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KOUREMENOS, T.
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THEOKRITOS KOUREMENOS / COLUMBUS, OHIO

Aristotle's Mathematical Matter and Eudoxus' Proportion Theory

Aristotle's mathematical matter, the ὅλη νοητή of mathematical objects, is the starting point of this paper, whose purpose is not only to propose a new interpretation of this notion but also to elucidate certain points in Aristotle's philosophy of mathematics: whether he thought of mathematics as axiomatizable¹ in accordance with his theory of science², and whether

¹ J. Hintikka, Reply to Dorothea Frede, *Synthese* 28 (1974), 94 denies the application of Aristotle's axiomatic model to mathematics because the only medium of deduction is the syllogism; cf. W. Leszl, *Mathematics, Axiomatization, Hypotheses*, in: E. Berti (ed.), *Aristotle on Science. The Posterior Analytics*, Padova 1981, 284, who proposes that the axiomatization of mathematics necessitates recourse to technical mathematical proof. Hintikka does not explain his thesis and, although I. Mueller, *Greek Mathematics and Greek Logic*, in J. Corcoran (ed.), *Ancient Logic and Its Modern Interpretations*, Synthese, Historical Library 9 (Dordrecht-Boston, 1974), 48–57, and J. Barnes, *Proof and the Syllogism*, in Berti 17–59, have shown the limitations of categorical syllogism as the logic of mathematical proof, this does not imply that Aristotle did not think of his syllogistic theory as the underlying logic of mathematics in some restricted sense; cf. J. Corcoran, *Aristotle's Natural Deduction System*, in Corcoran 92, D. Frede, *Reply on Hintikka's Paper 'On the Ingredients of an Aristotelian Science'*, *Synthese* 28 (1974), 88 and Barnes 41/42. A way to justify anachronistically Aristotle's apparent conviction that syllogistic is the underlying logic of mathematics seems to be provided by Corcoran and Smiley's (*What is a Syllogism?*, *JPhL* 2, 1973, 136–154) interpretation of Aristotle's syllogistic as a natural deduction system. J. Hintikka and U. Remes, *Ancient Geometrical Analysis and Modern Logic*, in R. S. Cohen, P. K. Feyerabend and M. W. Wartofsky, *Essays in Memory of Imre Lakatos*, Dordrecht-Boston 1976, 253–275 have shown the close relationship between ancient geometrical analysis and modern natural deduction systems, especially the method of semantic tableaux, developed by E. W. Beth, *Semantic Entailment and Formal Derivability*, *Mededelingen van de Koninklijke Nederlandse Akademie van Wetenschappen Afdeling Letterkunde N. R.* 18. 13 (1955). Now, the method of semantic tableaux can be used for the logical analysis of mathematical proofs and has certain similarities with Aristotle's „proof by ex-

such an axiomatization would entail the unification of geometry and arithmetic into a single science of mathematics³. A considerable part of the paper is devoted to the relationship between Eudoxus' proportion theory and Aristotle's answers to the above problems. Related to this topic is the mathematical background of Aristotle's concept of *συνεχές*⁴ and, last but not least, his often questioned mathematical competence⁵, as it is indicated by his *Physics*.

1. ὅλη νοητή and συνεχές

Aristotle's references to ὅλη νοητή⁶, the material substratum of mathematical objects, are usually understood as implying the abstract notion of

position"; see E. W. Beth, *The Foundations of Mathematics*, New York 1966, 190–192.

² For a negative answer to this question see W. Kullmann, *Die Funktion der mathematischen Beispiele in Aristoteles' Analytica Posteriora*, in Berti (above, n. 1), 253/254.

³ H. G. Apostle, *Aristotle's Philosophy of Mathematics*, Chicago 1952, claims that Aristotle conceived of a general science which incorporates proportion theory and the properties of the category of quantity. A similar approach is taken by C. B. Crawley, *Universal Mathematics in the Aristotelian-Thomistic Tradition: The Hermeneutics of Aristotelian Texts Relative to Universal Mathematics*, Washington 1980; for criticism see H. R. Mendell, *Aristotle and the Mathematicians: Some Cross-Currents in the Fourth Century* (Diss.), Stanford 1985, 229ff. However, it is not easily understandable what scientific value could such a general science of quantity, or „posology“, have. As it will be shown below (ch. 10 and 11), Aristotle was familiar with the mathematical research of his day and it seems quite improbable that he could have entertained a notion like „posology“, whose concrete mathematical meaning, if any, is hard to grasp.

⁴ The mathematical background of Aristotle's *συνεχές* is examined by H. J. Waschki, *Von Eudoxos zu Aristoteles. Das Fortwirken der Eudoxischen Proportionentheorie in der Aristotelischen Lehre vom Kontinuum*, Amsterdam 1978, and M. Caveing, *Quelques Remarques sur le Traitement du Continu dans les Éléments d'Euclide et la Physique d'Aristote*, in *Penser les Mathématiques*, Paris 1982, 145–166. Waschki and Caveing take the mathematics of Euclid's *Elements* as the background of Aristotle's *συνεχές*, which is thus put in an anachronistic historical context; see below n. 69.

⁵ See especially G. Milhaud, *Aristote et les Mathématiques*, AGPh 16 (1902/1903), 367–392.

⁶ The term ὅλη νοητή occurs only in Met. 1036a9/10, 1037a4, 1045a34 and 36. The first two passages deal with the problem in what sense a semicircle is part of a mathematical circle and Ps.-Alexander of Aphrodisias In Met. 510,3–5; 515,26–28 identifies ὅλη νοητή with extension. In Met. 1045a34 and 36 ὅλη νοητή is used for the generic element in a definition. For an attempt to connect the two uses of the term see W. D. Ross, *Aristotle's Metaphysics 2*, Oxford 1924, 199/200.

extensionality which is arrived at by means of mental abstraction and underlies the properties of mathematical objects. Thus indeterminate extension becomes the ultimate product of mathematical abstraction, the „material“ substratum upon which mathematical properties are superimposed⁷. Such an interpretation of ὅλη νοητή admits three genera of extensionality, indeterminate extension in one, two and three dimensions⁸. However, the distinction between one-, two- and three-dimensional genera, which are the matter of lines, planes and solids respectively, is problematic. Aristotle considers the ὅλη νοητή as an object of metaphysical inquiry (Met. 1059b14–21). If the mathematical matter is the abstract notion of indeterminate extension, why should metaphysics deal with it⁹? No apparent relationship between indeterminate extension in one, two or three dimensions and metaphysics suggests itself easily. Moreover, one of Aristotle's objection to the Academic mathematical matter is that, if mathematical objects have as their matter „the great and the small“, then two absurd consequences follow: either each dimension is irrelevant to the others or the distinction

His solution, that ὅλη νοητή „in its widest conception is the thinkable generic element“ an instance of which is extension, is accepted by T. L. Heath, *Mathematics in Aristotle*, Oxford 1949, 214 and elaborated by I. Mueller, *Aristotle on Geometrical Objects*, AGPh 52 (1970), 156–171; see also the extensive discussion by H. Happ, Hyle, Berlin-New York 1971, 581ff., J. Annas, *Aristotle's Metaphysics Books M and N*, Oxford 1976, 30–34, S. Gaukroger, *Aristotle on Intelligible Matter*, Phronesis 28 (1980), 187–197, and E. Hussey, *Aristotle's Physics Books III and IV*, Oxford 1983, 184. For a different interpretation of ὅλη νοητή see J. Lear, *Aristotle on Mathematics*, PhR 91 (1982), 179–183, who relates the notion of ὅλη νοητή to the constructions involved in geometrical proofs. Frère Augustin-Gabriel, S. G., *Matière Intelligible e Mathématique*, A, Laval Théologique e Philosophique 17 (1961), 173–196, esp. 191–196, relates the notion of ὅλη νοητή to continuity.

⁷ The notion of indeterminate extension is problematic. There is no indication that Aristotle conceived indeterminate extension as abstract mathematical space, although he lacked the notion of physical space, as Happ (above n. 6) 604/605 assumes; cf. H. G. Zekl, *Topos, Paradeigmata* 10, Hamburg 1990, 32/33. Phys. 204b20–22 does not support the assumption of an Aristotelian indeterminate extension in three dimensions; on the contrary, this passage denies the possibility of an infinitely extending body. It is important to note in this context that any assumption about mathematical space is absent from the Euclidean postulates; see A. Seidenberg, *Did Euclid's Elements, Book I, Develop Geometry Axiomatically?*, AHES 14 (1974/75), 272/273.

⁸ This interpretation, which depends heavily on Alexander's understanding of ὅλη νοητή (cf. the comments in Annas [above, n. 6] 30), is proposed by Mueller (above, n. 6).

⁹ Ross 2 (above, n. 6) 309 explicitly equates the ὅλη νοητή with space and offers to answer to this question; cf. Happ (above, n. 6) 612 and Annas (above, n. 6) 33 who simply state the fact without any explanation.

between dimensions is meaningless (Met. 1085a9–20)¹⁰. Aristotle's objection presupposes that dimensions are species of „the great and the small“ but, since according to the received interpretation of ὅλη νοητή the dimensions correspond to genera of indeterminate extension, the question whether Aristotle's ὅλη νοητή is immune to his objection against the Academic theory arises naturally. What is the relationship between the genera of indeterminate extension in one, two or three dimensions?

The obvious answer is that all of them encompass συνεχῇ μεγέθη and this suggests a modification in the received interpretation of ὅλη νοητή: the focus must shift from dimensionality to συνεχές¹¹. Aristotle talks about the „plane“ and „solid“ as genera of plane and solid figures (Met. 1024a36–b4) but taking these genera as the ὅλη νοητή, the ultimate product of mathematical abstraction, would deprive the ὅλη νοητή of its required generality as a material substratum¹², because, although all of them are συνεχῇ (DC 268a6–10), each is clearly distinct from the others¹³. Aristotle's description of mathematical abstraction in Met. 1061a28–b3 and 1061b21–25 puts the emphasis on συνεχές as the product of abstraction while dimensionality, especially in Met. 1061a28–b3, appears as a further characterization of συνεχές. However, there is no reason to assume that such a dimensional characterization plays any major role in the definition of συνεχές. ποσόν and συνεχές, the final product of abstraction in Met. 1061a28–b3, is defined in Met. 1020a7–11 without any reference to dimensionality which is treated in 1020a11/12 as a further characterization of συνεχές, albeit irrelevant to its definition. Moreover, the discussion of μέγεθος in DC 268a1–b10 shows that dimensionality is defined in terms of συνεχές. Thus συνεχές seems to be a notion prior to dimensionality and therefore it is reasonable to assume that συνεχές, without any reference to dimensionality, must have been considered by Aristotle as ὅλη νοητή; indeed, in Met. 1061b21–25 the οἰκεία ὅλη of mathematics are one-, two- and three-dimensional μεγέθη qua συνεχῇ.

2. ὅλη νοητή and the Genus of Mathematics

According to this interpretation, there is no need to distinguish among three genera corresponding to the three dimensions. ὅλη νοητή is only one

¹⁰ On this passage see Annas (above, n. 6) 184.

¹¹ Cf. Frère Augustin-Gabriel (above, n. 6) 190–193.

¹² Happ (above, n. 6) 612 recognizes that ὅλη νοητή must be „etwas überindividuelles“.

