

Determination of the Sun's Orbit

Hipparchus, Ptolemy, al-Battānī, Copernicus, Tycho Brahe

Y. MAEYAMA

Communicated by J. NORTH

Contents

Introduction	1
Symbols	2
1. Preliminary remarks	3
2. Deviation of Hipparchus' solar hypothesis from the modern theory and its influence on the determination of the two orbital elements	4
3. Observational errors and the determination of the orbital elements: Derivation of the formulae	9
4. Errors in the underlying observations and the orbital elements of the Sun	13
5. Explanatory remarks on the underlying observations of the Sun	26
Summary	44
Bibliography	47

Introduction

Much has been said and written about historical determinations of the Sun's orbit; the subject is indeed an essential part of every planetary theory and world-system, both geocentric and heliocentric.

The historical records show us that the solar eccentricity – a parameter which, after Hipparchus, no astronomer could avoid discussing – decreased linearly from Hipparchus and Ptolemy (ca. 0.0416) through al-Battānī (0.0346...) and many other Islamic astronomers down to Copernicus (0.0323). This decrease, together with other similar phenomena, led Copernicus to his most bizarre theory of the 1717-year cycle. Barely 70 years later, the heavens, as rigorously observed by Tycho, forced him to take the historical way back and to adopt a much greater value (0.03584),¹ just between Copernicus' and Hipparchus' values. This in turn induced him, or made him the more determined to measure the Universe himself, independently of all his predecessors.

The problem of all these numerical values for the Sun and their deviations from the modern values has already been studied and explained in terms of the errors in the

1090 (pdf)
MAEYAMA, Y.
1998

underlying observations.² Our present study aims at finding the cause of those errors. We shall follow as closely as possible the range of the immediate observations, which are assumed to be normally distributed about the true value of each observed object. We shall thus deal mainly with:

1. the theoretical deviation of Hipparchus' solar hypothesis³ from modern theory (ch. 2),
2. the mathematical derivation of the historical solar elements by means of the underlying observations and the modern values (ch. 3),
3. the analysis of the underlying observational errors (ch. 4).

The results obtained are entirely new, and our main achievements are easily explained with the help of few equations and figures. Particularly interesting is the fact that our results could be compared with each other, based on my early study on the length of the seasons (1988), the determination of which was the first step in deriving the solar elements.

This paper has a long history. I first noticed the problem presented here in 1972 whilst working on Tycho Brahe and Kepler. I published its partial solution in my paper on Tycho's solar theory (1974). In 1986–87 I solved the problem, but in order to present it better, I first finished my study on the length of the seasons (1988). Now at last I have been able to finish the task I first undertook 20 years earlier.

Symbols

$A(^{\circ})$	longitude of the apogee of the Sun's orbit
$d(^{\circ})$	declination
e	solar eccentricity according to the modern theory
$h(^{\circ})$	(Sun's meridional) altitude
k	solar eccentricity according to Hipparchus' solar hypothesis, $\approx 2e$
$K(^{\circ})$	Sun's true longitude at t_2 , $= l_2 - l_1 = l_2$
$l(^{\circ})$	Sun's true longitude at t , mostly used as $l_{1,2,3}$ at $t_{1,2,3}$
$L(^{\circ})$	Sun's mean longitude at t , mostly used as $L_{1,2,3}$ at $t_{1,2,3}$
p	mean equatorial horizontal parallax of the Sun, $8.80''$
p'	mean equatorial horizontal parallax of the Sun, e.g. $2'51''$ (Ptolemy), $3'$ (al-Battānī, Copernicus, Tycho)
$P(^{\circ})$	longitude of the perigee of the Sun's orbit
r	coefficient of refraction, $\approx 60''$ at $h > 10^{\circ}$ (cf. e.g. Expl. Suppl. 54–56, Woolard/Clemence, 79–96)
r'	atmospheric refraction for the Sun after Tycho (Op. om. II, 64), $= f(h)$
t	time, mostly used as $t_{1,2,3}$
$v(^{\circ})$	true anomaly
$v_1(^{\circ})$	Sun's true anomaly at t_1 , $l_1 = 0$

z (°)	Sun's zenith-distance
α (°)	mean anomaly
γ, δ (°)	equation of the center at $l_{1,3}$ and l_2 respectively
ε (°)	obliquity of the ecliptic
θ (°)	angle between the solar longitude at t_2 and the apogee, cf. Fig. 2.1
φ (°)	geographical latitude
$\bar{\varphi}$ (°)	colatitude, $= 90 - \varphi$
ω (°)	Sun's mean daily motion (e.g. tropical, $0.9856...^\circ/\text{d}$)

Symbols for historical values are marked by (') and the deviations by (Δ) like $\Delta l = l' - l$.

