

894
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You need not be a musician
to appreciate this application
of mathematics to music

PYTHAGOREAN PAPER FOLDING: A Study in TUNING and TEMPERAMENT

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IN TUNING the tones of a scale the ear wrestles with some fascinating problems concerning vibration ratios. This paper describes a convenient method for displaying to the eye the mathematical relationships involved.¹

The proportions embodied in musical intervals and the essential features of three tuning systems—Pythagorean, Just (or Pure), and Equal Temperament—can be demonstrated by a succession of paper-folding operations. Musicians may be delighted—and mathematicians amused—to discover that even such fabulous numbers as the “Pythagorean comma” of ratio 524288, 531441 or “the twelfth root of 2” ($^{12}\sqrt{2}$)—generally simplified to 1.059463+—will exhibit their meaning with as much accuracy as anyone deserves—that is, with as much as he can hear—after a relatively few folds. The procedure outlined below approaches the elegance of geometry by relying upon proportional division rather than on mathematical computation.

The folding procedure is a variant of the old monochord demonstration in which frequency is inversely proportional to string length; a strip of paper—calculating machine tape proves convenient—is substituted for the vibrating string. If a

monochord is available—and one is easily improvised—the results can be demonstrated to the ear by adjusting the bridges to the lengths determined by folding and then plucking the string to sound the tones.² The longer the strip of paper employed, the smaller the percentage of error; a length of three to four feet is recommended. If a modicum of care is exercised in folding, errors should be subliminal when tested on the string. But even if a monochord is not used, paper-folding should dramatize the ideas involved.

The operations described below are carried through five stages. Important insights are gained at each level, so that failure to complete the whole series does not diminish satisfaction earned along the way. The first demonstration aims at establishing a major scale by eight successive folding maneuvers little more complicated than halving a length by folding it or quartering it with a double fold. The second continues the folding pattern through five more tones to display the Pythagorean comma which lies in wait if the series of pure fourths and fifths is continued. The next demonstration introduces the “pure” or “Just” third and displays its incompatibility with other values. The fourth establishes the tones required for a typical scale in Just intonation and provides *syntonic commas* at strategic locations for the following dem-

1. A mathematician's treatment of tuning problems can be found in an article by H. S. M. Coxeter, “Music and Mathematics,” in the March 1968 issue of this journal (vol. 61, no. 3, pp. 316-19).

2. A thorough exposition of tuning principles and instructions for the use of the monochord will be found in *Tone* by Siegmund Levarie and Ernst Levy (Kent State University Press, 1968).

Excellence in Mathematics Education—For All 233

MATH. TEACHER 63 (1970) p. 233-237

onstration. The last operation splits the syntonic commas to estimate the position of several tones in equal temperament, making clear their relation to the tones of Just and Pythagorean tuning.

Demonstration 1

Pythagorean Tuning: The C Major Scale

A scale can be "tuned" by folding a succession of perfect fourths, ratio 3:4, and perfect fifths, ratio 2:3, analogous to the sequence employed by piano tuners. It proves convenient, as will be understood later, to call the tone associated with the whole "paper string" length "B" and to label one end of the paper zero (0) and the other B. Folding always proceeds from zero; if it is placed to the left the resultant scale will appear in descending order; if it is to the right, the same scale will ascend.

For the first fold, which locates the upper octave, B', at one-half the length for B, bring the two ends of the paper together and crease in the middle. (As each tone is located, place a mark in the crease and label for future reference.)

The next tone, E, is located at three-fourths of the original length. Fold the doubled length of paper in half (i.e., in quarters); E lies on the third crease, a perfect fourth higher than B, at a ratio of 3:4 (i.e., $3/4$ to $4/4$) (fig. 1, below).

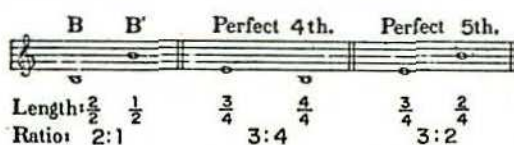


FIG. 1. BASIC RATIOS

Alternately—and this is worth noting—E also lies a perfect fifth below B', at a ratio of 3:2 (i.e., $3/4$ to $2/4$); it could have been reached by first halving the half required for B' and then folding the doubled section over onto the remainder to measure a third equal segment.

