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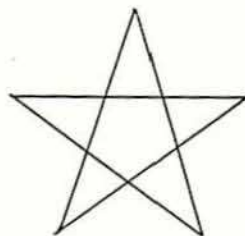
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The Pythagorean Legacy

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"a figure and a step forward, not a figure to gain three oboli" [2]*



PYTHAGORAS, THE MAN

"Number rules the universe" [3]

Of all the interesting figures in the history of ancient mathematics, Pythagoras easily ranks first, partly from the mysticism surrounding his life, partly from his own mysticism, partly from the Brotherhood which he established, and partly from the unquestioned ability of the man himself [7]. As the introduction of geometry into Greece is by common consent attributed to Thales, so are all agreed that to Pythagoras of Samos is due the honor of having raised the mathematics of everyday life to the rank of a science [1].

The exact date and place of his birth are both unknown, although much speculation has been made. Pythagoras seems to have been born between the 50th and 52nd Olympiads, to use the Greek system of chronology, or between 580 and 568 B.C. of our calendar. Although called a Samian, we are not certain that he was born on the island of Samos. Suidas, a late medieval writer (c. 1000), says Pythagoras was born in Italy, and migrated to Samos with his father. Nevertheless, the weight of authority favors his Samian birth, since a number of coins of the island, struck some centuries after his time, bear his name and figure. This would hardly have been the case had he merely spent his boyhood there [1].

But in whatever land he was born, and in whatever year, and of whatever parentage, Pythagoras lived in stirring times and was himself one of the great makers of the civilization. Samos was just becoming the center of Greek art and culture. Polycrates was just ascending the throne, and Anacreon was beginning to write his famous lyrics in the Samian court. Pythagoras was therefore brought up amid scenes

*The numerals in brackets refer to the bibliography at the end of this paper.

that could hardly fail to stimulate a youth of his native powers and urge him to a high intellectual life. Moreover, the spirit of the times was active in great works. Buddha was just promulgating his doctrine in India, and Confucius and Lao-tze were laying the foundation for their philosophic cults in China [7].

A note of historic importance is the fact that arithmetic and geometry took a notable step forward at this time, due in no small way to the introduction of Egyptian papyrus in Greece. This event occurred about 650 B.C., during the reign of King Psammetichus. The invention of printing in the 15th century did no more to effect a revolution in thought than did the introduction of this invention on the northern shores of the Mediterranean Sea [7]. It was certainly true that Pythagoras lived when the world was ripe for great movements [7].

Our knowledge of the life of Pythagoras is very limited, the early writers having vied with each other in the invention of fables relating to his travels, his miraculous powers, and his teachings. He seems to have sought out Thales and to have been his pupil. Tradition says that he was initiated by the master into the secrets of Zeus on Mount Ida, and was then told that if he would have further light he must seek it in Egypt [7].

In spite of the varied assertions of many writers, the evidence derived from the philosophy of Pythagoras points to his contact with the Orient. The mystery of the East appears in all his teachings. His mysticism of numbers is quite like that found earlier in Babylon, and indeed his whole philosophy savors much more of the Indian than the Greek civilization in which he was born [7].

Returning home from his somewhat mysterious travels, Pythagoras found Samos under the tyranny of Polycrates and Ionia under the dominion of the Persians, and, accordingly, he migrated to the Greek seaport of Crotona, located in Southern Italy. There he founded the famous Pythagorean school, which in addition to being an academy for the study of philosophy, mathematics, and natural science, developed into a closely knit brotherhood with secret rites and observances [4].

In time, the influence and aristocratic tendencies of the Brotherhood became so great that the school was broken up, the property confiscated, and Pythagoras exiled. The next years he lived in Tarentum, but even there the democratic party gained the upper hand, and he was forced to flee again, this time to Metapontus. Democracy triumphed there also; the school was burned, many disciples died deaths of torture, and Pythagoras himself suffered and died soon after [5]. The Brotherhood, although scattered, continued to exist for at least two centuries more [4].

At the present time, Pythagoras is thought of primarily as a mathematician. In Amsterdam a street is named after him, in the neighborhood with streets also named after Archimedes, Newton, and Copernicus. Today, to mention the name of Pythagoras immediately brings to mind the famous Pythagorean theorem. It was quite different in antiquity. Herodotus calls him "an important Sophist," and his contemporaries looked upon him as a religious prophet and a performer of miracles [8].