¹³ The implication that each dimension has its own „noetic“, i.e. material, substratum is clearly stated by Gaukroger (above, n. 6) 189.

genus, that of συνεχῇ μεγέθη. This modification avoids a further problem of the received interpretation which admits three genera as material substrata of geometrical objects but posits a single genus of geometry¹⁴. For Aristotle the genus of geometry comprises μεγέθη (An. Post. 76a42–b2) which are defined as συνεχῇ (Met. 1020a10–12, DC 268a1ff.) and it is noteworthy that in Met. 1061b2/3 Aristotle emphasizes the single genus of geometry after an enumeration of geometrical properties (Met. 1061a35–b2), the πάθη of this genus as πάθη of its objects qua ποσόν and συνεχές¹⁵, and the definition of the ultimate product of mathematical abstraction as ποσόν and συνεχές (Met. 1061a29–34). The context of Met. 1061a28–b3 shows a close relationship between the single genus of geometry and the definition of mathematical matter as ποσόν and συνεχές. Geometry is the paradigm of the science of being qua being which is possible because, although its objects belong to distinct genera, they are subsumed under the single description qua ὄντα (Met. 1061b3–17; cf. 1003a33ff.) which effects the single genus necessary for an Aristotelian science¹⁶. The analogy between geometry and the science of being presupposes that geometry's genus comprises objects belonging to distinct genera, i.e. the dimensional genera of Met. 1061a33/34, but nevertheless it is a single genus, and geometry a single science like the science of being (cf. Met. 1060b33–35), because the generically distinct objects fall under the single description qua συνεχές which, paralleling qua ὄν, does not take into account their differentiae, i.e. dimensionality. The fact that geometry studies the properties of generically, i.e. dimensionally, distinct objects, as the analogy between geometry and the science of being requires, is clearly shown by Met. 997a25–30 (cf. 997b14/15) and DC 268a30–b2. Moreover, the analogy implies that it is συνεχές, the mathematical matter, that generates the genus of geometry, since all dimensionally distinct objects are examined qua συνεχῇ, exactly as the science of being examines generically distinct objects qua ὄντα. But, if „qua συνεχές“ func-

¹⁴ See Mueller (above, n. 6) 167/168. Mueller accommodates the dimensionally distinct geometrical objects within the single genus of geometry by appealing to the general notion of μέγεθος that encompasses all geometrical objects. However, he fails to account explicitly for the common characterization of all geometrical objects as μεγέθη.

¹⁵ On the *qua* operator see W. Wieland, *Die Aristotelische Physik*, Göttingen 1962, 197–202, and Lear (above, n. 6), 162–175.

¹⁶ On the single genus of metaphysics effected by means of focal meaning see G. E. L. Owen, *Logic and Metaphysics in Some earlier Works of Aristotle*, in I. Düring and G. E. L. Owen, *Aristotle and Plato in the Mid-Fourth Century*, Göteborg 1960, 163–190; see also M. T. Ferejohn, *Aristotle on Focal Meaning and the Unity of Science*, *Phronesis* 25 (1980), 117–128.

tions like „qua ὄν“, then the mathematical matter has to be „something common“ (ἐν τι καὶ κοινόν, Met. 1061b12), apparently to the dimensionally distinct genera of geometrical objects. This is a requirement that the genera of indeterminate extension in one, two and three dimensions cannot meet. Thus, contrary to the received interpretation, the genus of geometry not only implies a single mathematical matter but it is also effected by it.

Why does Aristotle emphasize that geometry has a single genus in relation to the mathematical matter? Apart from the analogy he draws between geometry and the science of being, his comment bears on his conception of geometry and mathematics in general. The question is ultimately related to the meaning of συνεχές as the matter of mathematical objects which Aristotle relates to the unity of geometry's genus.

Up to this point, the product of mathematical abstraction in Met. 1061a28–b3 has been referred to almost exclusively as συνεχές, although Aristotle links συνεχές with ποσόν¹⁷. This conjunction appears contradictory. In Met. 1020a7–12 Aristotle distinguishes συνεχές as a sub-category of ποσόν thus precluding the synonymy of these terms. It is reasonable to assume that ποσόν in Met. 1061a33 and 34 means not συνεχές ποσόν¹⁸, but the latter holds for numbers whereas συνεχές applies to geometrical μεγέθη only and, despite this fundamental distinction, the πάθη in Met. 1061a35–b2 are exclusively geometrical. The same contradiction appears reversed in Met. 1061b21–25, where συνεχές alone characterizes ποσά, which include not only geometrical μεγέθη but also numbers¹⁹. Moreover, Aristotle's emphasis on συνεχές and ποσόν is striking; he defines συνεχές as διαιρετόν εἰς αἰεὶ διαιρετά²⁰ and ποσόν as simply διαιρετόν, but what is the point of reducing geometrical, and mathematical in general, properties to properties of objects qua συνεχή and ποσά in the above sense?

¹⁷ There is no reason to assume with Happ (above, n. 6) 604 n. 217 that ποσόν does not play any role in the understanding of this passage.

¹⁸ = πλῆθος according to Met. 1020a8–10; cf. Frère Augustin-Gabriel (above, n. 6) 191.

¹⁹ Qua συνεχές must pick out an essential characteristic of both numbers and μεγέθη, as qua ὄν does for the objects of metaphysics (the two terms are contrasted in Met. 1061b24), and thus it cannot imply the „mimetic“ characterization of numbers in terms of μεγέθη referred to in Met. 1020b3–6; on „linear“, „plane“ and „solid“ numbers see Heath (above, n. 6), 209.

²⁰ According to Waschkies (above, n. 4) 237–242 this is Aristotle's second characterization of the continuum developed in Phys. Γ ch. 4–8 and Phys. Z ch. 1/2. For Aristotle's first characterization of continua „als geordnete Strukturen“ in Top. Δ ch. 2 and Phys. E ch. 3 see Waschkies (above, n. 4) 198–220. On Aristotle's definition of the continuum see D. Bostock, Aristotle on Continuity in Physics VI, in L. Judson (ed.), Aristotle's Essays: A Collection of Essays, Oxford 1991, 180–188.

Particularly interesting is Aristotle's characterization of the entire mathematics in terms of συνεχές (Met. 1061b21–25). The context of this characterization is the paradigmatic role of mathematics for the science of being (Met. 1061b17–33) and, therefore, it implies that mathematics, like geometry in Met. 1061b2/3, has a single genus despite the distinct genera of its objects²¹. Aristotle's common characterization of geometrical μεγέθη and numbers as συνεχή is apparently to be understood in relation to the unity of the genus of mathematics. However, he distinguishes between the primitive terms and the πάθη of geometry and arithmetic respectively (An. Post. 76b1–9). Moreover, he emphasizes that the difference between geometrical and arithmetical proof results from the different genera of geometry and arithmetic; however, he allows for certain points of contact by stipulating that, in these limited cases, the μεγέθη are characterized as numbers (An. Post. 75a38–b20). Aristotle's stipulation implies that there are proofs, in which the primitive terms of geometry and arithmetic are identical, but, according to An. Post. 87a38/39, such an identification implies that geometry and arithmetic have the same genus. It seems that Aristotle is ambivalent about whether there exists a single genus of mathematics. Thus a common characterization of μεγέθη and numbers in terms of mathematical matter²² and, by implication, a common science of mathematics effected by it, appears problematic²³.

3. ὅλη νοητή and Eudoxus' Proportion Theory

In both Met. 1061a28–b3 and 1061b21–25 Aristotle states that geometry and mathematics examine mathematical properties of objects qua συνεχή. Among these properties he includes proportion-theoretic properties

²¹ The passage is not examined by Heath (above, n. 6) 222–224 under the heading „Universal Mathematics“ and it is passingly referred to by Mendell (above, n. 3) 237, although it is extremely important for the development of any argument about universal mathematics in Aristotle.

²² Frère Augustin-Gabriel (above, n. 6) 193 posits two kinds of intelligible matter, one for numbers and one for μεγέθη. This division corresponds to the division of mathematics into number theory and geometry and presupposes Frère Augustin-Gabriel's conviction that a unified mathematics, incorporating both arithmetic and geometry, would be impossible according to Aristotle's standards for demonstrative science (Matière Intelligible e Mathématique, B, Laval Théologique e Philosophique 18 [1962], 207–210; see, however, ch. 8 below).

²³ W. Fiedler, Analogiemodelle bei Aristoteles. Untersuchungen zu den Vergleichen zwischen den einzelnen Wissenschaften und Künsten, Amsterdam 1978, 57ff., and Kullmann (above, n. 2) 253, wrongly understand Met. 1026a26/27 as referring to „allgemeine Mathematik“. For the correct interpretation of this passage see Mendell (above, n. 3) 237/238.

(Met. 1061b1/2), which help understand the above two passages. In An. Post. 74a17–25 a proportion-theoretic theorem is proven for all mathematical objects as a property they have qua τοδί, irrespectively of their dimensional genus or whether they are μεγέθη or numbers. Aristotle refers to Eudoxus' reformulation of proportion theory and it is unlikely that his reference in Met. 1061b1/2 to proportion-theoretic properties proven as properties of objects qua συνεχῆ does not imply his reference in An. Post. 74a17–25 to the same properties proven as properties of objects qua τοδί, especially given that both qua συνεχῆς and qua τοδί do not take dimensionality into account. Indeed, συνεχῆς, like τοδί in An. Post. 74a21, makes all distinct geometrical μεγέθη and numbers ὀνομασμένον τι αὐτὰ πάντα ἔν. Leaving aside for the time being the meaning of συνεχῆς and τοδί²⁴ (see below § 10), the obvious analogy between An. Post. 74a17–25 and Met. 1061a28–b3 and 1061b21–25 explains two problematic features of the last two passages: first, the importance of qua τοδί consists in the absence of any dimensional distinction between geometrical μεγέθη, and thus it supports the above interpretation of συνεχῆς as a non-dimensional notion; second, since qua τοδί does not distinguish between numbers and geometrical μεγέθη, Met. 1061a35–b2 and 1061b21–25 do not involve any contradiction because συνεχῆς, as its conjunction with ποσόν indicates, covers both geometrical μεγέθη and numbers²⁵. If this is so, why are arithmetical properties properties of numbers qua συνεχῆ in Met. 1061b21–25? In what way does An. Post. 74a17–25 explain Aristotle's emphasis on the single genus of geometry and mathematics in Met. 1061b2/3 and 1061b21–25 respectively?

In An. Post. 74a17–25 the alternando property of proportion is said to have been proved by Eudoxus for lines, planes, solids and numbers as a property of these mathematical objects qua τοδί. Aristotle contrasts Eu-

²⁴ It is very important to determine the exact meaning of τοδί. Heath (above, n. 6) 44 and 223 proposed that τοδί stands for ποσόν or μέγεθος (cf. Waschkies [above] 143/144). This terminological interpretation of τοδί goes back to Philoponus who identified it with ποσόν; cf. J. P. Zabarella, *Opera Logica*, Hildesheim 1966, 727. However, as W. Knorr, *The Evolution of the Euclidean Elements*, Synthese Historical Library 15 (Dordrecht-Boston 1975), 291 n. 32 rightly observes, An. Post. 99a10 implies that Eudoxus' universal proofs require a common property that characterizes both μεγέθη and numbers; cf. Lear (above, n. 6) 167. The whole issue is not simply terminological but related to Eudoxus' new formulation of proportion theory and thus τοδί must imply a basic characteristic of it; cf. Mueller (above, n. 1) 46.