1. Preliminary remarks: Determination of two orbital elements of the Sun, the position of the apsidal line and the eccentricity

In order to derive two orbital elements of a planet revolving about its central body we need two equations, which involve two sets of two fundamental quantities relating to the transition of the planet in time and position, which are given by three observations. Since the observation of the planet gives us only its celestial direction but not its orbital position, we further need a hypothesis which defines the celestial direction into the orbital point such that the planetary orbit can be defined by the three points.

Hence, to determine our two orbital elements we need:

1. a hypothesis on the planetary motion,
2. three longitudinal observations l_1, l_2, l_3 at times t_1, t_2, t_3 .

We shall deal only with Hipparchus' hypothesis concerning the apparent solar motion around the Earth (Alm. III, 4). This hypothesis was virtually the only one which was constantly employed for the solar determination; it remained relevant at least up to the time of Vieta's new invention (1600) and Kepler's *Astronomia nova* (1609).

For all the astronomers we consider, the three solar observations at $t_{1,2,3}$ can be defined as

$$l_1 = 0 \quad (\text{vernal equinox}),$$

$$l_2 = l_1 + K, \quad K = 90^\circ (\text{summer solstice; Hipparchus, Ptolemy, al-Battānī}) \\ = 45^\circ n \quad [n = 1, 3, 5, 7 (\text{fuṣūl, sections});$$

$$\text{Islamic astronomers after c. + 830,}^{1,1} \text{ Copernicus, Tycho],}$$

$$l_3 = 180^\circ (\text{autumnal equinox}).$$

Hipparchus' method differs from other determinational methods of the orbital elements such as those of Ptolemy with the punctum equans, of Vieta (1600) and Kepler

(1609) which are all iterative and approximative. Hipparchus' method is mathematically accurate and it yields, given the year length and three observational values, one single answer. Since all historical values for the relevant tropical year can be considered sufficiently accurate for our present problem, the errors in the solar elements can only arise from two sources:

1. inaccuracy of the hypothesis,
2. inaccuracy of the underlying observations.

The values in the historical records deviate from the modern ones. The deviations are due to a complex agglomerate of many different factors. In order to distinguish between these factors we are obliged to make use of the expansion in series. This is particularly useful, since the deviations are small. Therefore, with only a few terms in each equation, all basic problems become visible to any desired order of accuracy. We shall follow the above two items separately.

2. Deviation of Hipparchus' solar hypothesis from the modern theory and its influence on the determination of the two orbital elements, the position of the apsidal line and the eccentricity

If the mean anomaly α is known, we can roughly compare the solar motion according to Hipparchus and the solar motion according to Kepler's laws:

$$\begin{aligned} \text{Hipparchus: } v_H &= \alpha + k \sin \alpha + \frac{k^2}{2} \sin 2\alpha + \dots, \\ \text{Kepler: } v_K &= \alpha + 2e \sin \alpha + \frac{5}{4}e^2 \sin 2\alpha + \dots, \end{aligned} \quad (2.1)$$

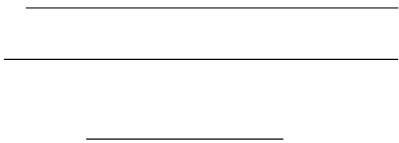
where k is the eccentricity in Hipparchus' solar model (Fig. 2.1). If we now put $v_H = v_K$ (in connection with some longitudinal observations of the Sun), k will correspond nearly to $2e$, the double of the modern eccentricity, their difference amounting only to the e^2 -order:

$$k = 2e - \frac{3}{2}e^2 \cos \alpha + \dots \quad (2.2)$$

The theoretical error in the position of the apogee is supposed to be correspondingly small. For our quantitative analysis of the two orbital elements below we shall therefore take as our basis the double eccentricity and apogee, $2e$ and A , both modern parameters. It is therefore convenient to start with the modern parameters at the epoch t . From the geometrical construction given in Fig. 2.1 we have the relations

$$\begin{aligned} 2e \sin \theta &= \sin \delta, \\ 2e \sin(K + \theta) &= \sin \gamma, \end{aligned} \quad (2.3)$$

hence we obtain the eccentricity and the position of the apogee $A(= K + \theta)$ as functions of γ and δ :



$$\begin{aligned}
k &= \frac{\sqrt{\sin^2 \gamma' + \sin^2 \delta' - 2 \sin \gamma' \sin \delta' \cos K}}{\sin K} = f(\gamma', \delta') \\
\theta' &= \tan^{-1} \left(\frac{\sin \delta' \sin K}{\sin \gamma' - \sin \delta' \cos K} \right) = f(\gamma', \delta').
\end{aligned} \tag{2.5}$$