Pythagoras based his philosophy upon the postulate that number is the cause of the various qualities of man and matter. This led him to exalt arithmetic. It also led him to dwell upon the mystic properties of number and to consider arithmetic as one of the four degrees of wisdom: arithmetic, music, geometry, and spherics (astronomy), forming the quadrivium [7]. This quadrivium was long associated with the program constituting the necessary and sufficient course of study for a liberal education throughout the Middle Ages [2].

From various early writers, we judge that Pythagoras asserted that unity is the essence of number, the origin of all things, the divine; that he had the idea of the limited and the unlimited. Diogenes Laertius (2nd century A.D.) says that Pythagoras was interested in number, and that the part of mathematics to which Pythagoras applied himself above all others was arithmetic [1].

To be sure the dictum, "Number rules the universe" might bring a condescending smile to the lips of a modern scientist. But if we forget the lofty form in which these words were put and conceive numbers in the broad sense of the term, is there anything in the dictum to which a modern scientist could not and would not then subscribe? Number reigns as firmly in the new physics as it did in the old. The argument that "the study of any phenomenon has not been consummated until the phenomenon has been made mathematically articulate," is as convincing today as in the time of Pythagoras. The conjecture that physical properties may exist that are beyond the powers of numbers to express, would be as ridiculous to the man of science today as it was to Pythagoras [3].

THE PYTHAGOREAN BROTHERHOOD

"Every man builds upon his predecessors" [5]

When Pythagoras reappeared after his years of wandering, he sought a favorable place for a school and finally settled upon Crotona, a town on the southeastern coast of Italy, in a territory called by the Greeks at that time Great Greece [7]. Here the school that he opened was crowded with enthusiastic audiences; citizens of all ranks attended,

especially those of the upper classes, and even women broke a law which forbade their going to public meetings [2].

Pythagoras spoke captivantly, and it is for this reason that his orations brought about a change in the thinking of Crotona's inhabitants. Crowds of listeners streamed to him. Besides the youth who listened all day to his teachings, some 600 of the worthiest men of the city came to hear. Matrons and maidens came together at his evening lectures. Among them was the young, gifted, and beautiful Theana, who thought happiness was becoming the wife of the sixty year old teacher [5]. Theana wrote a biography of her husband, but, unfortunately, it has been lost [2].

Pythagoras divided those who attended his lectures into two classes, that might be classified as probationers and Pythagoreans. The majority were probationers, and it was only to the Pythagoreans that the teacher revealed his chief discoveries. The latter formed the Brotherhood [2] which served as a model for many of the secret societies throughout Europe and the new world [7].

The Brotherhood held all things in common, sharing the same philosophical and political beliefs, engaged in the same pursuits, and were bound by oath not to reveal the teachings or secrets of the school. Their food was simple, their discipline severe, and their mode of life arranged to encourage self-command, temperance, purity, and obedience. This strict discipline and secret organization gave the Brotherhood a temporary supremacy in the state which brought upon it the hatred of various classes [2].

Though the political influence of the Brotherhood was destroyed, they seemed to have re-established themselves at once as a philosophical and mathematical society, with Tarentum as their headquarters. There they continued to flourish for more than a hundred years [2]. The triple interwoven triangle or pentagram—a star shaped regular pentagon—was used as the symbol or sign of recognition. It was called by the Pythagoreans "Health" (υγεια) [1].

Pythagoras never embodied his doctrine in any treatise. Like Thales, and those Oriental teachers from whom he probably learned, he transmitted his theories by word of mouth. This he did through the Brotherhood, thus making known his doctrines freely to all who were deemed worthy to receive them [7]. These disciples of Pythagoras proved themselves worthy of their mission. They inherited noble self-renunciation from their master. The moral dignity of these men further shown by their maxim—a maxim conceived in the spirit of true social philosophers [1]. It was their boast that they sought knowledge and not wealth, as told in their language, "a figure and a step; but not a figure and three oboli." [2]. Such then were the