²⁵ Cf. Lear (above, n. 6) 167.

doxus' removal of any homogeneity restrictions²⁶ with the pre-Eudoxean proof in which dimensions, determining distinct εἶδη of lines, planes, solids and numbers, had to be treated separately. That the pre-Eudoxean approach implies a fragmentation of the genus of proportion theory into εἶδη of lines, planes, solids and numbers²⁷ is shown by An. Post. 85a37–b1, where proportion theorems are said to be proved as properties of mathematical objects not qua lines, planes, solids and numbers but qua παρὰ ταῦτά τι (An. Post. 85b1)²⁸. An. Post. 85a37–b1 simply restates An. Post. 74a17–25 and both passages clearly state a metamathematical result: the proof of the alternando theorem for all distinct mathematical objects qua τοδί is a universal (καθόλου) proof²⁹. But, if in An. Post. 74a17–25 and 85a37–b1 the unity of the genus of proportion theory is due to a metamathematical result about universal proofs of properties of objects qua τοδί, the unity of the genus of geometry in Met. 1061a28–b3 must entail an analogous metamathematical result³⁰: the properties of generically distinct mathematical objects are considered as properties of these objects qua ποσόν and συνεχῆς, which necessitates universal proofs of theorems about these properties³¹.

Met. 1005a19ff. and 1061b17ff. substantiate the above interpretation. In the former passage the set of common axioms is the κοινόν of the distinct genera of departmental sciences. The set of common axioms is the κοινόν of these genera qua beings. In the latter Aristotle focuses only on the category of ποσόν whose genera share the common axioms³². Aristotle contrasts the science of being with mathematics; mathematics is concerned with lines, angles and numbers qua συνεχῆ, not qua beings. The implication is that

²⁶ See I. Mueller, *Homogeneity in Eudoxus' Theory of Proportion*, AHES 7 (1970), 1–6.

²⁷ Cf. Mueller (above, n. 26) 1, T. L. Heath, *A History of Greek Mathematics*, Oxford 1965, 384 and (above, n. 6) 223.

²⁸ Which, however, must not be conceived as a Platonic independent entity; see Mueller (above, n. 6) 157 and Lear (above, n. 6) 165.

²⁹ On metamathematical results in Aristotle's logic see Corcoran (above, n. 1) 112–122.

³⁰ In An. Post. 87a38–b4 Aristotle relates explicitly the unity of the genus and proofs within this certain genus as conditions for the unity of a science. On the relationship between the genus' homogeneity and indemonstrable/demonstrable sentences referring to the same domain of objects see H. Scholz, *The Ancient Axiomatic Theory*, in J. Barnes, M. Schofield and R. Sorabji (eds.), *Articles on Aristotle 1*, London 1975, 53 and 62.

³¹ It is probable that Aristotle's argument for the unity of metaphysics involved a metaprinciple dealing with provability of statements within a genus and thus necessitating the existence of this genus; see Ferejohn (above, n. 16), 124–127.

³² See Ross 2 (above, n. 6) 314.

mathematical proofs are about mathematical objects not qua beings, because in this case mathematical proofs would not be mathematical as An. Post. 75b37–76a3 shows³³, but qua συνεχῆ. However, since proofs by means of the common axioms are universal proofs for all generically distinct beings qua beings, the complete analogy between the science of being and mathematics requires that mathematical proofs for all generically distinct mathematical objects qua συνεχῆ are universal proofs. Moreover, if qua συνεχῆ, like qua τοῦ, implies a metamathematical result which necessitates universal proofs, since in the science of being it is the set of common axioms that effects the universal proofs for objects qua beings, in mathematics the universal proofs for objects qua συνεχῆ must imply some indemonstrable principle(s) that characterize all dimensional genera and numbers.

4. Postulates and the Genus of Mathematics

According to the above interpretation, in Met. 1061a28–b3 and 1061b21–25 Aristotle generalizes a metamathematical result concerning particularly Eudoxus' proportion theory over the entire genus of geometry and mathematics in general. The proof of the alternando theorem within the pre-Eudoxean anthyphaeretic proportion theory³⁴ explains Aristotle's reasons for such a generalization. Modern reconstructions of the anthyphaeretic proof show that it must have treated the case of areas as primary and then a proof for the case of lines was derived from the case of areas³⁵. In Aristotle's terminology, the anthyphaeretic proof was neither κατὰ παντός nor καθ' αὐτό, as these notions are developed in An. Post. 73b25–74a3³⁶: if S^k and S^i are the species (An. Post. 74a20–21) of lines and numbers respectively in the Porphyrean tree of proportion theory's genus, the anthyphaeretic proof proves the alternando property separately, or derivatively (An. Post. 73b32–74a3), for S^k and S^i , and not universally for a S^i with $i < k$ and $i < l$. For Aristotle the anthyphaeretic proof, since it is not universal

³³ For a comprehensive discussion of Aristotle's reference to Bryson's attempt to square the circle see I. Mueller, Aristotle and the Quadrature of the Circle, in N. Kretzmann (ed.), *Infinity and Continuity in Ancient and Medieval Thought*, Ithaca 1982, 152–154 and 160–164.

³⁴ For the general characteristics of the anthyphaeretic proportion theory see Knorr (above, n. 24) 270/271.

³⁵ For the reconstruction of the anthyphaeretic proof see O. Becker, *Eudoxos-Studien I. Eine Voreudoxische Proportionslehre und ihre Spuren bei Aristoteles und Euklid*, OS B.2 (1933), 311–333, B.L. van der Werden, *Science Awakening*, Groningen 1954, 177/178, and Knorr (above, n. 24) 340–342.

³⁶ The following analysis is based on J. Barnes, *Aristotle's Posterior Analytics*, Oxford 1975, 120/121.

and καθ' αὐτό, lacks necessity, an indispensable characteristic of scientific demonstration (An. Post. 71b9–16, 73b25–28), and, therefore, the meta-mathematical importance of Eudoxus' proportion theory lies in establishing the required necessity of proportion-theoretic proofs. But, as EE 1222b23–41 shows, the necessity of mathematical propositions depends on certain indemonstrable principles, hypotheses, and, therefore, the necessity of proportion-theoretic propositions must imply such a principle, which characterizes all μεγέθη and numbers³⁷.

That Aristotle considered the anthyphaeretic proof not necessary but accidental, is clearly born out by another reference to Eudoxus' proportion theory in An. Post. 99a1–10, where, in mathematical terms, Aristotle discusses the question whether two proofs of the same theorem can be given. In such a context a reference to the anthyphaeretic and the Eudoxean proof of the alternando theorem is indeed fitting: the alternando theorem can be demonstrated separately for lines qua lines and numbers qua numbers. A universal demonstration of the alternando property can also be given as a property of all dimensionally distinct geometrical μεγέθη and numbers qua ἔχοντα αὐξήσιν τοιανδί. However, since for Aristotle only the universal proof qualifies as a scientific proof, according to the distinction he makes in An. Post. 99a1–6 the anthyphaeretic proof must have been considered by him as an accidental proof.

An. Post. 74a17–25 shows that the S^i , necessary for Eudoxus' universal proofs, must be ἀνώτερον παρὰ τὸ καθ' ἕκαστον (An. Post. 74a7–9), i. e. superordinate to all dimensional εἶδη of geometrical μεγέθη and the genus of numbers. But, since τὰ καθ' ἕκαστα are the εἶδη of lines, planes, solids and numbers, S^i must be the entire genus of proportion theory³⁸. Thus Eudoxus' universal proofs of proportion theoretic results presuppose a single genus of proportion theory superordinate to the particular genera of mathematical objects. This genus is required for the necessity of mathematical

³⁷ On the necessity of mathematical propositions see Leszl (above, n. 1) 293–305; cf. the brief comment in Heath (above, n. 6) 280.

³⁸ Pace Mendell (above, n. 3) 229–238. See also W. D. Ross, *Aristotle's Prior and Posterior Analytics*, Oxford 1949, 524. Philoponus assumes that the genus of proportion theory was identified as ποσόν; his conjecture is accepted by Heath (above, n. 6) 223 and elaborated by Apostle (above, n. 3) and Crawley (above, n. 3). It is obvious, however, that any relationship between the genus of proportion theory and ποσόν requires a formal specification of the meaning of ποσόν from a mathematical point of view. Otherwise, ποσόν is just an empty term which does not explain anything. For my interpretation of ποσόν and its relationship to Eudoxus' proportion theory see below ch. 7 and 8.

proofs, but one should also take into account the special condition for mathematical necessity laid out in EE 1222b23–41. Indeed, *qua* τοδί, which, paralleling *qua* ποσόν and συνεχές, effects *S*¹, is said in An. Post. 74a24/25 to be posited as a hypothesis (ὁ καθόλου ὑποτίθενται ὑπάρχειν). The Aristotelian hypotheses are equivalent to Euclid's postulates and thus it remains to examine in what way a postulate can effect *S*¹, the single genus of Eudoxus' proportion theory and, consequently, the single genus of geometry and mathematics in general³⁹.

In An. Post. 99a7–10 Aristotle considers all geometrical μεγέθη and numbers *qua* ἔχοντα αὐξήσιν τοιανδί⁴⁰ as ὥς ἐν γένει. From a comparison with An. Post. 74a17–25 it is clear that Aristotle refers to *S*¹, the single genus of Eudoxus' proportion theory in contrast with the fragmented genus of anthyphaeretic proportion theory. The same problem is brought up in An. Post. 75b5/6, where Aristotle discusses the applicability of arithmetical proofs to geometrical μεγέθη. He is aware that the genus of arithmetic is distinct from the genus of geometry (An. Post. 75b3–5 and 7/8); however, he does allow for such an applicability in some cases and in these cases

³⁹ For the relationship of Aristotelian hypotheses and the Euclidean postulates see H. P. D. Lee, *Geometrical Method and Aristotle's Account of the First Principles*, CQ 29 (1935), 113–124; cf. B. Einarson, *On Certain Mathematical Terms in Aristotle's Logic*, *AJP* 57 (1936), 33–54 and 151–172, and Heath (above, n. 6) 53–57. On the significance of hypotheses in Aristotle's conception of science see J. Hintikka, *On the Ingredients of an Aristotelian Science*, *Nous* 6 (1972), 66–69. For an extensive discussion of hypotheses pertaining to the axiomatization of mathematics see Leszl (above, n. 1) 305–315. Any relationship between the Aristotelian hypotheses and the Euclidean postulates has been denied by Ch. H. Kahn, *The Role on nous in the Cognition of First Principles in Posterior Analytics ii 19*, in Berti (above, n. 1) 391–393; cf. also Ross (above, n. 38) 59, who complains that Aristotle should have recognized proper principles corresponding to the Euclidean postulates. Kahn thinks that Aristotle's distinction between definitions and hypotheses, which are assertions of existence, does not leave room for postulates of the Euclidean type. K. von Fritz, *Die APXAI in der griechischen Mathematik*, *Archiv für Begriffsgeschichte* 1 (1955), 59ff. discusses extensively the relationship between Aristotle's hypotheses and Euclid's postulates and shows that the required link between the two is the conception of geometrical existence as constructibility; since Euclid's first three postulates are construction postulates that assert the constructibility of basic μεγέθη, whose existence is assumed by Aristotle's hypotheses, an interpretation of hypotheses as postulates is consistent with a basic characteristic of Greek geometry. On Aristotle's hypotheses as existence assumptions see B. Landor, *Definitions and Hypotheses in Posterior Analytics 72a19–25 and 76b35–77a4*, *Phronesis* 26 (1981), 308–318.

