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The Pythagoreans

THE speculative thinkers of the sixth and fifth centuries are collectively known as the Presocratic philosophers, but the fact that we apply the same term 'philosopher' to all these men should not be allowed to obscure the important differences between them, for they had very different aims and interests and indeed very different social roles. There are several striking contrasts between the Milesians and the thinkers we must next consider, the so-called Pythagoreans, and the Pythagoreans themselves were far from being a homogeneous group.

Very little is known for certain about Pythagoras himself. We gather that he was born in Samos some time before the middle of the sixth century and that he later moved to Croton in Magna Graecia¹ to escape the tyranny of Polycrates in Samos. The followers of Pythagoras tended out of piety to ascribe their own ideas to the founder himself, and when our late sources do the same, they must be treated with caution. Nevertheless we have it on good authority that Pythagoras taught a way of life—for that is what Plato tells us in the *Republic* (600ab). The early Pythagoreans were not only, and not even primarily, interested in the inquiry concerning nature. They were a group held together by religious beliefs and practices. Thus they believed in the immortality and transmigration of souls, and they practised certain ritual abstentions, for example from certain types of food. Moreover they acted together as a political force in several cities in Magna Graecia in the late sixth century.

Here, then, was one way in which the Pythagoreans

¹ This is the term applied to the area of what is now southern Italy that was colonised and controlled by the Greeks from the late eighth century.

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differed from the Milesians. And another is in the type of cosmological theory that some of them put forward. Where Aristotle represents the Milesians as speculating about the 'material cause' of things, he has this to say about the chief doctrines of the Pythagoreans (as his opening words show, he is referring to Pythagoreans of the fifth century rather than to Pythagoras' own contemporaries):

Contemporaneously with these philosophers [Anaxagoras, Empedocles and the atomists] and before them, the so-called Pythagoreans, who were the first to engage in mathematics, advanced this study, and being trained in it they thought that its principles were the principles of all things. But of these principles numbers are by nature the first, and in numbers they seemed to see many resemblances to the things that are and come to be—more than in fire and earth and water . . . ; and again they saw that the modifications and the ratios of the musical scales were expressible in numbers. Therefore, since all other things seemed in their whole nature to be modelled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number (*Metaphysics* 985 b 23 ff).¹

According to Aristotle, these Pythagoreans found the principles of all things in numbers. Where the Milesians had chosen material substances as the primary things—for even Anaximander's Boundless is material, just as much as Thales' water or Anaximenes' air—the Pythagoreans may be said to have focused attention on the formal aspect of phenomena. Whether or not they were the first to recognise the numerical ratios of musical harmonies, this certainly provided one of their chief examples to illustrate the role of number. The intervals of an octave, fifth and fourth could all be expressed in

¹ Based on the Oxford translation, *The Works of Aristotle translated into English*, edited by W. D. Ross (Oxford, Clarendon Press), *Metaphysics*, W. D. Ross (Vol. VIII, 2nd ed., 1928).

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terms of simple numerical ratios, 1:2, 2:3 and 3:4. Here was a startling instance of phenomena that had no obvious connection with numbers exhibiting a structure that could be expressed mathematically, and it seemed to the Pythagoreans that if this applied to musical intervals, it might well be true of other things too, if only their mathematical relations could be discovered.

The importance of this search for numbers in things is clear. The Pythagoreans were thus the first theorists to have attempted deliberately to give the knowledge of nature a quantitative, mathematical foundation. This places them at the head of what was to be a development of the greatest importance for science. But to put their achievement into perspective we must add two things. The first is that the Pythagoreans held not merely that the formal structure of phenomena is expressible in numbers, but also that things consist of numbers: many of them assumed that things are made of numbers, the numbers themselves being conceived as concrete material objects.

Secondly, many of the resemblances that the Pythagoreans claimed to find between things and numbers were quite fantastic and arbitrary. Thus we are told that they equated justice with the number four (the first square number) and marriage with the number five (this represents the union of male—identified with the number three—and female—two). Opportunity, apparently, was identified with the number seven, and the special significance attached to this number evoked some sharp criticisms from Aristotle:

Why need these numbers be causes? There are seven vowels, the scale consists of seven strings, the Pleiades are seven, at seven animals lose their teeth (at least some do, though some do not), and the champions who fought against Thebes were seven. Is it then because the number is the kind of number it is, that the champions were seven or the Pleiad consists of seven stars? Surely the champions were seven because there were seven gates or for some other

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reason, and the Pleiad *we* count as seven, as we count the Bear as twelve, while other peoples count more stars in both. . . . These people are like the old-fashioned Homeric scholars, who see small resemblances but neglect great ones (*Metaphysics* 1093 a 13 ff).¹

Obviously while the search for numerical ratios proved fruitful in such fields as the analysis of musical harmonies, and mathematics itself, it also and more often led to mumbo-jumbo and crude number-mysticism.