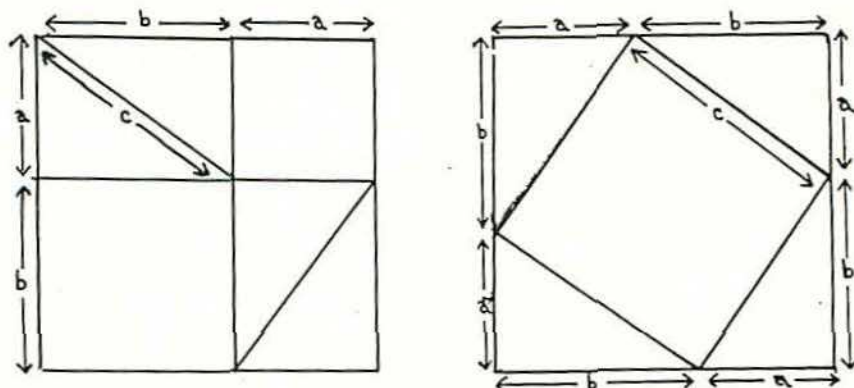
men by whom the first steps in mathematics—the first steps ever the most difficult—were made [1].

THE PYTHAGOREAN THEOREM

Who does not think, when he hears the name of Pythagoras, of the famous theorem showing the relation between the sides of a right triangle [8], which we express today in the formula $a^2 + b^2 = c^2$ where a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse? No other proposition of geometry [3] has exerted so much influence on so many branches of mathematics as has this simple formula. Indeed, much of the history of classical mathematics, and of modern mathematics as well, could be written around this proposition [3].

It is known in history as the 47th proposition, its number in the first book of Euclid's *Elements* [5]. Although the practical application of this theorem was known long before the time of Pythagoras, he doubtless generalized it from an Egyptian rule of thumb ($3^2 + 4^2 = 5^2$), and first demonstrated it about 540 B.C. Since Pythagoras' time, many different proofs of the theorem have been supplied [4]. In the second edition of *THE PYTHAGOREAN PROPOSITION*, E. S. Loomis has collected and classified 371 demonstrations, including 109 algebraic proofs, 256 geometric proofs, 4 quaternionic proofs, and 2 dynamic proofs [5].

There has been much conjecture as to the proof Pythagoras might have offered, and it is generally felt that it was probably a dissection type of proof similar to the following.



Let a , b , and c denote the lengths of the legs and the hypotenuse, respectively, of the given right triangle, and consider the two squares in the figure above, each having $a + b$ as length of the sides. The

first square is dissected into six pieces, namely the two squares on the legs and four right triangles congruent to the given triangle. The second square is dissected into five pieces, namely the square on the hypotenuse and four right triangles congruent to the given triangle. By subtracting equals from equals, it now follows that the square on the hypotenuse is equal to the sum of the squares on the legs [4].

To prove that the central piece of the second triangle dissection is actually a square of side c , we need to employ the fact that the sum of the measures of angles of a right triangle is equal to the sum of the measures of two right angles. The *Eudemian Summary* attributes this theorem for the general triangle to the Pythagoreans. Since the proof requires some knowledge of the properties of parallels, the early Pythagoreans are also credited with the development of that theory [4].

The significance of the famous Pythagorean theorem can be seen in the numerous names by which it has been called; some are [5]:

- The Carpenter's Theorem
- The Hecatomb Proposition
- The Pons Asinorum (erroneously)
- The 47th Proposition
- The Pythagorean Proposition
- The Bride's Chair

The influence of this theorem has been far reaching in a variety of areas. To begin with, the theorem is the point of departure for most metric relations in geometry, i.e. of those properties of configurations that are reducible to magnitude and measure. Such figures that are amenable to study by classical methods are either polygons or limits of polygons; and whether the method be congruence, areal equivalence, or similitude, it rests ultimately on the possibility of resolving a figure into triangles.

The Pythagorean equation being non-linear, has numerical applications leading to irrational numbers. In this way mathematics, almost since its inception, was confronted with the perplexing problem of incommensurable magnitudes, and this exerted a profound influence on the evolution of the number concept. The introduction of infinitesimal methods led to further extensions of the formula's scope. In the guise of a differential form, it became the measure of the length of the arc of a plane curve. The idea was eventually extended to space curves, then generalized to curved surfaces [4].