⁴⁰ For the mathematical meaning of this phrase see Knorr (above, n. 24) 291 n. 32.

geometrical μεγέθη are characterized as numbers (An. Post. 75b3–6)⁴¹ paralleling the characterization of numbers as συνεχή, i.e. μεγέθη, in Met. 1061b21–25. There can be no doubt that the cases Aristotle has in mind are, or at least include, Eudoxus' universal proportion-theoretic proofs but, since μεγέθη and numbers are distinct genera, the applicability is allowed only if μεγέθη and numbers are „one genus in some sense” (An. Post. 75b8/9), i.e. if there is a *S*¹, as in An. Post. 99a7–10. Since Aristotle stipulates that a demonstrative transition from one genus to another presupposes that the two genera share something κοινόν (An. Post. 75b20), which is their proper principles (An. Post. 76a15), it is reasonable to assume that, as An. Post. 87a38–b4 shows, *S*¹, „one genus in some sense” over distinct genera (= ὥς ἐν γένει), presupposes that *S*¹, *S*¹ and all other genera share proper principles. The proper principles of a science include hypotheses (An. Post. 93b23) and thus a universal proof requires a postulate which establishes its necessity according to EE 1222b23–41.

The set of principles that necessitate *S*¹, a requirement for the demonstrative transition of a universal proof, supports the above interpretation of mathematical matter as causally related to the single genus of mathematics, its οἰκεία ὕλη, and implying a set of indemonstrable principles that characterize the distinct genera of geometrical μεγέθη and numbers. If *qua* ποσόν and συνεχές (Met. 1061a34/35; 1061b24) = *qua* τοδί (An. Post. 74a24) = *qua* ἔχον αὐξήσιν τοιανδί (An. Post. 99a10; see below § 10), the κοινόν of the distinct genera of geometrical μεγέθη and numbers according to An. Post. 75b20, imply a postulate which characterizes all these genera, these genera are ὥς ἐν γένει in the sense that all are true interpretations of the same postulate⁴². Thus in Met. 1061a28–b3 and 1061b21–25 the distinct

⁴¹ See Knorr (above, n. 24) 310. In view of Eudoxus' proportion theory I cannot agree with Barnes (above, n. 36) 129 that Aristotle's characterization of geometrical μεγέθη as numbers is „a jocular hypothesis”. Heath (above, n. 6) 45 does not relate this passage to Eudoxus' proportion theory. Kullmann (above, n. 2) 253/254 construes this passage as excluding the possibility of a common characterization of μεγέθη and numbers; thus he understands Aristotle's reference to Eudoxus' universal proofs as an indication that for Aristotle mathematics is compatible with his theory of science, which depends on the homogeneity of the genus.

⁴² On interpretations in Aristotle's logic see Corcoran (above, n. 1) 103–107. For numbers and all classes of geometrical μεγέθη as isomorphic models of an axiom system in Eudoxus' proportion theory see F. Beckmann, *Neue Gesichtspunkte zum 5. Buch Euklids*, *AHES* 4 (1967), 31; Beckmann, however, considers the fifth book of Euclid's *Elements* as representing Eudoxus' proportion theory (see below n. 69). A universal proof of the alternando theorem for all true interpretations of an axiom is a mathematically more palpable notion than the „proof by analogy” to which Apostole (above, n. 3) 65 and Mendell (above, n. 3) 232/233 appeal.

genera of geometrical and mathematical objects belong to a single genus, that of geometry and mathematics respectively, as true interpretations of a postulate⁴³. Moreover, since the single genus of these sciences is causally related to the product of mathematical abstraction, the mathematical matter, the latter must have been conceived by Aristotle as a true interpretation of a postulate: that geometry and mathematics examine lines, planes, solids and numbers qua ποσόν and συνεχές, means that all these genera are true interpretations of a postulate which is implied by ποσόν and συνεχές. These interpretations are mathematical matter in the sense of 'explanatory' or 'demonstrative' matter for mathematical theorems, because, as An. Post. 74a17–25 shows, it is these interpretations which necessitate the universal proofs of theorems⁴⁴. Thus it turns out that Aristotle's generalization of the metamathematical property of Eudoxus' proportion theory over the entire genus of geometry, and especially mathematics, suggests a foundational unification of geometry and arithmetic under an axiomatic system that necessitates universal proofs of the type Eudoxus introduced into proportion theory⁴⁵ (see below § 9).

5. ὅλη νοητή and Metaphysics

If S' is a model-theoretic structure⁴⁶, then the notion of mathematical matter is a fitting subject for the science of being. Because, since the genera of departmental sciences qua beings are characterized by the set of common axioms whose true interpretations are all genera, a model-theoretic structure similar to that of mathematics underlies the science of being. Aristotle is, therefore, right in saying that the mathematical matter does not belong to

⁴³ Cf. Scholz (above, n. 30) 59 n. 26. Corcoran (above, n. 1) 103/104 emphasizes the absence of any mention of alternative interpretations in Aristotle's logical theory but one should also take into account sciences like mathematics and metaphysics whose genus comprises distinct genera; see also below ch. 6.

⁴⁴ On material explanation see Barnes (above, n. 36) 215/216.

⁴⁵ It is usually assumed that Aristotle's conception of axiomatic science influenced Euclid's *Elements*; see von Fritz (above, nr. 39) 45–63 and 99–103, A. Szábo, *Anfänge des Euklidischen Axiomen-Systems*, AHES 1 (1960), 104/105, K. Berka, *Aristoteles und die Axiomatische Methode*, *Altertum* 9 (1963), 200–205, and I. Mueller, *Euclid's Elements and the Axiomatic Method*, *British Journal for the Philosophy of Science* 20 (1969), 289–309. See, however, below n. 90.

⁴⁶ The validity of the alternando theorem for all distinct mathematical objects is guaranteed by the „Principle of Form“; see the comments in Scholz (above, n. 30) 59 n. 26 and the discussion in Corcoran (above, n. 1) 105/106 (cf. also 107/108). It is interesting to note that Euclid does not employ the Principle of Form; see Mueller (above, n. 1), 43/44.

the science dealing with „demonstration and knowledge“ (Met. 1059b14–20). All references to Eudoxus' proportion theory, and thus to the mathematical matter, in the *Analytics* are from a model-theoretic point of view because they are concerned with universal proofs. Although Aristotle realizes that universal proofs require a model-theoretic structure that subsumes distinct genera under „one genus in some sense“, it is in the *Metaphysics* that this requirement becomes of primary importance for the science of being⁴⁷. Moreover, Aristotle's usual example of a common axiom, which belongs to the domain of metaphysical inquiry, is an axiom pertaining only to the 'mathematical' category of quantity (equals subtracted from equals leave equals)⁴⁸, and, since the mathematical matter implies a postulate common to all branches of mathematics, this notion is an appropriate subject for metaphysics.

Although in Met. 1061a28–b33 Aristotle draws an analogy between the unified genus of mathematics and the unified genus of the science of being, it is impossible that he would have considered a unified genus of mathematics as a completed result⁴⁹. In An. Post. 76b3–6 he restricts the demonstrative transition of an arithmetical proof to geometrical μέγεθη in some cases only, which definitely include proportion-theoretic results like the alternando theorem, and elsewhere he maintains the formal distinction between μέγεθη and numbers⁵⁰. The common characterization of μέγεθη and numbers in Met. 1061a28–b33 can be explained by assuming that, under the influence of Eudoxus' reformulation of proportion theory, Aristotle envisaged a unification of all branches of mathematics under an axiomatic system that characterizes all their genera in accordance with his theory of science⁵¹. The start had been made by Eudoxus whose proportion

⁴⁷ Cf. Mendell (above, n. 3) 238.

⁴⁸ This is Euclid's third Common Notion. The fact that the third Common Notion is more at home in the fifth book of the *Elements*, a development of Eudoxus' proportion theory (see below n. 69), than in the first leads van der Waerden (above, n. 35) 183 and Seidenberg (above, n. 7) 273–279 to the conclusion that the third Common Notion was originally formulated within Eudoxus' proportion theory. This hypothesis is attractive: it provides a further relationship between Eudoxus' proportion theory and Aristotle's metaphysics and shows that Eudoxus had an interest in establishing general principles.

⁴⁹ Cf. Aristotle's awareness that his metascientific argument in favor of the unified genus of metaphysics cannot be unqualifyingly conclusive; see Ferejohn (above, n. 16) 126.

⁵⁰ Cf. Knorr (above, n. 24) 310.

⁵¹ There are, however, indications that Aristotle considered the majority of the sciences complete or almost complete; see J. Barnes, *Aristotle's Theory of Demonstration*, in Barnes, Schofield and Sorabji (above, n. 30) 85 n. 88.

theory proved theorems for all distinct genera of mathematical objects as true interpretations of at least one axiom. Moreover, Eudoxus' proportion theory most probably resulted from his reexamination of theorems from different branches of mathematics⁵² and thus it was a theory with a wide range of applications that could have inspired foundational speculations. It is quite possible that Eudoxus himself might have shared, or even initiated, such speculations⁵³. But one cannot venture further than that.

6. Aristotle's notion of interpretation

The modern notion of interpretation should not be foisted upon Aristotle. A single science is one which has a single genus and a single genus is defined by the primitive terms (An. Post. 87a38–b1). If the genera of geometry and arithmetic coalesce into a single genus of mathematics, their primitive terms, *μεγέθη* and numbers respectively (An. Post. 76a43–b5), have to be identical⁵⁴. According to An. Post. 76a31–34, this implies the identification of proper geometrical and arithmetical principles that assume the existence and meaning of *μεγέθη* and numbers respectively⁵⁵. This is Aristotle's only implicit definition of interpretation and his concept of a single genus of mathematics must necessarily conform with it; his stipulation that the demonstrative transition requires a set of proper principles common to both genera suggests that this is the case and, therefore, Aristotle's characterization of *μεγέθη* and numbers qua *συνεχῆ* in Met. 1061b21–25 must imply such a set (see below § 8). A further desideratum is to explain Aristotle's misgivings about the feasibility of a foundational unification of arithmetic and geometry (see below § 9). Moreover, the historical background of Ar-

⁵² See Knorr (above, n. 24) 261–288.

⁵³ For supporting evidence see above, n. 48. On Eudoxus and the formalization of proportion theory see Knorr (above, n. 24) 275. A drive to axiomatic mathematics could have been related to Eudoxus' stay in Plato's Academy; see G. E. L. Owen, *The Platonism of Aristotle*, in Barnes, Schofield and Sorabji (above, n. 30) 26–28.

⁵⁴ See the analysis of An. Post. 87a38–b4 in Barnes (above, n. 36) 183; cf. J. van Rijen, *Aspects of Aristotle's Logic of Modalities*, Dordrecht-Boston-London 1989, 177–179.

