

THE PYTHAGOREAN THEOREM OF MUSICAL MATHEMATICS

He really wasn't stringing his fellow Greeks along; he picked it out in intervals between philosophizing.

IF we listen to melody and harmony all over the world, and through history, we find so many things that turn up over and over that we are entitled to suspect the existence of physical causes for them. The Greek philosopher Pythagoras was one of the earliest we know of who thought about this in an orderly way. He based his theory of music on the fact that some of the musical intervals (that is, *differences* in pitch) between notes in the Greek scales seemed to be related in a direct way to the sounds given out by a string when it is plucked with some simple fraction of its length left free to vibrate. In more modern language, we say that a half-length string sounds the important interval of one octave in pitch above the full-length string, and a string which is constrained to vibrate over only one third of its length sounds higher by the musical interval of a "twelfth," and so on, making what musicians call a harmonic series of notes.

A certain amount of number juggling can show that almost all the musically important intervals of modern western music (and of most other forms as well) can be more or less deduced from the whole-number fractions of the lengths of a vibrating string. Now, it must be confessed that in order to do this one used to need not only a great deal of faith in the magic properties of numbers, but also a good knowledge of the musical customs which were to be explained. To the Greeks, and to many people who came later, the magic of pure number was so potent that they were quite willing to settle for the numerical relations they found, without ever asking *why* they came out that way. Some musicians with more logical correctness than physical knowledge have seized upon the Pythagoreans'

need to know the musical answer beforehand as evidence that the numerical method of organizing music is a sort of peculiar accident. They are tempted to assert that all musical rules are a matter of custom and agreement, and that there is no absolute right or wrong way to go about choosing notes in music. My own belief is that music is *very* strongly affected by the manner in which things vibrate, and by the manner in which our ears work. It is certainly easy enough to show mechanical reasons why certain pitch intervals are special and distinct from their neighbors, in a musical sense.



I shall make use only of some mechanical facts of life. I shall say nothing about all the complex things which go on between the mechanical ear and the brain. These also have a big effect on music, but would lead us out of our depth and away from physics.

Let us start by asking what happens when we play successively two simple harmonic sounds of differing frequency. If the first note is played quite loud, we hear not only the frequency of that note, but also integral multiples of that frequency, $2f$, $3f$, $4f$, and so on. The response of the ear mechanism is responsible for this behavior, and the higher multiples are relatively weak in comparison with the lower ones. If now the

second note played happens to be the same frequency as *one* of the family of sounds produced by the first note, we recognize it (unconsciously) as a repetition of something heard before, and so consider a note of this second frequency as being in a special relation to the first.

Everyone who has fooled with a guitar, or played a violin, knows that the pitch rises as the vibrating length of a string is decreased, and many of you will have heard in school that the frequency of vibration of a string is inversely proportional to the length of the string. This means that cutting down the length to one half will double the frequency, reducing it to one-third will triple the frequency, and so on. We can now see very clearly why Pythagoras' experiment with strings might be expected to correlate with his knowledge that there are certain especially interesting musical pitch relations.

You may feel that this little demonstration is not enough to lay the scoffing ghosts of untrammelled artistic freedom, and you are right. But it is a step in the proper direction, and a bigger step than it may appear at first. I am being a little rude here toward people who make a hobby of not knowing enough about their subject of specialization. Composers who, for example, preach the extremist dogma of the "twelve-tone scale," in which every note is on an even footing with every other one, are no better in their disregard of the physical world than are sculptors in wood who try to ignore effects of the grain on the texture, working, and strength of their carvings.

—ARTHUR H. BENADE
in "Horns, Strings and Harmony"

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