Putting

$$\gamma' = \gamma + \Delta\gamma, \quad \delta' = \delta + \Delta\delta, \tag{2.6}$$

we obtain the deviation of the solar eccentricity Δk as a function of the two modern equations of the center γ and δ and the deviations $\Delta\gamma$ and $\Delta\delta$ to the first approximation:

$$\begin{aligned}
\Delta k &= k - 2e \\
&= \frac{\Delta\gamma \cos \gamma (\sin \gamma - \sin \delta \cos K) + \Delta\delta \cos \delta (\sin \delta - \sin \gamma \cos K)}{2e \sin^2 K} \\
&= f(\gamma, \delta, \Delta\gamma, \Delta\delta).
\end{aligned} \tag{2.7}$$

In the following we shall express the above quantities in modern parameters.

1. γ, δ

Simplifying the formulas by putting $v_1 = 180 - (K + \theta)$, we insert $\sin \gamma, \cos \gamma, \sin \delta$ and $\cos \delta$ according to (2.3) into (2.7) and express Δk as a function of $\Delta\gamma$ and $\Delta\delta$:

$$\begin{aligned}
\Delta k &= k - 2e \\
&= \frac{1}{\sin K} \{ \Delta\gamma [-\cos(v_1 + K) + \dots] + \Delta\delta [\cos v_1 - \dots] \} \\
&= f(\Delta\gamma, \Delta\delta).
\end{aligned} \tag{2.8}$$

Likewise from (2.4) and (2.5):

$$\begin{aligned}
\Delta\theta &= (\theta' - \theta) = A' - A \\
&= \frac{1}{2e \cos \theta} \left(\Delta\delta \cos \delta - \Delta k \frac{\sin \delta}{2e} \right) \\
&= - \frac{\Delta\gamma \sin 2(v_1 + K) + 2\Delta\delta [\sin K - \cos v_1 \sin(v_1 + K)]}{4e \sin K \cos(v_1 + K)} \\
&= f(\Delta\gamma, \Delta\delta).
\end{aligned} \tag{2.9}$$

2. $\Delta\gamma, \Delta\delta$

Unlike the lengths of the seasons the historical values for the length of the year can be considered as being mostly accurate and the small errors can be neglected as far as our present problem is concerned.

According to Hipparchus' method the two equations of the center γ' and δ' are given by the observational quantities as

$$\begin{aligned}\gamma' &= \frac{1}{2}[(L_3 - L_1)] - 90 \\ \delta' &= \frac{1}{2}[(L_3 + L_1)] - L_2 + K - 90.\end{aligned}\quad (2.10)$$

Since the mean and true longitude are given by

$$\begin{aligned}L &= P + \alpha, \\ l &= P + v,\end{aligned}\quad (2.11)$$

and since the mean anomaly α can be expressed as a function of the true anomaly of the Keplerian motion v as^{2,1}

$$\alpha = v - 2e \sin v + \frac{3}{4}e^2 \sin 2v - \frac{1}{3}e^3 \sin 3v + \dots = f(v), \quad (2.12)$$

we can express γ' and δ' , for the case where no observational errors are involved, by means of the modern values:

$$\begin{aligned}\gamma' &= 2e \sin v_1 + \frac{1}{3}e^3 \sin 3v_1 + \dots, \\ \delta' &= 2e \sin(v_1 + K) - \frac{3}{2}e^2 \cos(2v_1 + K) \sin K \\ &\quad + \frac{1}{3}e^3 \sin 3(v_1 + K) + \dots,\end{aligned}\quad (2.13)$$

where according to our definition above (2.11)

$$\begin{aligned}P_2 + v_2 &= l_2 = l_1 + K = P_1 + v_1 + K \\ P_3 + v_3 &= l_3 = l_1 + 180 = P_1 + v_1 + 180.\end{aligned}\quad (2.14)$$

Now we obtain from (2.3)

$$\begin{aligned}\gamma &= 2e \sin v_1 + \frac{4}{3}e^3 \sin^3 v_1 + \dots, \\ \delta &= 2e \sin(v_1 + K) + \frac{4}{3}e^3 \sin^3(v_1 + K) + \dots,\end{aligned}\quad (2.15)$$

hence, from (2.13) and (2.15)

$$\begin{aligned}\gamma' - \gamma &= \Delta\gamma = e^3 \left(-\sin v_1 + \frac{2}{3} \sin 3v_1 \right) + \dots, \\ \delta' - \delta &= \Delta\delta = -\frac{3}{2}e^2 \sin K \cos(2v_1 + K) + \dots\end{aligned}\quad (2.16)$$

These are the theoretical deviations of the two equations of the center γ' and δ' according to Hipparchus' hypothesis from the modern values which yield the accurate solar parameters $2e$ and $A (= K + \theta)$.