From the above operations we derive two folding rules.

1. To ascend a perfect fourth, fold the

length required for the given tone in fourths (i.e., fold in half twice); the new tone lies on the third crease.

2. To descend a perfect fifth, fold the length for the given tone in half and employ this result to measure an equal segment below it.

As each tone in the series is located, its length serves in turn as the base from which the following tone is generated (fig. 2) in the order B', E, A, D, G, C, and F.

Halve these lengths by folding, then fold the doubled section over onto the remainder and mark how far it reaches to locate the next tone a fifth lower, ratio 3:2.



Quarter these lengths by a double fold: the next tone lies at the third crease, a fourth higher, ratio 3:4.

FIG. 2. THE SEVEN TONES:
ORDER OF GENERATION

When the above tones have been located, halve C to produce C' an octave higher and fold the end segment required for low B back out of sight—as a kind of "added note" lying outside our present interests, but waiting to be rediscovered if we need it, much as the lowest monochord tone was treated in earlier Greek thinking before it was incorporated into their Greater Perfect System. (We have chosen to work in the C octave because of its congenial familiarity.)

In a scale in Pythagorean tuning all the major seconds are of the same size, with a ratio of 8:9, but the minor seconds are less than a "semitone"—with a ratio of 243:256, or approximately 19:20.³ The Greeks called this small interval a *lemma*.

3. The whole tone ratio was first established between B' at $8/16$ and A at $9/16$ by folding the following sequence: $1/2 \times 3/2 \times 3/4 = 9/16$. The lemma, or semi-tone, results from subtracting two such intervals $(8/9)^2$ from the perfect fourth (3:4) as follows: $3/4 \div (8/9)^2 = 3/4 \times 81/64 = 243/256$.

—“left-over,” that is, when two major seconds are subtracted from a perfect fourth. This undersized semitone is far from obsolete. Performers who “stretch” intervals to intensify upward or downward leading-tones may employ it on their way to the tone of resolution.

In one sense the Pythagorean tuning system is obsolete; in another sense it is implicit within every other tuning we have used in the Western world. Insofar as tonality is a ruling principle in music—and it is not a conscious one in all music—the most powerful line of relationships extends through the dominant order of fourths and fifths. These intervals, even when offensively out of tune, remain, together with the octave, the strongest anchors for a mind seeking “shape” within the musical flux.

Demonstration 2

The Pythagorean Comma

Tones generated via perfect fourths and fifths unfold endlessly without duplicating an earlier tone. For instance, in this tuning C_b is distinctly lower than B. A pianist may dispute this until, in tuning his own instrument, he discovers that the 13th tone in the tuning series disagrees violently with the first one, or that the 12th tone refuses to make a perfect fourth or fifth with the first one. Paper-folding carried through five more tones displays this discrepancy vividly with the Pythagorean comma, a micro-interval with a ratio of $524288/531441$ (roughly 73:74) lying between B' and C_b' (fig. 3).

In Pythagorean tuning all sharpened tones are a comma higher than the flatted ones which equal-temperament treats as enharmonically equivalent ($F\sharp-G\flat$, $C\sharp-D\flat$, $G\sharp-A\flat$, etc.).⁴ The comma is wide enough to constitute an obstacle to the

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Quarter these lengths by a double fold: the next tone lies at the third crease, a fourth higher, ratio 3:4.

A summary of the operations by which the Pythagorean comma is demonstrated is given below, beginning with B'.



FIG. 3. THE PYTHAGOREAN COMMA

rigorous use of perfect fourths and fifths in tuning a keyboard instrument, but it is also small enough—fortunately, for the convenience of musicians—to become subliminal when distributed over twelve intervening fourths and fifths.