One of the examples that Aristotle gives of the arbitrary manipulation of numbers by the Pythagoreans is from astronomy, and their speculations in this field deserve more detailed consideration. Here too they were much influenced by religious and ethical motives. They believed that the whole heaven is 'a musical scale and a number', and according to the famous doctrine of the harmony of the spheres,² the movements of the heavenly bodies give rise to concordant, though inaudible, sounds: the reason why we do not hear them, according to one report, is that we have been used to them since birth. Moreover the soul was also conceived as a *harmonia* or attunement, and its welfare depends on its being well-tuned and orderly, *kosmios*, like the world-order or cosmos itself.

Yet these doctrines certainly did not prevent, and probably even encouraged, Pythagorean speculation about the relations between the heavenly bodies. Several different theories are attributed either to the Pythagoreans as a whole, or to different groups or individuals among them. Thus in one doctrine which is generally

¹ From the Oxford translation, *The Works of Aristotle translated into English*, edited by W. D. Ross (Oxford, Clarendon Press), *Metaphysics*, W. D. Ross (Vol. VIII, 2nd ed., 1928).

² The Pythagoreans and many later Greek astronomers imagined the visible heavenly bodies as situated on, and carried round by the movement of, concentric spheres that are themselves invisible. There is one sphere to each of the planets, the sun and the moon, and a single sphere for the stars (often called, in Greek astronomy, the 'fixed' stars to contrast them with the 'wandering' stars or planets).

taken to represent an early Pythagorean tradition, the earth is at the centre of the universe and it contains a fiery core, 'Hestia', the central 'hearth'. But a second theory is also reported and attributed by some of our post-Aristotelian sources to Philolaus of Croton, a late fifth-century Pythagorean, in particular. In this, Hestia, the central fire, is not within the earth, but is a separate body, and the earth itself is imagined as circling round it like the other heavenly bodies, the planets, sun and moon. This system is, then, neither geocentric, nor yet heliocentric. The centre is an invisible body of fire, and the doctrine is further complicated by the introduction of a second invisible body, the 'counter-earth', which circles the central fire underneath the earth. Reading from the centre outwards, then, we have the central fire, next the counter-earth, then the earth itself, and outside the earth, the moon, the sun and the planets.

The main evidence for this theory comes from two passages in Aristotle which severely criticise the grounds on which it was put forward. In *On the Heavens* (293 a 17 ff) he says:

Concerning the position [of the earth] there is some divergence of opinion. Most of those who hold that the whole universe is finite say that it lies at the centre, but this is contradicted by the Italian school called Pythagoreans. These affirm that the centre is occupied by fire, and that the earth is one of the stars, and creates night and day as it travels in a circle about the centre. In addition they invent another earth, lying opposite our own, which they call by the name of 'counter-earth', not seeking accounts and explanations in conformity with the appearances, but trying by violence to bring the appearances into line with accounts and opinions of their own.¹

Another highly critical comment on the Pythagorean theory occurs in a passage in the *Metaphysics* (986 a 3 ff):

All the properties of numbers and scales which they

¹ From the Loeb translation by W. K. C. Guthrie (Cambridge, Mass., Harvard University Press; London, Heinemann, 1939).

could show to agree with the attributes and parts and the whole arrangement of the heavens, they collected and fitted into their scheme; and if there was a gap anywhere, they readily made additions so as to make their whole theory coherent. For example, as the number ten is thought to be perfect and to comprise the whole nature of numbers, they say that the bodies which move through the heavens are ten, but as the visible bodies are only nine [that is the sphere of the fixed stars, counted as one, plus the five planets, sun, moon and earth], to meet this they invent a tenth—the 'counter-earth'.¹

Aristotle dismissed the doctrine of the counter-earth as a piece of fanciful number-mysticism, but another passage in *On the Heavens* (293 b 23 ff) suggests that this is not quite the whole story, for there he indicates that the theory was brought to bear on a genuine difficulty, namely why eclipses of the moon are more frequent than those of the sun. Although, if we take the earth as a whole, solar eclipses are more common, only a small proportion of these can be observed from any particular place. On an average, eclipses of the moon visible at any one place are about twice as frequent as eclipses of the sun, and the Pythagoreans apparently tried to account for this by suggesting that not only the earth, but also the counter-earth, intervenes between the moon and its source of light. However, the details of this theory remain, like much else in their astronomy, both vague and obscure, and they evidently made no attempt to give a precise mathematical account of the relations between the heavenly bodies.

Undoubtedly the most interesting feature of the system we have outlined is that it removed the earth from the centre of the universe. Moreover it did so, in large part, for symbolic reasons. According to yet another passage in Aristotle (*On the Heavens* 293 a 30 ff),

¹ From the Oxford translation, *The Works of Aristotle translated into English*, edited by W. D. Ross (Oxford, Clarendon Press), *Metaphysics*, W. D. Ross (Vol. VIII, 2nd ed., 1928).