In our general computation to the e^2 -order we therefore put $\Delta\gamma = 0$ and $\Delta\delta$ above into (2.8) and (2.9) and obtain the deviations expressed only by the two modern parameters, e and v_1 :

$$\begin{aligned}
 \Delta k_h &= (k - 2e)_h = -\frac{3}{2}e^2 \cos v_1 \cos(2v_1 + K), \\
 &= f(e, v_1), \\
 |\Delta k_h|_{\max} &= \frac{3}{2}e^2 (\approx 0.0004), \\
 \Delta\theta_h &= (\theta' - \theta)_h = (A' - A)_h \\
 &= \frac{3 \cos(2v_1 + K)}{4 \cos(v_1 + K)} e [\sin k - \cos v_1 \sin(v_1 + K)] \\
 &= f(e, v_1). \tag{2.17}
 \end{aligned}$$

These are the theoretical deviations of the two solar elements, as determined by Hipparchus' hypothesis from the modern parameters if the observations contain no errors. If we roughly put $v_1 = 90^\circ$ in (2.17) for $t = 1250 \pm 450 = 800 \sim 1700$ ($v_1 = 97 \sim 83^\circ$), we have

$$\begin{aligned}
 k &\approx 2e, & \Delta k_h &= f(e^3) \approx \pm 0.000005, \\
 \Delta\theta_h &= \frac{3}{4}e \cos K, & |\Delta\theta_h|_{\max} &\approx 0.7^\circ; (v_1 = 90^\circ). \tag{2.18}
 \end{aligned}$$

If we further put $K = \pm 90^\circ$ [Hipparchus, Ptolemy, al-Battānī ($v_1 = 113.5^\circ, 108.9^\circ, 96.2^\circ$)], the apogee is also obtainable accurately:

$$\Delta\theta \approx 0; \quad (v_1 = 90^\circ, K = \pm 90^\circ). \tag{2.19}$$

Putting now in (2.17) only $K = \pm 90^\circ$ (Hipparchus, Ptolemy, al-Battānī) we obtain

$$\begin{aligned}
 \Delta k_h &= \pm \frac{3}{2}e^2 \sin 2v_1 \cos v_1 \\
 \Delta\theta_h &= \pm \frac{3}{2}e \sin^2 v_1 \cos v_1. \tag{2.20}
 \end{aligned}$$

For accurate observations the deviation of the eccentricity from $2e$ and the error of the apogee are bounded by an approximately quantity of ± 0.0003 and of $\pm 0.6^\circ$ respectively.

Thus, for ca. 1800 years between Hipparchus and Kepler the theoretical error of Hipparchus' solar hypothesis had, as far as it exceeded the values above, nothing whatsoever to do with the well-known numerical errors in the historical records on the solar orbit (Fig. 2.2). The deviations of the two solar elements from the modern values must therefore be attributed to the problem of observations, immediate or corrected.