⁵⁵ Scholz (above, n. 30) 62 is mistaken in asserting that „we ... have to extend the notion of a domain and characterize it ultimately by the totality of the arguments whose substitution for the free variables in a formalized axiom-system makes these axioms into true sentences“. Scholz understands such an extension as a characterization of primitive terms but An. Post. 87a38–b4 shows that Aristotle starts with the identification of the primitive terms of these domains in which propositions are true; cf. Corcoran (above, n. 1) 103/104 on the notion of intended interpretation in Aristotle's logic.

istotle's hints at a single science of mathematics should not be lost sight of. The validity of the above interpretation requires that *ποσόν* and *συνεχές* in Met. 1061a28–b2 and 1061b21–25 bear the same meaning with *τοδί* in An. Post. 74a24 (see below § 10); *ποσόν* and *συνεχές* and *τοδί* must be shown to imply a basic characteristic of Eudoxus' proportion theory, a postulate that necessitates the universal proofs of his theory. According to the Aristotelian theory this postulate has to be an existence postulate and it must also yield a definition of the primitive terms of both geometry and arithmetic, *μεγέθη* and numbers respectively (see below § 8).

7. *ποσόν*, *συνεχές* and Eudoxus' 'Bisection Principle'

Aristotle defines both *ποσόν* and *συνεχές* in terms of division (Met. 1020a7–11). However, whereas *ποσόν* is simply *διαιρετόν*, *συνεχές* is defined as *τὸ εἰς ἄπειρα διαιρετόν* (Phys. 200b17–20) and in Phys. 206b7–20 and 266b10–20 Aristotle expresses infinite divisibility by means of a principle that allows to exceed *παντός ὁρισμένου [μεγέθους]* (Phys. 206b19/20; cf. 206b18/19, 266b3/4 and 266b19/20) by successive removal of a proportional part, which in Phys. 266b11/12 is the half. The principle Aristotle employs to express the infinite divisibility of *συνεχές* has been identified as 'the Bisection Principle' (henceforth BP) which played a cardinal role in Eudoxus' proportion theory⁵⁶. Unlike the Euclidean convergence lemma (El. Prop. 10, 1), Eudoxus' BP was an unproven assumption⁵⁷ and thus Aristotle's mathematical matter implies indeed a postulate which, according to An. Post. 74a17–25, has the significant foundational advantage of effecting universal proofs of proportion-theoretic theorems: indeed, it is BP that allows Eudoxus to dispense with the anthyphaeretic definition of proportion⁵⁸, which, as shown above, not only is problematic from a mathematical point of view but also conflicts with Aristotle's requirement of formal necessity for scientific propositions⁵⁹.

⁵⁶ For a comprehensive discussion of BP and its relation to Eudoxus' proportion theory see W. R. Knorr, *Archimedes and the Pre-Euclidean Proportion Theory*, *AIHS* 28 (1978), 200–205 (see also 210/211 for Aristotle's use of BP). Aristotle's consistent use of the verb *ὑπερβάλλειν* and the phrase *παντός ὁρισμένου [μεγέθους]* in his enunciations of BP indicates that he refers to the wording of a particular formulation of the principle.

⁵⁷ See Knorr (above, n. 56) 205ff. and *Infinity and Continuity: The Interaction of Mathematics and Philosophy in Antiquity*, in Kretzmann (above, n. 33) 124/125.

⁵⁸ See Knorr (above, n. 56) 193 and cf. (above, n. 24), 272/273.

⁵⁹ According to my interpretation of *ὑλη νοητή* as a true interpretation of BP which necessitates a formal proof, a semicircle is indeed the *ὑλη νοητή* of a circle, as Aristotle takes it in Met. 1036a9/10 and 1037a4, because certain theorems about

There is, however, one fundamental difference between the Aristotelian formulation of BP and its use by Eudoxus in his exhaustion method, where BP most probably originated from⁶⁰: whereas the Eudoxean formulation of BP avoids any reference to infinite continuation of division⁶¹, Aristotle's formulation in Phys. 206b7–20 and the other passages of the Physics presuppose the potentially infinite continuation of division. But according to Met. 1020a11 the potentially infinite divisibility of *συνεχές* characterizes only *μεγέθη*, whereas in Met. 1061b23–25 it extends to numbers, too. For Aristotle the axiomatization of mathematics depends on this common characterization of *μεγέθη* and numbers. The problem that arises is to explain in what sense *μεγέθη* and numbers are characterized by BP and can be analyzed into two particular questions: what formal characterization of both *μεγέθη* and numbers is entailed by BP and what is the foundational importance of potentially infinite divisibility.

In Met. 1061b23–25 numbers and *μεγέθη* are *ποσά*. According to Met. 1020a7/8, this means that both are simply divisible, without any implication of finite or infinite divisibility that distinguishes numbers and *μεγέθη* respectively. Thus *ποσόν*, the sheer assumption of divisibility, seems to have the distinctive mark of Eudoxus' formulation of BP as is applied in exhaustion proofs; indeed, since *συνεχές* is based on BP and, as Met. 1020a7–11 shows, the only difference between *ποσόν* and *συνεχές* is that the latter implies potentially infinite divisibility, and moreover, *συνεχές* is *ποσόν* but not necessarily vice versa, *ποσόν* must hark back to the original mathematical formulation of BP as a principle of division to which Aristotle's potential infinity is irrelevant. In order now to solve the problem of the common characterization of numbers and *μεγέθη* by BP, it remains to see if Aristotle operates with the Eudoxean formulation of BP to which the Aristotelian notion of potential infinity is adjoined and, moreover, what is the link between Eudoxus' BP and its reformulation by Aristotle as a characterization of *συνεχές* that covers both numbers and *μεγέθη*.

the circle are proved by application of BP, which divides the arc of a circle. It cannot be accidental that, as Archimedes notes in the preface to his *Quadrature of the Parabola*, Eudoxus was the first to prove formally the theorem that circles are to one another as the squares on their diameters (El. Prop. 12, 2) by means of BP, which successively divides arcs of a circle by bisecting the sides of an inscribed polygon; see Knorr (above, n. 56) 200 and I. Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, Cambridge, Mass.-London 1981, 200/201.

⁶⁰ See Knorr (above, n. 56) 200–205 and (above, n. 57) 123ff.

⁶¹ See Knorr (above, n. 57) 127.

If both numbers and *μεγέθη* qua *ποσά* are characterized by BP, BP must not appeal to infinite divisibility. From Phys. 206b3–12 it is clear that finite and infinite divisibility are distinguished by means of a finite and infinite sequence respectively, depending on the common ratio⁶². Aristotle's infinite sequence is not an arbitrary sequence but an approximation to a *μέγεθος* given beforehand (Phys. 206b16–20), and thus it exhibits a basic feature of Eudoxus' exhaustion method⁶³. Now, BP allows Eudoxus to dispense with any reference to the infinity of such a sequence of proportional terms. If Aristotle operates with a formulation of BP that does not refer to infinite divisibility which characterizes only *μεγέθη*, he must follow the exhaustion method in his formal handling of divisibility.

Indeed, in Phys. 232b20–233a12, although he sets out to prove that time is *συνεχές*, i. e. *ad infinitum* divisible (Phys. 232b23–26), and his aim calls for the construction of the infinite sequence in Phys. 206b7–9, he performs only a finite number of divisions and then he generalizes by appealing to the proportionality of these divisions (Phys. 233a3–5; for a detailed analysis of this proof see below, § 11). In Phys. 207b27–34 Aristotle refers to proportional division as a recent mathematical development that makes the assumption of actual infinities unnecessary in mathematical proofs: by means of proportional division, given any *μέγεθος* divided in a certain ratio, the existence of another, arbitrarily small, *μέγεθος* is guaranteed⁶⁴. The indirect handling of infinite sets of proportional terms by means of their proportionality alone, without any direct appeal to their infinity, is a mark of the exhaustion method and shows that Aristotle operates with a formulation of BP which *per se* does not imply either actual infinity or potentially infinite divisibility⁶⁵. For obvious reasons Aristotle's proof in Phys. 232b20–233a12 cannot imply a finite formulation of BP; the same holds for Eudoxus' exhaustion method, because in that case his reasoning

⁶² See the discussion of this passage in Heath (above, n. 6) 108–110.

⁶³ See Mueller (above, n. 59) 235/236.

⁶⁴ Pace J. Hintikka, *Aristotelian Infinity*, PhR 75 (1976), 197–218, and Knorr (above, n. 57) 122 n.22, Heath, (above, n. 6) 111/112, is right in relating Aristotle's comment to Eudoxus' exhaustion method. *vũv* in Phys. 207b30 must refer to a recent mathematical development; cf. *vũv* in An. Post. 74a23 which introduces Eudoxus' new proof of the alternando theorem. Since Aristotle says that mathematical proofs avoid any assumption of infinity by means of a technique that combines division and proportionality, he can only refer to Eudoxus' exhaustion method.

⁶⁵ The case against any philosophical influence on Eudoxus' introduction of BP and his method of exhaustion has been argued plausibly by Knorr (above, n. 57) 123–135; cf. Mueller (above, n. 59) 123–135.

would fall victim to the geometrical naiveté of Antiphon's attempt to square the circle by employing an intuitive precursor of BP⁶⁶.

Thus Aristotle seems to admit two formulations of BP: one that avoids any assumption of infinite divisibility by means of proportionality and is related to ποσόν (Met. 1020a7/8), and one that implies potentially infinite divisibility, applicable only to μεγέθη and related to συνεχές (Met. 1020a10/11). Aristotle's conjunction of ποσόν and συνεχές in Met. 1061a28–b3 implies the two formulations of BP which, coupled together as in the proof at Phys. 232b20–233a12, exemplify two diverse Aristotelian tendencies: on the one hand mathematical finitism⁶⁷, which according to the above interpretation finds a parallel in Eudoxus' exhaustion method and is thus related to formal mathematical proof, and on the other admission of infinity in a certain (potential) sense required for the geometrical analysis of physical processes among other things⁶⁸. As it will be shown below in §

⁶⁶ For an analysis of Antiphon's argument see Knorr (above, n. 57) 130–132 and Mueller (above, n. 33) 154–156. A conception of BP that bypasses considerations of finite or infinite continuation of division explains Aristotle's problematic reference to general mathematical proofs in Met. 1077b17–20. According to this passage universal mathematical proofs treat μεγέθη and numbers neither as „possessing μέγεθος“ nor as „being divisible“. As Ross (above, n. 6) 416 observes, it is odd that Aristotle seems to do away with divisibility, although it can be an essential property of all mathematical objects; cf. Heath (above, n. 6) 223. Indeed, Aristotle does characterize both μεγέθη and numbers in terms of divisibility, which, as ποσόν and συνεχές, becomes of paramount importance for the axiomatization of mathematics; he simply filters out the specific kind of divisibility that characterizes only numbers or only μεγέθη. „Being divisible“ must be a peculiar characteristic of numbers, in contrast to „possessing μέγεθος“. But, since μέγεθος implies infinite divisibility, according to Met. 1020a7–11 „being divisible“ must be a lax way of referring to the finite divisibility of numbers. According to this reading, Met. 1077a17–20 does not undermine the interpretation of universal mathematical proofs as related to divisibility obtained by a formulation of BP that avoids any assumption about finite or infinite divisibility.

⁶⁷ Aristotle's mathematical finitism is extensively discussed by Hussey (above, n. 6) 93–96 and 178/179. Hussey, however, mistakenly considers Eudoxus' proportion theory and his method of exhaustion as involving assumptions about infinitely large totalities; see Knorr (above, n. 57) 123–135 and Mueller (above, n. 59) 233/234. It is more plausible that Aristotle's finitism did not provide an alternative basis for the mathematics of his day but was indebted to the mathematical practice of his day and especially to Eudoxus' methods which avoid any assumption about infinite totalities.