Demonstration 3

Just Intonation: Pure Thirds

Harmony is based on the triad with ratios 4:5:6, but Pythagorean tuning misplaces the middle tone by an appreciable amount. This can be seen by establishing a "pure" third, ratio 4:5, between C and A $\flat$; fold the segment required for C in quarters, then use the quadrupled section to measure a fifth equal segment along the remainder. The Just A $\flat$ lies substantially higher than the Pythagorean one by a micro-interval known as the syntonic or Didymic comma, with a ratio of 80:81. If this tuning is adopted for A $\flat$ then its "perfect" fifth with D $\flat$ becomes a comma too large; if D $\flat$ is retuned—and any number of tones beyond it—we shall still be left with an atrocious fifth somewhere in the system, and the ear is more offended by a bad fifth than by a bad third. Endless controversy has raged over this pure third; the adoption of equal temperament has ruled it out except for

4. It is interesting to estimate the position of another Pythagorean comma at $F\sharp$, two-thirds of the length required for the low B with which we started. This requires an awkward and logically inadmissible triple fold ($\cong$), but it helps to make clear that $F\sharp$, lying on the second crease, is higher than $G\flat$.

singers and for instrumentalists whose pitch is variable.⁵ In practice, wider—thus less euphonious—thirds are often preferred to a pure one because they intensify melodic tension and emphasize the distinction between major and minor.

Another interesting problem with the pure third is that if it is pursued rigorously it plays havoc with the octave. If, for instance, a hypothetical $F\flat$ is established below the Just $A\flat$ (by quartering the length used for $A\flat$ and measuring a fifth segment below it) and a hypothetical $D\flat\flat$ is then established another third lower (by performing the same operation on $F\flat$), we find a gap of dramatic proportions—nearly a quarter of a tone ($42/100$ semitone)—between the third major third, $D\flat\flat$, and the octave, C (fig. 4).⁶

Quarter each length in turn, then measure a fifth equal segment along the remainder to locate the following tone.

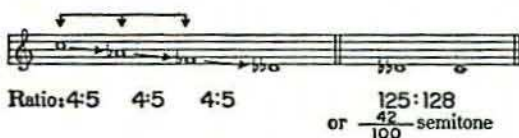


FIG. 4. SEQUENCE OF PURE THIRDS

Demonstration 4

Just Intonation: The Scale

An octave scale in Just intonation can be established by using four tones, C , $B\flat$, F , and G , from the Pythagorean series and then tuning the other three tones by pure thirds at $C-A\flat$, $G-E\flat$, and $F-D\flat$. The Just $A\flat$ was established in demonstration 3; tune $E\flat$ and $D\flat$ by analogous operations. (Reminder: quarter the length for G and measure a fifth segment below it to locate $E\flat$; quarter the length for F and measure a fifth segment below it to locate $D\flat$.) All three Just thirds are higher in

5. For an historical survey of the various tuning systems and an extensive treatment of the problem of the pure third see *Tuning and Temperament* by J. Murray Barbour (Michigan State College Press, 1951).

6. In equal-temperament the major thirds are oversized enough for three of them to reach the octave; this demonstration exposed the distortion that results.

pitch—at a shorter string length—than their Pythagorean counterparts by the interval of the syntonic comma.

The resulting scale is the inverse of the major scale, that is, the pattern of tones and semitones is reversed. This *Phrygian*

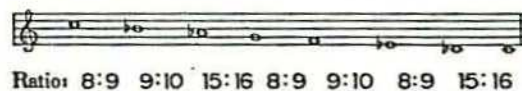


FIG. 5. THE SCALE IN JUST INTONATION

mode is shown (fig. 5) instead of major because it is easier to achieve by paper folding; in C major in Just intonation the tones E , A , and B would be lowered by a comma.

In the Just scale shown above in figure 5 there are three large wholetones, ratio 8:9, two smaller ones, ratio 9:10, and two somewhat "oversized" semitones with a ratio of 15:16. This tuning system has more theoretical than practical value.

Demonstration 5

Equal Temperament

Paper-folding cleverness cannot demonstrate the idea of a number like "the twelfth root of 2" (approximately 1.059463⁺) which provides the theoretical basis for an equal tempered scale. But, by the happy coincidence that equal temperament tones lie near the middle of syntonic commas, we can estimate the location of several of them with almost as much accuracy as the ear can appreciate. Interpolate the tempered $A\flat$, $E\flat$, and $D\flat$ into the middle of the syntonic commas established in demonstrations 3 and 4 (fig. 6).

