is also $3(2\alpha_1 + \delta_1)^2 = 12\alpha_1^2 + 12\alpha_1\delta_1 + 3\delta_1^2$, and because $\delta_1^2=3\alpha_1^2$,

$$3(2\alpha_1 + \delta_1)^2 = 9\alpha_1^2 + 12\alpha_1\delta_1 + 4\delta_1^2 = (3\alpha_1 + 2\delta_1)^2.$$

The law of formation of the corresponding side- and diagonal-numbers is evidently. Side numbers $\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$, diagonal numbers $\delta_n = 3\alpha_{n-1} + 2\delta_{n-1}$.

When	$\alpha_1 = 1$	$\delta_1 = 1$	When	$\alpha_1 = 1$	$\delta_1 = 3$
	$\alpha_2 = 3$	$\delta_2 = 5$		$\alpha_2 = 4$	$\delta_2 = 7$
	$\alpha_3 = 11$	$\delta_3 = 19$		$\alpha_3 = 15$	$\delta_3 = 25$
	$\alpha_4 = 41$	$\delta_4 = 71$		$\alpha_4 = 56$	$\delta_4 = 97$
	$\alpha_5 = 153$	$\delta_5 = 265$		$\alpha_5 = 209$	$\delta_5 = 362$
				$\alpha_6 = 780$	$\delta_6 = 1351$

and $\delta_n^2 = 3\alpha_n^2 - 2$ and $\delta_n^2 = 3\alpha_n^2 + 1$.

$$\frac{1}{1} < \frac{5}{3} < \frac{19}{11} < \frac{71}{41} < \frac{265}{153} < \dots < \sqrt{3} < \dots < \frac{1351}{780} < \frac{362}{209} < \frac{97}{56} < \frac{26}{15} < \frac{7}{4} < \frac{3}{1}$$

II. In the following we start from the identity (I).

1. In the figure 3 we take $AB=\alpha_1$, $BF=2\alpha_1$, $AF=\delta_1$. Then $\delta_1^2=5\alpha_1^2$, $\delta_1:\alpha_1=\sqrt{5}$, and $5(2\alpha_1 + \delta_1)^2 = 20\alpha_1^2 + 20\alpha_1\delta_1 + 5\delta_1^2$. Because $\delta_1^2=5\alpha_1^2$, is $5(2\alpha_1 + \delta_1)^2 = 20\alpha_1^2 + 20\alpha_1\delta_1 + 25\alpha_1^2 = (5\alpha_1 + 2\delta_1)^2$.

Side numbers $\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$, diagonal numbers $\delta_n = 5\alpha_{n-1} + 2\delta_{n-1}$, $\delta_n^2 = 5\alpha_n^2 + (5-4)^n \cdot (-1)^n$.

When $\alpha_1=1$, $\delta_1=2$, $\frac{2}{1} < \frac{38}{11} < \dots < \sqrt{5} < \dots < \frac{161}{72} < \frac{9}{4}$.

2. In the same figure 3 we take $AB=\alpha_1$, $BF=3\alpha_1$, $AF=\delta_1$. Then $\delta_1^2=10\alpha_1^2$, $\delta_1:\alpha_1=\sqrt{10}$, and $10(2\alpha_1 + \delta_1)^2 = 40\alpha_1^2 + 40\alpha_1\delta_1 + 10\delta_1^2$. Because $\delta_1^2=10\alpha_1^2$, is $10(2\alpha_1 + \delta_1)^2 = 40\alpha_1^2 + 40\alpha_1\delta_1 + 40\alpha_1^2 = (10\alpha_1 + 2\delta_1)^2$.

Side numbers $\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$, diagonal numbers $\delta_n = 10\alpha_{n-1} + 2\delta_{n-1}$, $\delta_n^2 = 10\alpha_n^2 + (10-4)^n \cdot (-1)^n$. When $\alpha_1=1$, $\delta_1=2$, $\frac{2}{1} < \frac{63}{22} < \dots < \sqrt{10} < \dots < \frac{356}{112} < \frac{14}{4}$.

3. In the same figure 3 we take $AB=\alpha_1$, $BF=4\alpha_1$, $AF=\delta_1$. Then $\delta_1^2=17\alpha_1^2$, $\delta_1:\alpha_1=\sqrt{17}$, and $17(2\alpha_1 + \delta_1)^2 = 68\alpha_1^2 + 68\alpha_1\delta_1 + 17\delta_1^2$. Because $\delta_1^2=17\alpha_1^2$, is $17(2\alpha_1 + \delta_1)^2 = 68\alpha_1^2 + 68\alpha_1\delta_1 + 4\delta_1^2 = (17\alpha_1 + 2\delta_1)^2$.