⁶⁸ Knorr (above, n. 57) 121/122 considers Aristotle's potential infinity as an attempt to save the concept of infinity in the face of a movement among his contemporary mathematicians, like Eudoxus, to give up that concept. As Aristotle's mathematized definition of physics in Phys. 202b30ff. shows, the only reason he had

8, the admission of infinity is required not only for physical applications but also for purely mathematical considerations that pertain directly to the Aristotelian conception of the foundational unity of mathematics.

Similarly, in Met. 1061b21–25 all μεγέθη and numbers are ποσά, true interpretations of BP in its Eudoxean formulation, but mathematics examines them qua συνεχῆ, i.e. true interpretations of BP in its Aristotelian formulation applicable to μεγέθη only. Leaving aside potential divisibility, the only difference between Eudoxus' and Aristotle's formulation of BP, for the time being, ποσόν and συνεχῆ must presuppose a common characterization of both μεγέθη and numbers. The infinite sequence of proportional terms in Phys. 206b7–9 and Aristotle's proof in Phys. 232b20–233a12 show that συνεχές presupposes the notion of proportionality; his brief reference to Eudoxus' exhaustion method in Phys. 207b27–34 shows that the Eudoxean application of BP depends on proportionality, too. Since BP plays a central role in Eudoxus' proportion theory, which treats commensurable μεγέθη as primary, proving the same result for incommensurable μεγέθη by means of an indirect argument that leads to a contradiction⁶⁹, it seems that the required meeting point of the Aristotelian and the Eudoxean formulation of BP must be Eudoxus' characterization of proportionality. Eudoxus' effective definition of proportionality is Euclid's parts-conception of proportion among integers (El. 7, Def. 20)⁷⁰ and thus it accounts for the homogenous treatment of numbers and geometrical μεγέθη as μεγέθη in an extended sense of the term that Aristotle requires in An. Post. 75b5/6. Now if it is taken for granted that, as will be shown in § 10, Aristotle operates with Eudoxus' arithmetical notion of proportionality, the concomitant application of BP is bound to divide the proportional μεγέθη into commensurable parts, so that the μεγέθη can be treated as either geometrical μεγέθη or numbers (see below § 10). The above interpretation of ποσόν as implying Eudoxus' formulation of BP that does not commit one to an infinite sequence, shows that, on these premisses, Aristotle's characterization of numbers and μεγέθη as ποσόν is justified.

to retain the infinite as potential infinite was the analysis of physical processes; for συνεχές and infinite as a prerequisite of motion see Wieland (above, n. 15) 278ff. For Aristotle's mathematical physics see Hussey (above, n. 6) 185–200 and Aristotle's *Mathematical Physics: A Reconstruction*, in Judson (above, n. 20) 213–242.

⁶⁹ For these and other characteristics of Eudoxus' proportion theory see Knorr (above, n. 56) 190/191. Based on certain theorems of Archimedes and Pappus, Knorr has plausibly reconstructed Eudoxus' proportion theory and has shown that the proportion theory of Euclid's *Elements* Book V is an evolution of Eudoxus' proportion theory.

⁷⁰ See Knorr (above, n. 56) 190 and 218.

The above discussion accounts for Aristotle's use of $\pi\alpha\sigma\acute{o}\nu$ in Phys. 237b23–238a29. In Phys. 237b30/31 and 238a31 $\pi\alpha\sigma\acute{o}\nu$ is used for the number of times a $\mu\acute{\epsilon}\rho\omicron\varsigma$ measures a $\mu\acute{\epsilon}\gamma\epsilon\theta\omicron\varsigma$; the same expression is used in Met. 1020b5/6 for plane and solid numbers. In Met. 1023b12–17 Aristotle defines $\mu\acute{\epsilon}\rho\omicron\varsigma$ ⁷¹ as the product of the division of $\pi\alpha\sigma\acute{o}\nu$ and the arithmetical nature of both $\pi\alpha\sigma\acute{o}\nu$ and the notion of measurement is shown by Met. 1052b20ff. and Phys. 221a26ff. (esp. 221b14–20). Moreover, in Phys. 250a17, 30, 31 and b2 $\pi\alpha\sigma\acute{o}\nu$ is used for $\mu\acute{\epsilon}\gamma\epsilon\theta\eta$ that enter into proportional relationships⁷². The term is used in the same sense in Phys. 237a30/31 and 238a23, in proofs which presuppose the proportionality of the uniform motion theorem, most probably established by Eudoxus in his work *On Speeds* (see below § 11), and employ techniques compatible with Eudoxus' proportion theory⁷³. Besides Aristotle's arithmetical treatment of $\mu\acute{\epsilon}\gamma\epsilon\theta\eta$, it is interesting to note that Aristotle simply assumes the existence of a $\mu\acute{\epsilon}\rho\omicron\varsigma$ that measures a given $\mu\acute{\epsilon}\gamma\epsilon\theta\omicron\varsigma$. This assumption, the taking of the m-part of a $\mu\acute{\epsilon}\gamma\epsilon\theta\omicron\varsigma$, is tacitly used by Euclid in his proof of El. 5, 5⁷⁴ and its use by Aristotle is justified, if he presupposes Eudoxus' formulation of BP, which, as Phys. 207b27–34 shows, guarantees the existence of any m-part, however small; indeed, in Eudoxus' proportion-theoretic proofs BP is used to derive the existence of a required $\mu\acute{\epsilon}\gamma\epsilon\theta\omicron\varsigma$ assumed to be commensurable with a given $\mu\acute{\epsilon}\gamma\epsilon\theta\omicron\varsigma$ ⁷⁵.

A way, albeit derivative, in which Aristotle feels free to characterize numbers as $\sigma\upsilon\nu\epsilon\chi\acute{\epsilon}\varsigma$ becomes now apparent: $\sigma\upsilon\nu\epsilon\chi\acute{\epsilon}\varsigma$ depends on the Aristotelian version of BP, which, besides Eudoxus' BP, captures his arithmetical notion of proportionality and, moreover, introduces potential infinity in lieu of actual infinity that the Eudoxean formulation of BP avoids. As it is, at the root of Aristotle's $\sigma\upsilon\nu\epsilon\chi\acute{\epsilon}\varsigma$ lies an arithmetical characterization of proportionality. The analysis of the proof in Phys. 232b20–233a12 (below

⁷¹ $\mu\acute{\epsilon}\rho\omicron\varsigma$ can also be used in the arithmetical sense of „an aliquot part“ (cf. El. Def. 7, 3); see Ch. Mugler, *Dictionnaire Historique de la Terminologie Géométrique des Grecs*, Paris 1959, sub voce $\mu\acute{\epsilon}\rho\omicron\varsigma$. On the Euclidean notion of $\mu\acute{\epsilon}\rho\omicron\varsigma$ see Mueller (above, n. 59) 61/62; cf. D. H. Fowler, *The Mathematics of Plato's Academy: A New Reconstruction*, Oxford 1990, 226–228.

⁷² On the proportionality in Phys. 249b7–250a25 see Hussey (above, n. 68) 215f. Crawley (above, n. 3) 168/169 proposes that $\pi\alpha\sigma\acute{o}\nu$ might mean „quantum proportional in a proportionality“.

⁷³ See Knorr (above, n. 56) 211 n. 80.

⁷⁴ See Mueller (above, n. 59) 122.

⁷⁵ See Knorr (above, n. 56) 187 and 189, for further applications of this Eudoxean technique in Aristotle's *Physics* see 233a35–b3 (cf. Knorr (above, n. 56) 197 n. 57) 237b28, 238a6/7, 238a11/12 and a21, 266b21–24.

§ 11) shows that Aristotle employs both the BP and proportionality, exactly as Eudoxus does in his exhaustion method, in order to derive potentially infinite divisibility, required for the analysis of physical processes, by means of a formal mathematical method that avoids any assumption of actual infinity. However, the mathematical method is arithmetical.

8. Numbers and $\sigma\upsilon\nu\epsilon\chi\acute{\epsilon}\varsigma$

In Phys. 207a33–b21 Aristotle also implies a much closer connection between numbers and $\sigma\upsilon\nu\epsilon\chi\acute{\epsilon}\varsigma$ that demonstrates the importance he attaches to his own formulation of BP for the foundational unification of numbers and geometrical $\mu\acute{\epsilon}\gamma\epsilon\theta\eta$ into the single genus of mathematics. For Aristotle, a number is a finite set of units and the possibility of finding a greater set is grounded in the potentially infinite divisibility of $\mu\acute{\epsilon}\gamma\epsilon\theta\eta$. If the BP is represented as a function ϕ satisfying the conditions⁷⁶

$$\phi(1, y) = y \text{ and } \phi(i+1, y) = \frac{1}{2}(\phi(i, y))$$

(y is a $\mu\acute{\epsilon}\gamma\epsilon\theta\omicron\varsigma$), the Aristotelian formulation of BP requires the modal operator $\Diamond$ ⁷⁷ which accounts for potentially infinite divisibility:

$$\forall \phi(1, y) = y, \Diamond \exists \phi(i+1, y) = \frac{1}{2}(\phi(i, y)) \quad (1)^{78}$$

Thus Aristotle's characterization of $\sigma\upsilon\nu\epsilon\chi\acute{\epsilon}\varsigma$ can be understood as Eudoxus' BP, to which the semantics of a modal operator is adjoined. Apparently it is $\phi(i+1, y) > \phi(i+2, y)$ and $i+1$ and $i+2$ correspond to the multitude of $\frac{1}{2}(\phi(i+1, y))$'s and $\frac{1}{2}(\phi(i+2, y))$'s that make up the $\mu\acute{\epsilon}\gamma\epsilon\theta\omicron\varsigma$ y at the two successive divisions. According to Phys. 207a33–b21 a collection of units n greater than the collection m has to be introduced by a modal operator, since its existence depends on the potential divisibility of the $\mu\acute{\epsilon}\gamma\epsilon\theta\omicron\varsigma$ y in (1):

$$\forall m \Diamond \exists n (n > m) \quad (2)^{79}$$

But for Aristotle (2) is a restatement of (1) in terms of $i+1$ and $i+2$, i.e. the aggregates of $\frac{1}{2}(\phi(i+1, y))$'s and $\frac{1}{2}(\phi(i+2, y))$'s:

$$\forall(i+1) \Diamond \exists(i+2) ((i+2) > (i+1)) \quad (3).$$

Each $\frac{1}{2}(\phi(i+1, y))$ and $\frac{1}{2}(\phi(i+2, y))$ can be conceived as a unit and thus

⁷⁶ For this formulation see Mueller (above, n. 59) 140.

⁷⁷ For the semantics necessary for the introduction of the modal operator see Hussey (above, n. 6) 178/179.

⁷⁸ This characterization of potential infinite divisibility agrees with claim (5) in W. Charlton, *Aristotle's Potential Infinities*, in Judson (above, n. 20) 141. In (1) Charlton's R should be understood as proportionally divided.

⁷⁹ Cf. M. Mignucci, *Aristotle's Arithmetic*, in A. Graeser (ed.), *Mathematik und Metaphysik bei Aristoteles*, Bern 1987, 209, who offers a stricter formulation for the successor of m.

Aristotle can use the BP to derive the existence of a collection of units n as in (3).