Last, but not least, was the influence of the Pythagorean theorem on noneuclidean geometries. When the axioms of geometry began to be subjected to critical analysis, it was soon realized that the Pythagorean relation between the sides of a right triangle was equivalent

to the Euclidean postulate of parallels. Thus, if one were to reject this postulate but retain the others, one would have to replace the Pythagorean relation by another form. These considerations led mathematicians to the epoch-making idea of defining space structures by means of quadratic forms, an idea which, when extended to space-time manifolds, became the foundation of the mathematical theory of relativity [4].

Thus the Pythagorean theorem is rightly regarded as the most fascinating theorem of Euclidean geometry, so much so, that thinkers from all classes and nationalities, from the aged philosopher in his armchair to the young soldier in the trenches next to no-man's-land, have wiled away hours seeking a new proof of its truth [5].

PYTHAGOREAN TRIPLES

Clearly allied with the Pythagorean theorem is the problem of finding integers a , b , and c that represent the legs and hypotenuse of a right triangle, or, to state it in algebraic form [4], to determine all integer sets which satisfy the equation $x^2 + y^2 = z^2$. Such sets are called Pythagorean triples. In the equation $x^2 + y^2 = z^2$, x and y represent the lengths of the legs of the right triangle and z represents the length of the hypotenuse. Later Pythagoreans have been credited with the formula

$$m^2 + \left(\frac{m^2 - 1}{2}\right)^2 = \left(\frac{m^2 + 1}{2}\right)^2$$

the three terms of which, for any odd value for m , yield a Pythagorean triple [4].

Because of the homogeneous character of the Pythagorean relation, the triples can be classified as primitive and nonprimitive. A triple is primitive if its terms have no common divisors other than 1. Examples of primitive triples are (3, 4, 5), (5, 12, 13), and (8, 15, 17); examples of nonprimitive triples are (9, 12, 15), (10, 24, 26) and (80, 150, 170).

Associated with every primitive triple (x , y , z), is an infinitude of nonprimitive triples (nx , ny , nz) n a natural number. On the other hand, it is always possible to determine a primitive triple, the terms of which are proportional to x , y , and z . This is done by the simple expedient of dividing every term of the triple by the greatest common divisor of the three elements of the triple. This operation is sometimes called contraction [3].

Thus, whether any given triple is primitive or nonprimitive, depends on the value of the greatest common divisor of the elements of the triple. The labor incident to calculating this divisor is greatly facilitated

by the following theorem: Any integer that divides two terms of a Pythagorean triple, also divides the third term [3].

The theorem stated above has two practical corollaries: (1) To determine the greatest common divisor of a triple it is sufficient to calculate the greatest common divisor of any two terms of the triple; and (2) If any two terms of a triple are relatively prime, then the triple is primitive [3].

An analysis of Plimton 322 offers fairly convincing evidence that the ancient Babylonians knew how to calculate such triples [4]. There are many allusions to triples to be found in Diophantus' *Arithmetica*. Perhaps the greatest step was taken when Fibonacci sought to extend the area of inquiry to determining primitive triples, given the difference between the hypotenuse and the even side, and discovered that the problem had no solution unless the stipulated difference was a perfect square [3].

This much can be affirmed with certainty. The Pythagoreans were fully aware of the importance of the concept of primitivity. They knew that one of the sides of a primitive triple was even, the other odd, and the hypotenuse was always odd. They knew how to generate certain types of triples in number indefinite, and concluded from this that the aggregate of primitive triples in number indefinite, and concluded from this that the aggregate of primitive triples was infinite. In the final analysis the proposition is a study of integers [3].

OTHER CONTRIBUTIONS OF PYTHAGORAS

"All roads lead back to Greece" [3]

With the possible exception of Aristotle, no other philosopher of antiquity received as much publicity as Pythagoras. The spectacular character of the man, the fact that he was the titular head of a semi-religious cult, and the acknowledged fountainhead of the Platonist School, coupled with the extravagant claims made for him by his followers, may explain his widespread fame. These claims were not confirmed to the realm of mathematics [3].

Music, harmony, and numbers are indissolubly united according to the doctrine of the Pythagoreans. All three are among the essential elements of the Pythagorean system of education and of its path for the elevation of the soul [8]. In the realm of music, Pythagoras is said to have discovered that the fifth and the octave of a note can be produced on the same string by stopping at $\frac{2}{3}$ and $\frac{1}{2}$ of its length. It is thought that this harmony gave rise to the name of the "harmonic proposition". Although Pythagoras seems to have derived some knowledge of music from Egypt, he is generally called the inventor of musical science or the harmonic cannon [8].