Side numbers $\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$, D. N. $\delta_n = 17\alpha_{n-1} + 2\delta_{n-1}$, $\delta_n^2 = 17\alpha_n^2 + (17-4)^n \cdot (-1)^n$.

When $\alpha_1=1$, $\delta_1=2$, $\frac{2}{1} < \frac{110}{29} < \dots < \sqrt{17} < \dots < \frac{713}{169} < \frac{21}{14}$.

4. In the figure 4 we take $AB=\alpha_1$, $BF=2\alpha_1$, $AF=\delta_1$. The angle $ABF=120^\circ$. Then $\delta_1^2=7\alpha_1^2$, $\delta_1:\alpha_1=\sqrt{7}$, and $7(2\alpha_1 + \delta_1)^2 = 28\alpha_1^2 + 28\alpha_1\delta_1 + 7\delta_1^2$. Because $\delta_1^2=7\alpha_1^2$, $7(2\alpha_1 + \delta_1)^2 = 28\alpha_1^2 + 28\alpha_1\delta_1 + 4\delta_1^2 = (7\alpha_1 + 2\delta_1)^2$.

S. N., $\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$, D. N., $\delta_n = 7\alpha_{n-1} + 2\delta_{n-1}$, $\delta_n^2 = 7\alpha_n^2 + (7-4)^n \cdot (-1)^n$.

When $\alpha_1=1$, $\delta_1=2$, $\frac{2}{1} < \frac{50}{19} < \dots < \sqrt{7} < \dots < \frac{233}{55} < \frac{11}{4}$.

5. In the same figure 4 we take $AB=\alpha_1$, $BF=3\alpha_1$, $AF=\delta_1$. Then $\delta_1^2=13\alpha_1^2$, $\delta_1:\alpha_1=\sqrt{13}$, $13(2\alpha_1 + \delta_1)^2 = 52\alpha_1^2 + 52\alpha_1\delta_1 + 13\delta_1^2$. Because $\delta_1^2=13\alpha_1^2$, is $13(2\alpha_1 + \delta_1)^2 = 52\alpha_1^2 + 52\alpha_1\delta_1 + 4\delta_1^2 = (13\alpha_1 + 2\delta_1)^2$.

S.N., $\alpha_n = 2\alpha_{n-1} + \delta_{n-1}$, D. N., $\delta_n = 13\alpha_{n-1} + 2\delta_{n-1}$, $\delta_n^2 = 13\alpha_n^2 + (13-4)^n \cdot (-1)^n$.

When $\alpha_1=1$, $\delta_1=2$, $\frac{2}{1} < \frac{86}{25} < \dots < \sqrt{13} < \dots < \frac{487}{136} < \frac{17}{4}$.

In the same way we take the side- and the diagonal-numbers for the $\sqrt{6}$, $\sqrt{8}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{17}$, $\sqrt{19}$. (We mention Theaetetus of Plato 147 D).

III. If $\lambda \geq 5$, integer no square number and $\delta_1^2 = \lambda\alpha_1^2$, then $(\lambda-4)\delta_1^2 = (\lambda-4)\lambda\alpha_1^2$, and according to the identity (I) $\lambda(2\alpha_1 + \delta_1)^2 = 4\lambda\alpha_1^2 + 4\lambda\alpha_1\delta_1 + \lambda\delta_1^2$,

$$= 4\lambda\alpha_1^2 + (\lambda-4)\lambda\alpha_1^2 + 4\lambda\alpha_1\delta_1 + \lambda\delta_1^2 = (\lambda\alpha_1 + 2\delta_1)^2$$

Side numbers

$$\begin{aligned} \alpha_1 &= 2\alpha_1 + \delta_1 \\ \alpha_2 &= 2\alpha_2 + \delta_2 \\ \alpha_3 &= 2\alpha_3 + \delta_3 \\ &\vdots \\ \alpha_n &= 2\alpha_{n-1} + \delta_{n-1} \end{aligned}$$

Diagonal numbers

$$\begin{aligned} \delta_1 &= \lambda\alpha_1 + 2\delta_1 \\ \delta_2 &= \lambda\alpha_2 + 2\delta_2 \\ \delta_3 &= \lambda\alpha_3 + 2\delta_3 \\ &\vdots \\ \delta_n &= \lambda\alpha_{n-1} + 2\delta_{n-1} \end{aligned}$$

$$\frac{\delta_1}{\alpha_1} < \frac{\delta_2}{\alpha_2} < \frac{\delta_3}{\alpha_3} < \dots < \sqrt{\lambda} < \dots < \frac{\delta_n}{\alpha_n} < \frac{\delta_{n+1}}{\alpha_{n+1}}$$

and $\delta_n^2 = \lambda\alpha_n^2 + (\lambda-4)^n \cdot (-1)^n$. We take here always $\alpha_1=1$, $\delta_1=2$, $\lambda=2$ and $\lambda=3$ are special cases.