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the earth was not considered *noble enough* to occupy the most important position in the universe. Whereas religious considerations weighed with some Greek theorists *against* shifting the earth from the centre of the universe, they could and sometimes did weigh *for* that very conclusion. Whatever later astronomers felt, some of the Pythagoreans clearly had no compunction in removing the earth from the centre and making it subject to movement like a planet.

Two further aspects of the work of the Pythagoreans throw light on the methods of early Greek science, (i) the evidence concerning their empirical investigations in acoustics, including the use of simple experiments, and (ii) the development of deductive methods in mathematics. In both cases the information available from our sources leaves much to be desired, and in both cases it relates mainly to thinkers active in the late fifth or early fourth century.

Pythagoras' 'discovery' of the ratios of the musical harmonies was the subject of many legends in antiquity, several of which purport to describe how he arrived at his conclusion by carrying out observations or simple experiments, as for example by noticing the relation between the weights of hammers which made different notes when struck, or by filling jars with varying amounts of water and noting a relation between the quantity of the water and the sound the jar made when struck. Most of these stories must be rejected for the simple reason that the operations they describe do not, in fact, yield the results that are reported. But not all the accounts are fanciful. The stories that refer to his measuring the lengths of string that gave different notes, or to his making similar measurements of the columns of air in pipes, are more plausible and may well reflect the type of empirical investigations that were undertaken by Pythagoreans in the late fifth and early fourth centuries. Archytas of Tarentum, in particular, collected a variety of evidence in an attempt to establish his theory of the relation between the pitch of a note and its

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'speed': in fragment 1 one of his simpler examples refers to the different notes made by different lengths of pipe in a flute. And when Plato too refers to early experiments in acoustics, his testimony is all the more convincing as he himself disapproved so strongly of this method of dealing with the problems. In the *Republic* (531a-c) he makes Socrates speak contemptuously of those who 'measure the harmonies and sounds they hear against one another', who 'torture and rack the strings on the pegs' and 'look for numbers in these heard harmonies'. All this is far from showing that the Pythagoreans recognised the value of the experimental method in general. But it does suggest that some of them carried out certain simple experiments in one field, acoustics, at least. There is, however, little need to emphasise that the motive for which they carried out those experiments was a special one, namely to support the doctrine that 'all things are numbers' by revealing the numerical relations underlying the phenomena.

The history of mathematics in the period before Plato is obscure. The reliable first-hand evidence is scanty, and widely divergent views have been taken on the extent to which our first major mathematical text, the *Elements* of Euclid (composed about 300 B.C.), was based on earlier work. Down to the mid fifth century the Pythagoreans seem to have been chiefly interested in certain aspects of the theory of numbers. The classification of numbers as odd and even probably dates from that period, and so too does the association of certain numbers with geometrical figures of different kinds; thus 4 and 9 are 'square' numbers, 6 and 12 'oblong' ones (where the sides, i.e. factors, differ by 1) and so on (see diagram 1). No doubt the early fifth-century mathematicians were also familiar with certain simple geometrical theorems, including the theorem called after Pythagoras himself, namely that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides: indeed the truth of that theorem had long been known to the Babylonians, for series of

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'Pythagorean' numbers, such as 3, 4, 5, are recorded in cuneiform texts from the second millennium. In such cases the distinctive Greek contribution was not the discovery of the theorem, so much as its proof. But whether or to what extent such proofs were attempted before the middle of the fifth century is not at all clear: on the most

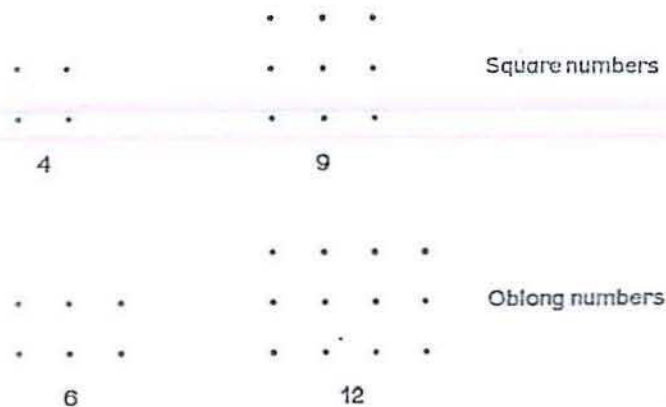


Diagram 1 Pythagorean 'square' and 'oblong' numbers.

likely interpretation of the evidence, the development of methods of mathematical demonstration is a product of the late fifth or early fourth century, and it is one that should undoubtedly be connected with other mathematicians besides those who may be considered Pythagoreans. While the detailed history of this development cannot be undertaken here, two examples may be mentioned briefly in order to illustrate the problems and methods of Greek mathematics before Plato.