Pythagoras seems to have believed that the interval between the heavenly bodies were determined by the laws of musical harmony, and hence rose the doctrine of the harmony of the spheres [7]. The heliocentric hypothesis is attributed to Pythagoras, and his teachings that the earth revolved around the sun persisted even after the more accurate contributions of Copernicus [3]. Pythagoras was correct in assuming the earth to be spherical in shape, and he knew the proper motions of sun, moon, and planets [8].

As magic and number magic belong together, so do mysticism and number mysticism. Every magician utilizes the magic power of words and of numbers. Every superstitious person knows sacred symbols and lucky numbers. These things had of old played an important role among the Babylonians, the Magi, and the Pythagoreans as well. For example, they looked upon even and odd as the roots of all things. The even numbers were called feminine, and the odd, masculine. The number 5, the sum of the first feminine and the first masculine numbers, was taken as a symbol for marriage [5].

The Pythagoreans discovered "perfect" numbers, that is numbers that are the sums of their proper divisors (e.g., $6 = 1 + 2 + 3$). The Pythagoreans also studied the amicable or friendly numbers. When Pythagoras was asked what a friend is, he is supposed to have replied, "A second I", and he mentioned the amicable numbers [6] 284 and 220, each of which equals the sum of the proper divisors of the other [8]. Such studies in number phenomenon account in part for the Pythagoreans interest in triangular, square, and other figurate numbers, often today regarded as insignificant [3].

One can discover in the Pythagorean speculations more than a mere germ of what we call scientific attitude. The representation of a physical law by means of a formula is so common today, that we accept it as though it were granted to man by Providence. But far from it being a gift from heaven, it was the culmination of a long and painful evolution [3].

Pythagoras was a religious mystic who viewed number as the key to the plan which the "Supreme Architect" used in fashioning the universe. He and his followers thought that the movement of the heavenly bodies, the composition of matter, the structure of thought, and the principles of human conduct were expressible in number because all was governed by number. It was Pythagoras' mission as a philosopher to interpret the work of the creator by deciphering, as it were, the intricate scroll of creation. To do this, he must first master the code in which this scroll was written, and this code was mathematics [3].

Did the principles of Pythagoras foreshadow the vast system of formulae and equations by means of which modern science links

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MISCELLEN

Antiphanes von Berge

Die Persönlichkeit des griechischen Münchhausen, über den bis vor Kurzem ziemlich verworrene Vorstellungen herrschten¹ ist durch Wilamowitz in ein helleres Licht gerückt worden. Er hat (Herm. XXXIX 149 f.) überzeugend nachgewiesen, dass der von einem ungenannten Schüler Platons, übrigens einem unzünftigen Philosophen, bei Plutarch de profect. in virt. 7 citirte Antiphanes: ὁ γὰρ Ἀντιφάνης ἔλεγε παίζων ἐν τινὶ πόλει τὰς φωνὰς εὐθὺς λεγομένας πηγνύσθαι διὰ ψῦχος· εἴθ' ὕστερον ἀνιεμένων ἀκούειν θέρους, ἃ τοῦ χειμῶνος διελέχθησαν kein anderer als der Bergäer sein kann, dessen Lebenszeit somit noch in das 4. Jahrhundert fällt. Man könnte sich bei diesem Ergebniss beruhigen, wenn nicht die sonstigen Zeugnisse bei eindringender Prüfung und genauer Interpretation über die Zeit und das Werk des Antiphanes neuen Aufschluss ertheilten.