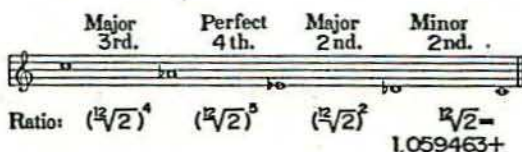


FIG. 6. EQUAL TEMPERAMENT

The ratio for an equal tempered semitone is one which carries us to the octave

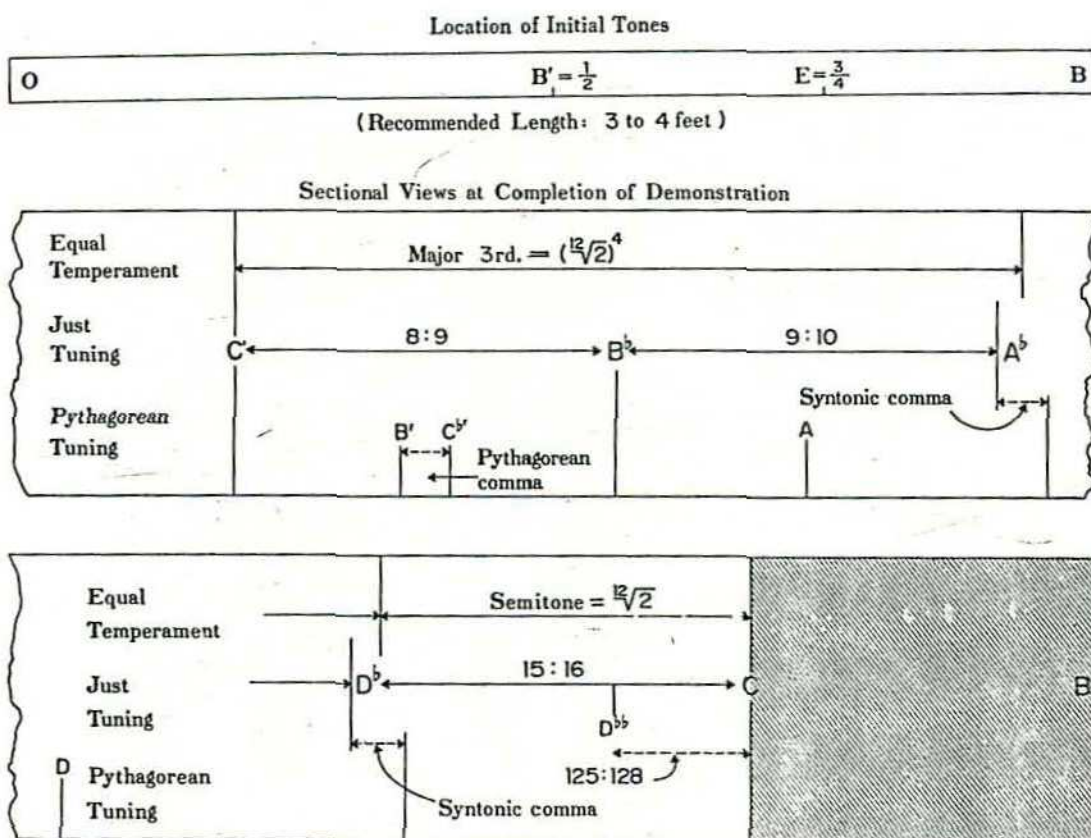


FIG. 7. THE FOLDED SCALE

after twelve identical operations. Vincenzo Galilei, father of a more famous son, first explained how to achieve this to a sixteenth-century lute maker, relying on a ratio of 18:17—an approximation at which he arrived not by calculation, we are told, but by intuition.⁷ Like Vincenzo we trust our intuition that the equal tempered tones we blithely located in the middle of syntonic commas are accurate enough for

7. Barbour, pp. 57-58.

practical purposes, enough—that is, to make clear the idea that temperament is a compromise between the conflicting claims of various ideal values. (See fig. 7)

In tuning a scale we are haunted by the perfection which lies just beyond our grasp and intrigued by the necessity for tempering even as much as lies within it. Paper-folding demonstrates the ear's problem to the eye: cyclic repetition at the octave requires that all other intervals be robbed of some measure of their perfection.

Letter to the Editor

DEAR EDITOR:

The following letter is the reply I made to two seventh-grade students in Omaha who wrote asking me about the *fifth* dimension. Apparently to the youngsters of the late 60s, the fourth dimension is no longer a problem.

Some of your readers may find my answer helpful if they come up against a similar question.

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