(3) explains Aristotle's statements that τὸ δ' ἐν ὅτι μέτρον σιγαίνει φανερόν (Met. 1078b33/34) and σιγαίνει γὰρ τὸ ἐν ὅτι μέτρον πλῆθος τινός, καὶ ὁ ἀριθμὸς ὅτι πλῆθος μετρημένον καὶ πλῆθος μέτρον (Met. 1088a4–6). According to Met. 1020a8/9 πλῆθος is defined as ἀριθμητὸν ποσὸν and μέτρον can be understood only in its technical geometrical sense, because any $\frac{1}{2}(\phi(i,y))$ is a measure of y . As it is, Aristotle's definition presupposes the fundamental undefined notion in Euclid's arithmetic, measurement.⁸⁰ But, since in Met. 1020a9/10 the ἀριθμητὸν ποσὸν is contrasted with μετρητὸν ποσόν, which is a συνεχές μέγεθος, Aristotle's characterization of numbers in terms of measurement as a πλῆθος μετρημένον καὶ πλῆθος μέτρον offers the link between συνεχές and numbers. συνεχές is grounded in BP, which, as Phys. 207a33–207b2 shows, Aristotle uses in order to deal with a fundamental point in arithmetic: whereas Euclid seems to assume the existence of an infinite number of units, any finite collection of which is a number⁸¹, Aristotle uses BP in order to account for the potential existence of any finite collection of units in agreement with his exclusion of actual infinities⁸². Thus a geometrical postulate accounts for the existence (τὸ εἶναι, An. Post. 75b6) of units, the primitive terms of arithmetic (An. Post. 76b3–5). Moreover, Met. 1088a4–6 shows that divisibility in the broad sense accounts for the τοῦ εἶναι (An. Post. 75b6) of a unit and, therefore, the BP, the axiomatic guarantor of divisibility, fully qualifies as a principle of the genus of arithmetic according to An. Post. 76a31–36. But BP does the same for the domain of geometrical μεγέθη. In Phys. 207a27–34 it is the BP that account for the εἶναι (Phys. 207b30) of any μέγεθος required in geometrical exhaustion proofs. Thus the BP is central for the existence of both numbers and some μεγέθη. Moreover, the τοῦ εἶναι of geometrical μεγέθη, given in Met. 1020a7–11 as μετρητὸν ποσόν, turns out to be exactly the same as the τοῦ εἶναι of numbers given in Met. 1088a4–6. It follows that the formal expression of divisibility is also a principle for the genus of geometrical μεγέθη and, since the genus of numbers and the genus of μεγέθη have the same principles, the primitive terms of both arithmetic and geometry can be unified into the single genus of mathematics as required in An. Post. 87a38/39. Thus συνεχές encapsulates both BP and potential infinite divisibility which yield common proper principles for the genera of both geometry and arithmetic and Aristotle's reference in

Met. 1061b21–25 to numbers and μεγέθη qua συνεχές as the οἰκεία ὕλη of mathematics does not involve any contradiction.

9. Some Historical Comments

Such a unification of the principles of arithmetic and geometry respectively into a single set of principles of mathematics agrees with some fundamental characteristics of Euclid's Elements. Euclid's treatment of μεγέθη is similar to his treatment of numbers⁸³, although Euclid never formulates a common characterization of μεγέθη and numbers like Aristotle does. Moreover, in (1) $(i+1)$ and $(i+2)$ are metalinguistic variables, whereas (3) which Aristotle must consider equivalent to (1), they are object language variables; the same formal situation occurs in Euclid's Def. 7, 16, his only definition of an arithmetical operation⁸⁴. As shown above, Aristotle, like Euclid, takes measurement as a fundamental notion for both numbers and μεγέθη. Thus Eudoxus' proportion theory could have provided a basis for an Aristotelian axiomatization of mathematics consistent with fundamental characteristics of Greek geometry and arithmetic. Given the fact that Eudoxus' handling of ratio could have led to a concept of real number⁸⁵, such an axiomatization could have made a significant contribution to Greek mathematics.

But, although Aristotle realizes the possibility of a unification of geometry and arithmetic consistent with his theory of science, he prefers to keep the two domains separate. For Aristotle a demonstrated proposition must belong to the same genus as the indemonstrable propositions, i. e. the same primitive terms must appear in both demonstrable and indemonstrable propositions (An. Post. 87b1–4)⁸⁶. It follows that, if there is a single genus of mathematics, no subset of its primitive terms⁸⁷ could be a counter-interpretation⁸⁸ either of an indemonstrable or demonstrable mathematical proposition. However, there are indemonstrable propositions which are true only for μεγέθη, like the assumption of the m -part and fourth proportional, both of which are used by Aristotle (see above § 7 and below § 10 respectively). In these cases, the primitive terms of arithmetic can be a counter-interpretation and the theory of incommensurable μεγέθη provides numerous de-

⁸³ See Mueller (above, n. 59) 121/122.

⁸⁴ See Mueller (above, n. 59) 59/60.

⁸⁵ See Knorr (above, n. 56) 219/220.

⁸⁶ See Scholz (above, n. 30) 62.

⁸⁷ Cf. Corcoran (above, n. 30) 62.

⁸⁸ On counter interpretations in Aristotle's logic see Corcoran (above, n. 1) 105.

⁸⁰ For measurement in Euclid's arithmetic see Mueller (above, n. 59) 61–64.

⁸¹ For Euclid's conception of unit and number see Mueller (above, n. 59) 58f.

⁸² See Hussey (above, n. 6) 97 and 178.

monstrable propositions whose counter-interpretation are the primitive terms of arithmetic.

Nevertheless, Aristotle does imply a unification of mathematics in *Met.* 1061b21–25. Eudoxus had achieved to formulate a proportion theory formally superior to the anthyphaeric proportion theory and Aristotle could have entertained the hope that future mathematical developments would eventually unify foundationally the entire field of mathematics.⁸⁹ It is somewhat ironic that Greek mathematics, as crystallized in Euclid's *Elements*, followed a direction that lacks basic advantages of a possible Aristotelian axiomatization⁹⁰: not only is a characterization of proportionality for both μέγεθος and numbers notably absent from the *Elements* (see below n. 101), but also non-Eudoxean considerations of homogeneity are introduced into the proportion theory of Euclid's fifth book.⁹¹

10. Aristotle's Characterization of Proportionality

Although Aristotle makes extensive use of proportions throughout the *Physics*, he nowhere explains how he understands the notion of proportion.⁹² However, in *DC* 273a21ff. he comes very close to an explicit characterization

⁸⁹ The fourth century was a period of intensive formal organization of the various branches of geometry; see Knorr (above, n. 56) 213ff.

⁹⁰ Beckmann (above, n. 42) offers a modern axiomatization of the Euclid's *Elements* V whose proportion theory is a development of Eudoxus' proportion theory (see above n. 69); cf. H. Hasse and H. Scholz, *Die Grundlagenkrisis in der Griechischen Mathematik*, Kant-Studien 33 (1928), 17. However, the axiomatization envisaged by Aristotle never took place in Greek mathematics. Against the usual assumption that Euclid developed geometry, and especially the first book of his *Elements* axiomatically, see Seidenberg (above, n. 7). In his axiomatization Beckmann (above, n. 42) 106/107 lists 24 tacit assumptions on which see Mueller (above, n. 1) 44 and 46–48; for a complete list of Euclidean assumptions see Mueller (above, n. 59) 312–314. For a modern axiomatization of *Elements* VII see N. Malmendier, *Eine Axiomatik zum 7. Buch der Elemente von Euklid*, Mathematisch-Physikalische Semestberichte 22 (1975), 240–254; cf., however, Mueller (above, n. 59) 113. A significant collection of articles on ancient axiomatics is found in J. Hintikka, D. Gruender and E. Agazzi (eds.), *Theory Change. Ancient Axiomatics and Galileo's Methodology*, Dordrecht-Boston-London 1978, Section II. Although I do not agree with all the theses expounded there, it is impossible for me to discuss any of them in detail because of space limitations.

⁹¹ For the introduction of homogeneity assumptions into the proportion theory of the fifth book of the *Elements* see Mueller (above, n. 26) 1ff.

⁹² Hussey (above, n. 6 and n. 68), although he rightly stresses the importance of proportion for Aristotle's mathematical physics, does not explain the way in which Aristotle understands the notion of proportion.

of proportion. In order to show that it is not possible for an infinite body AB to have finite weight Γ , (1) he subtracts from AB a finite μέγεθος BA with weight $E < \Gamma$ (*DC* 273a29–31). Aristotle assumes (2) that $\Gamma = nE$ (*DC* 273a31/32) and then proves his point by assuming (3) that there is a $BZ < AB$ such that BA is to BZ as E is to Γ (*DC* 273b1–3):

$$(E, \Gamma) = (BA, BZ) \quad (i).$$

(4) Therefore, BZ's weight is Γ . But this is absurd because Γ is AB's weight ex hypothesi and $BZ < AB$ (*DC* 273b3–6).⁹³

Of cardinal importance for the understanding of Aristotle's argument is that in (3) he assumes the existence of BZ that satisfies (i). The fundamental assumption behind Aristotle's explanation of his move, i.e. that from an infinite μέγεθος like AB any given μέγεθος can be subtracted (*DC* 273b2/3), is apparently the existence of fourth proportional, a basic assumption characterizing Eudoxus' proportion-theoretic proofs.⁹⁴ Another indication that Aristotle's proof is patterned on Eudoxus' method is the separate treatment of incommensurable Γ and E in *DC* 273b10–15 after the proof for commensurable Γ and E in (2).⁹⁵ Moreover, the fourth proportional BZ changes the inequality of *DC* 273b3–5 into the equality of *DC* 273b5/6. Again, the assumption of the fourth proportional, which changes a given inequality into an equality thus leading into an absurd consequence, is a technique employed by Eudoxus in his proportion-theoretic proofs.⁹⁶ Thus it is reasonable to assume that Aristotle's method of proof comes directly from Eudoxus' proportion theory.

Aristotle's proof is clearly based on the finitude of BZ which is required for the absurd consequence in (4). However, Aristotle seems to assume that BZ is finite without any proof or even any comment to this effect. Aristotle characterizes a finite μέγεθος in terms of multiples (*Phys.* 206b9–12, 273b28–33, 238a16–18, 266a33–b4)⁹⁷ and, since $\Gamma = nE$ in (2) and BZ is the fourth proportional in (i), then BZ is finite if:

$$\Gamma = nE \wedge BZ = nBA \rightarrow (E, \Gamma) = (BA, BZ) \quad (ii).$$

⁹³ Unlike Heath (above, n. 6) 165, who dismisses Aristotle's proof as „slipshod“, Knorr (above, n. 56) 198/199 treats it seriously and shows its affinity to Eudoxus' proportion-theoretic method of proof. The following discussion is a further elaboration of Knorr's analysis.

⁹⁴ See Knorr (above, n. 56) 198, who does not draw this analogy between Aristotle's proof and Eudoxus' proportion theoretic method.

⁹⁵ See Knorr (above, n. 56) 199.

⁹⁶ See Knorr (above, n. 56) 190. Again, Knorr does not notice this common element.

⁹⁷ Aristotle apparently employs the axiom of Archimedes; see Knorr (above, n. 56) 199 and esp. 210. His caveats, however, are unnecessary. There is no reason

Thus Aristotle does not state explicitly that BZ is finite because this is already implicit in the definition of proportionality appealed to in (3).