My first example illustrates both the uncertainties of the evidence relating to Greek mathematics and what I have referred to as one of their distinctive contributions. This concerns the irrationality of $\sqrt{2}$ —the fact that its value cannot be expressed as a proportion between two

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integers—or, to put it, as the Greeks generally did, in geometrical terms, the fact that the diagonal of a square is not commensurable with its side. Approximations to the value of $\sqrt{2}$ are already found in Babylonian mathematical texts. What the Greeks did, at some stage in the late fifth or early fourth century, was to demonstrate its irrationality. The traditional proof, which is alluded to by Aristotle (*Prior Analytics* 41 a 23 ff), proceeds by first assuming that the diagonal of a square is commensurable with its side, and then showing that this assumption leads to the impossible consequence that the same number is both odd and even (see the additional note at the end of this chapter). Unfortunately we have no means of determining when this proof was discovered, nor even when the fact of the irrationality of $\sqrt{2}$ was known to the Greeks. Most of the stories that we find in our sources on this topic are late fabrications, for example the legend that has it that an anonymous Pythagorean, usually taken to be Hippasus, divulged this secret and suffered death by drowning as divine retribution for doing so. We do not even know whether this discovery was made as a result of exploring the applications of Pythagoras' theorem, or whether—as has recently been thought more likely—it was prompted by philosophical problems relating to the idea of infinite divisibility. The only safe conclusion that our evidence allows us to draw is that the irrationality of $\sqrt{2}$ was known before the time of Plato. In the *Theaetetus* (147d) the mathematician Theodorus of Cyrene is described as 'showing that the sides [i.e. the roots] of the squares representing three square feet and five square feet are not commensurable in length with the line representing one foot', and taking all the cases up to the side of seventeen square feet in turn. While it is interesting that the problem of irrationals is still not treated here as a general problem, and is handled geometrically rather than arithmetically, the text clearly implies a familiarity with *some* demonstration of the incommensurability of the side and the diagonal of the square

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representing two square feet, since this fact is assumed as requiring no proof.

In my second example the evidence is more certain and the role of a Pythagorean mathematician more definite. One of the problems that exercised Greek mathematicians from the middle of the fifth century was that of the duplication of the cube: given a particular cube, how does one construct a cube of twice its volume? According to our sources Hippocrates of Chios, who is not to be confused with his contemporary and namesake, the great physician of Cos, recognised that this problem is equivalent to that of finding two mean proportionals (a, b) between two given lengths (x, y) such that $x:a=a:b=b:y$. This will provide the solution, since in the particular case where $y=2x$, the cube on a will be double the cube on x . But the first to solve the problem of finding two mean proportionals was the Pythagorean Archytas, whose work in acoustics has already been mentioned. His solution, which has come down to us in a commentary on Archimedes, is a geometrical one and remarkable for its ingenuity. Some idea of this may be gathered from the opening words of Heath's account.¹ Heath describes it as a

bold construction in three dimensions, determining a certain point as the intersection of three surfaces of revolution, (i) a right cone, (ii) a cylinder, (iii) a *tore* or anchoring with inner diameter *nil*. The intersection of the two latter surfaces gives (says Archytas) a certain curve ... and the point required is found as the point in which the cone meets this curve,

—whereupon Archytas demonstrates how the point so determined enables the two mean proportionals to be found. This example indicates the progress that had already been made in geometry in the early fourth century: Archytas' brilliant three-dimensional kinematic

¹ *A History of Greek Mathematics*, Vol. I, Oxford, Clarendon Press, 1921, pp. 246 ff.

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construction provides a foretaste of the methods that were to lead to one of the most remarkable achievements of early Greek science, the astronomical model of Eudoxus.

Additional Note

The 'traditional' proof that the diagonal of the square is incommensurable with the side is given in an appendix to Euclid, book X, and may be paraphrased as follows:

Let AC be the diagonal of the square, AB its side.

Suppose AC is commensurable with AB, and let $a:b$ be their ratio expressed in the lowest terms. Since $AC > AB$, $a > b$.

Then $AC:AB = a:b$

So $AC^2:AB^2 = a^2:b^2$

But (by Pythagoras' theorem) $AC^2 = 2AB^2$

Therefore $a^2 = 2b^2$

So a^2 , and therefore a , is even, and since $a:b$ is in its lowest terms, b is odd.

Since a is even, let $a = 2c$

So $4c^2 = 2b^2$

So $2c^2 = b^2$

From which it follows that b is even.

Since the assumption that AC is commensurable with AB leads to the impossible consequence that the same number (b) is both odd and even, the assumption must be false.

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