Eratosthenes, der älteste Zeuge, hat Euhemerus einen 'Bergäer' gescholten. Das bedeutet nicht schlechthin Lügner, trotz des von ihm abgeleiteten Verbums βεργαῖζειν ἀντὶ τοῦ μηδὲν ἀληθὲς λέγειν (Steph. Byz. s. Βέργη), sondern zielt auf etwas Besonderes, wie aus der von Polybios an Eratosthenes geübten Kritik (Strab. II 104) erhellt. Eratosthenes, so etwa lässt sich der Kritiker vernehmen, schenkt dem Pytheas Vertrauen und nennt den Euhemerus einen Bergäer, obwohl dieser nur nach dem einen Lande Panchaia gefahren zu sein behauptet, während jener den äussersten Norden Europas bis zu den Grenzen der bekannten Welt geschaut haben will. Der Vergleichspunkt ist also, dass Antiphanes wie Euhemerus einen lügenhaften Reisebericht verfasst hat, und deswegen steht er auch als Lügen-schriftsteller neben Pytheas und Euhemerus bei Strabon II 102 in einer gegen Poseidonios gerichteten Polemik. Zu diesem Ergebnisse stimmt das einzige Fragment: wie die angebliche Fahrt des Euhemerus von dem glücklichen Arabien nach dem fabelhaften Panchaia im äussersten Süden des Weltmeeres ging (φησὶν ἐκτοπισθῆναι κατὰ τὴν μεσημβρίαν εἰς τὸν Ὠκεανόν, frg. 2 Némethy), so die des Antiphanes nach dem äussersten Norden. Auch der Titel des Buches wird sich noch ermitteln lassen. Stephanos von Byzanz hat in dem Artikel Βέργη nach den üb-

¹ Susemihl Alex. Lit. Gesch. I 223. W. Schmid Art. Antiphanes in Wissowas Real-Encyklopädie I 2521 f. Uebrigens hat bereits Berger Die geogr. Frgm. des Eratosth. 43, 9 das bisher übereinstimmend dem Komiker Antiphanes gegebene Citat (Meineke Com. Gr. frg. III 160, Kock II 130) dem Bergäer überwiesen, ohne weitere Schlüsse zu ziehen.

the phenomena of nature? Certainly Galileo echoed the sentiments of Pythagoras when he wrote nearly 200 years later: "Mathematics is the alphabet with which God has written the universe" [3].

BIBLIOGRAPHY

1. ALLMAN, G. J., *Greek Geometry*, Dublin: Dublin University Press, 1889.
2. BALL, W. R., *A Short Account of the History of Mathematics*, London: Macmillan and Company, 1935.
3. DANTZIG, TOBIAS, *The Bequest of the Greeks*, Vol. I, New York: Charles Scribner's Sons, 1955.
4. EVES, HOWARD, *An Introduction to the History of Mathematics*, New York, Holt, Rinehart, and Winston, 1969.
5. LOOMIS, ELISHA, *The Pythagorean Proposition*, Washington, D. C.: National Council of Teachers of Mathematics, 1968.
6. MARKS, R. W., *The New Mathematics Dictionary and Handbook*, New York: Bantam Books, Inc., 1967.
7. SMITH, DAVID E., *History of Mathematics*, New York: Dover Publications, 1951.
8. WAERDEN, B. L., *Science Awakening*, New York: Oxford University Press, 1961.

MONK PARAKEET THREATENS U.S. CROPS

Handsome and charming—but a most destructive bird—the monk parakeet, which is a pigeon sized parrot, has escaped or been liberated from life as a pet and is now beginning to establish itself in the United States—cities, suburbs and woods.

An alarmed Federal Fish and Wildlife Service, which has learned that in Argentina the birds often ruin as much as 45 percent of crops of corn, sunflowers, millet, or fruit, is recommending a ban on importation of the bird, and is considering eliminating them wherever they can be found. More than 50,000 of them have been brought into the U.S. for sale as pets.

The Michigan Department of Natural Resources has joined the federal agency in the study and in a survey of the incidence and habits of the bird. Michigan is one of a half dozen states where the bird has been sighted.

The parakeet, a prolific breeder—one pair is reported to have raised 40 young in a single season—has a much higher potential for nuisance and damage than such imported pests as the starling and English sparrow.

These parakeets also have a history of being very aggressive toward other birds, and they might well decimate some American species.

Care must be taken in identifying monk parakeets. About a foot long, they are greenish gray above, with lemon yellow belly. Breast and forethroat are quaker gray (from which it may take its other name) with darker feather edges. Its wings are blue gray and its tail bluish green, long and pointed. Its more identifiable characteristics are the large hooked beak, and its resemblance to a parrot. Grosbeaks, which have heavy beaks like a cardinal, are smaller, but are often mis-identified as parakeets.

Parakeets have also been sighted in Massachusetts, Florida, Virginia, North Dakota, Puerto Rico, and New York City area.