(ii) is presupposed by Eudoxus' treatment of proportionality and, moreover, (ii) amounts to a formal characterization of proportionality that covers both *μεγέθη* and numbers and thus expands the notion of *μέγεθος* to include numbers⁹⁸, as Aristotle requires in *An.Post.* 75b5/6 for universal proofs⁹⁹. (ii) is provable within the proportion theory of Euclid's *Elements* V¹⁰⁰, and, since, according to Aristotle, the homogeneity in Eudoxus' universal proofs is effected by a hypothesis that characterizes both *μεγέθη* and numbers, there are strong reasons to believe that (ii) was axiomatically posited by Eudoxus as a definition of proportionality for the extended notion of *μέγεθος*¹⁰¹.

τοδί in *An. Post.* 74a24 presupposes this extended notion of *μέγεθος* and *ἔχον αὐξησιν τοιανδί*, the equivalent of *τοδί* in *An.Post.* 99a10, turns out to be an accurate description of proportional terms, as they are defined in (ii), that can be both *μεγέθη* and numbers. The fact that in *Met.* 1061a32–b3 proportion-theoretic properties are proven for all geometrical *μεγέθη* *qua ποσά* and *συνεχῆ*, which imply BP, does not contradict this result; because once it is realized that (ii) offers a definition of proportionality

to think that Aristotle implies Euclid's Def. 5, 4; on the axiom of Archimedes and Euclid's Def. 5, 4 see Mueller (above, n. 59) 139–145.

⁹⁸ (ii) explains why for Aristotle the *συνεχές* can be divided only in those ratios which are specified by the natural numbers. Pace Bostock (above, n. 20) 186/187, this restriction should not be considered as an intrinsic problem of the Aristotelian characterization of *συνεχές*, nor should Aristotle's *συνεχές* be criticized from a modern mathematical point of view. For an interpretation of physical continuum not as „Punktmenge“ but as „Teilbarkeitseigenschaften“ in the Aristotelian sense see C. F. von Weizsäcker, *Das Kontinuum*, in E. Rudolph (ed.), *Zeit, Bewegung, Handlung. Studien zur Zeitabhandlung des Aristoteles*, Stuttgart 1988, 144–167 esp. 146/147.

⁹⁹ See Mueller (above, n. 59) 137/138.

¹⁰⁰ For the proof see Mueller (above, n. 59) 137/138.

¹⁰¹ Mueller has proposed (ii) in order to account for the notable absence of a definition of proportionality for both *μεγέθη* and numbers from Euclid's *Elements*, although it is necessary for Def. 10, 5. According to my interpretation, Aristotle presupposes such a definition of proportionality, which agrees with Eudoxus' proportion-theoretic proofs and accounts for his removal of homogeneity assumptions. If this is so, Becker (above, n. 35) 311–333 is right in speculating about an original of Euclid's tenth book which made use of a treatment of proportionality not found in the *Elements*. Given the essential relationship between Eudoxus' proportion theory and his work on incommensurability (see Knorr [above n. 24] 261ff.), Euclid's tenth book can very well go back to Eudoxus' work on incommensurable *μεγέθη* which did contain a definition of proportionality for both *μεγέθη* and numbers.

for both *μεγέθη* and numbers, BP provides not only a way out of the anthyphaeretic definition of proportionality but the necessary condition that *μεγέθη* in the extended sense have to satisfy in order to enter into proportion relationships¹⁰².

Moreover, the term *αὐξην* occurs in *Phys.* 206b28 as successive addition of proportional terms that complements successive division by means of BP. This complementarity is formally expressed in *Phys.* 266b2–4 as the equivalence of BP and Archimedes' axiom¹⁰³. If *αὐξησιν τοιανδί* in *An. Post.* 99a10 recalls the corresponding term *αὐξην* from *Phys.* 206b28 and the axiom of Archimedes, then *ἔχον αὐξησιν τοιανδί* and *ποσόν* and *συνεχές* pick out two equivalent postulates, the Archimedean axiom and BP respectively, as properties that *μεγέθη* have to satisfy in order to enter into proportion relationships as defined in (ii)¹⁰⁴. In *Phys.* 207b27–32 divisibility, obtained by means of BP, is a property of *μεγέθη* that can enter into proportionalities; in the context of proportion theory, the emphasis on divisibility can be historically explained by the fact that Eudoxus' proportion theory evolved from his method of exhaustion¹⁰⁵. On the other hand, (ii) requires that the proportional *μεγέθη* be Archimedean. As a final point, one notices that, although *τοδί* = *ἔχον αὐξησιν τοιανδί* = *ποσόν* and *συνεχές* presuppose a definition of the mathematical entities, both *μεγέθη* and numbers, of proportion theory's genus, they actually amount to existence postulates. This situation is symptomatic of the close relationship between definitions and postulates that has been observed in both Aristotle and Greek mathematics¹⁰⁶.

11. Aristotle's *συνεχές* and Eudoxus' Exhaustion Method

Phys. 232b20–233a12 contains Aristotle's only proof that a *μέγεθος*, time, is *συνεχές*. Since *συνεχές* is mathematically defined in *Phys.* 206b7–9

¹⁰² Cf. the comments in Knorr (above, n. 56) 235. However, his thesis that a complete theory of proportion on the Eudoxean basis was never worked out completely is not convincing. Aristotle's references to Eudoxus' proportion theory and proofs for both *μεγέθη* and numbers imply the existence of certain postulates, which suggest a complete, or almost complete, formalization of Eudoxus' proportion theory.

¹⁰³ For this equivalence see Mueller (above, n. 59) 142/143.

¹⁰⁴ For a formulation of the axiom of Archimedes compatible with (ii) see D. Bostock, *Logic and Arithmetic, ii. Rational and Irrational Numbers*, Oxford 1979, 130–134.

¹⁰⁵ See Knorr (above, n. 56) 235.

¹⁰⁶ See von Fritz (above, n. 39) 391–393 and 412/413, and J. L. Ackrill, *Aristotle's Theory of Definition and Some Questions on Posterior Analytics* ii 8–10, in Berti (above, n. 1) 395–384.

by means of the BP as an infinite decreasing sequence of proportional terms, one expects that Aristotle's proof should aim at the construction of such an infinite sequence and Aristotle's definition of *συνεχές* in Phys. 232b24/25 seems to justify this expectation¹⁰⁷. His proof, however, seems to follow a quite different route. He considers a μέγεθος $\Gamma\Delta$, a temporal interval ZH and an object A moving faster than an object B which during the time ZH traverses the μέγεθος $\Gamma\Delta$. According to the result established in Phys. 232b5–20 A traverses $\Gamma\Delta$ in a time $Z\Theta < ZH$ and correspondingly B in $Z\Theta$ traverses a μέγεθος $\Gamma\Lambda < \Gamma\Delta$; again, A traverses $\Gamma\Lambda$ in a time $[Z\Lambda] < Z\Theta$. This divisions, which can be represented by the inequalities $\Gamma\Delta < \Gamma\Lambda$ and $ZH > Z\Theta > [Z\Lambda]$, must be shown to continue ad infinitum, if Aristotle is going to prove that time is infinitely divisible. How then can he claim in Phys. 233a5–7 that any temporal interval ZH can yield a subinterval $Z\Theta$? Before this generalization he rounds off the reciprocal divisions of $\Gamma\Delta$ and ZH with the remark that the divisions of $\Gamma\Delta$ and ZH are proportional (Phys. 233a3/4). The proportionality of time and μέγεθος has not been established before it is implied in Phys. 220b15–221a9, where it is said that time measures motion and motion measures time (Phys. 220b23/24; cf. 220b14–18) and, moreover, that time determines a motion that measures the entire motion (Phys. 220b32–221a4). Aristotle's comments clearly presuppose the definition of uniform motion which is referred to in Phys. 233b4–6 and, most probably, comes from Eudoxus' work On Speeds¹⁰⁸. The implied commensurability of time and μέγεθος means that $\Gamma\Delta = n\Gamma\Lambda$ and $ZH = nZ\Theta$; if one operates with (ii) as a definition of proportionality, the proportionality of time and μέγεθος results immediately. Eudoxus must have established as a theorem that in uniform motion the distance is proportional to time¹⁰⁹ and, since (ii) was his definition of proportionality, the proportional divisions of $\Gamma\Delta$ and ZH must appeal to the uniform motion theorem, which is the ἀποδεδειγμένον in Phys. 233a7¹¹⁰. Moreover, (ii) explains Aristotle's conclusion in Phys. 233a11/12 that time and μέγεθος have the same and equal divisions.

¹⁰⁷ Cf. Heath (above, n. 6) 130.

¹⁰⁸ See Knorr (above, n. 57) 120 n. 19.

¹⁰⁹ See Knorr (above, n. 57) 120 n. 19; for fragments of Eudoxus' work see F. Lasserre, *Die Fragmente des Eudoxos von Knidos*, Berlin 1966, 67–74 and 198–212.

¹¹⁰ This word cannot refer to the result established in Phys. 232b5–20, which is clearly referred to in Phys. 233a5–7; it remains that the ἀποδεδειγμένον can only be the proportionality in Phys. 233a3/4, which has nothing to do with Phys. 232b5–20 and thus must imply an independent result.

Thus Aristotle's proof runs as follows: by appealing to the result established in Phys. 232b5–20, he divides ZH proportionally to $\Gamma\Delta$. But for any submultiple of $\Gamma\Delta$, however small, the proportionality established by the theorem of uniform motion guarantees the existence of a corresponding submultiple of ZH. Thus the combined use of division and proportionality as established by a formally proven theorem eliminates any need for the non-constructive introduction of an actually infinite set. This technique is a distinctive characteristic of Eudoxus' exhaustion method and Aristotle's remark in Phys. 233a13–16 shows that in Phys. 232b20–233a12 he presents a formal proof and not a commonsensical argument that time is *συνεχές* because half the distance is always traversed in half the time and so on¹¹¹. The problem with such a Zenonian argument, which Aristotle immediately proceeds to refute in Phys. 233a21–b5, is obviously the introduction of actual infinity. But it would be mistaken to assume that it is the mathematics involved that derive the potentially infinite divisibility implicit in *συνεχές*¹¹². Nowhere does Aristotle imply that the mathematics of his age was in any way concerned with the notion of infinity and Phys. 207b27–34 answers negatively the question in Phys. 204a34–b1. As the discussion of divisibility in GC 1,2 shows, potential division is a matter not of mathematics per se but of the semantics of the modal operator adjoined to (1)¹¹³.

¹¹¹ For the formal character of book Z of the Physics see J. Jope, *Subordinate Demonstrative Science in the Sixth Book of Aristotle's Physics*, CQ 22 (1972), 279–292.

¹¹² This is not the place for a detailed refutation of Bostock (above, n. 20) 180, who claims that Physics Z does not know the potential infinity of Physics Γ. Besides the ἐνδέχεται in Phys. 232b21, which clearly implies potentiality, against Bostock's thesis speaks the fact that, although he is willing to recognize similar mistakes in the handling of infinity in both Z and Γ, he separates the two books as far as the notion of potential infinity is concerned.

¹¹³ See the discussion of GC 1,2 in Charlton (above, n. 78) 135ff. Note also the ἐνδέχεται in Phys. 232b21. As far as I know, the modalities in Aristotle's Physics is a topic still waiting detailed investigation. On the notion of chance, which figures twice in the important passage in Phys. 249b27–250b7, see M. Mignucci, 'Ὡς ἐπὶ τὸ πολὺ et Nécessaire dans la Conception Aristotélécienne de la Science', in Berti (above, n. 1) 173–203, and L. Judson, *Chance and Always or for the Most Part*, in Judson (above, n. 20) 73–99.