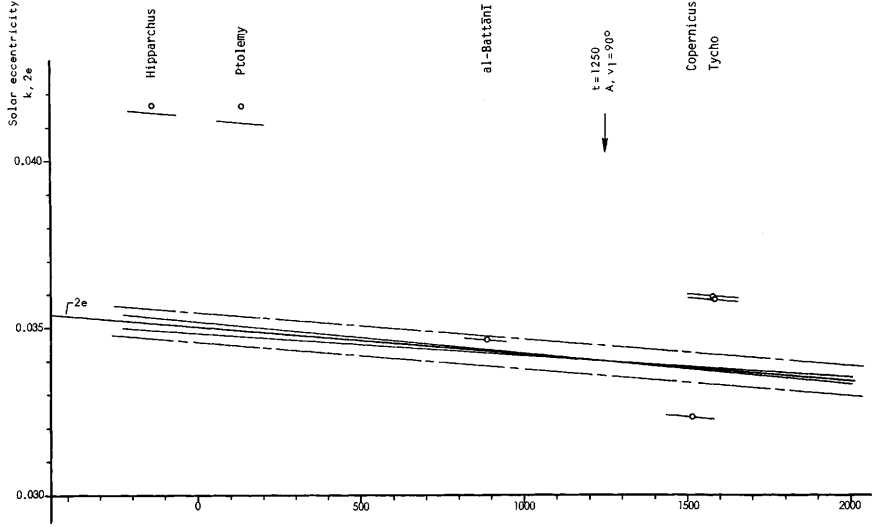


Fig. 2.2. The solar eccentricity as a function of time, $t = -500 \sim +2000$

The theoretical deviation of Hipparchus' solar hypothesis from the modern theory (cf. Tabs. 3.1., 4.1., Figs. 4.2–4.4)

$\text{---} \text{---} \text{---}$ modern value $2e$ with the maximum theoretical deviations of Hipparchus' determinational method as a function of time
 $k = 2e - \left[\frac{3}{2} e^2 \cos v_1 \cos(2v_1 + K) \right]_{\max, \min}; K = 0 - 360^\circ$, cf. (2.17).

If the observations and the corrections for solar parallax and refraction are accurate, the determined eccentricities are to be bounded within these two lines.

$\text{---} \text{---}$ maximum theoretical deviations, $2e \pm \frac{3}{2} e^2$; cf. (2.2), (2.17).
 $\text{---} \circ \text{---}$ historical records and our approximate reproductions by means of the historical observational data (Tab. 3.1., cols. o, p).

Note that the reproduced solar eccentricities agree well with the recorded ones. Note also al-Battānī's outstanding accuracy, due to his accurate observations and his neglect of the solar parallax; another reason for his accuracy was that because of the high colatitude of his observing place, the effect of refraction was small.

3. Observations and the determination of the orbital elements: Derivation of the formulae of the elements as a function of observational errors

After deducing the theoretical deviations of the solar elements determinable by Hipparchus' *hypothesis* from the modern parameters, Δk_h and $\Delta \theta_h$, we can complete the formulae (2.8), (2.9) by simply retaining $\Delta \gamma$ and $\Delta \delta$ as standing for observational errors only.

From (2.8), (2.9) and (2.17) we obtain the total deviations from the modern parameters as functions of $\Delta \gamma$ and $\Delta \delta$ to an approximate order of e^2 for the eccentricity Δk

and, for simplicity, to the first order in e for the apogee $\Delta\theta (= \Delta A)$:

$$\begin{aligned}
\Delta k &= k - 2e \\
&= \frac{1}{\sin K} [-\Delta\gamma \cos(v_1 + K) + \Delta\delta \cos v_1] - \frac{3}{2}e^2 \cos v_1 \cos(2v_1 + K) \\
&= f(\Delta\gamma, \Delta\delta), \\
\Delta\theta &= \theta' - \theta = \Delta A = A' - A \\
&= -\frac{\Delta\gamma \sin 2(v_1 + K) + 2\Delta\delta [\sin K - \sin(v_1 + K) \cos v_1]}{4e \sin K \cos(v_1 + K)} \\
&\quad + \frac{3}{4}e \frac{\cos(2v_1 + K)}{\cos(v_1 + K)} [\sin K - \sin(v_1 + K) \cos v_1] \\
&= f(\Delta\gamma, \Delta\delta). \tag{3.1}
\end{aligned}$$

In the final formulae above the two terms $\Delta\gamma$ and $\Delta\delta$ are now free from incorrectness of the adopted solar hypothesis and related solely to the quantities of observational errors:

$$\begin{aligned}
\Delta\gamma, \Delta\delta &= [\text{quantities deduced from historical records according to (2.10)}] \\
&\quad \text{minus } \gamma', \delta' [\text{theoretical quantities according to (2.13)}]. \tag{3.2}
\end{aligned}$$

Thus, the total deviations in (3.1) consist of two parts:

1. errors of observations ($\Delta\gamma$ -, $\Delta\delta$ -term),
2. errors of the *hypothesis* [Δk_h (e^2 -term), $\Delta\theta_h$ (the last e -term)].

If we put the observational errors according to (3.2) into $\Delta\gamma$ and $\Delta\delta$ and the modern values for the true anomaly $v_1 (= 180 - A)$ at the vernal equinox t_1 and the eccentricity e at the epoch, the Eqs. (3.1) give us the total deviations at the epoch, and by adding the modern parameters, we obtain the values corresponding to the attested solar elements. Tab. 3.1. shows the historical parameters and our reproduced values (cols. o/p, r/s). The agreement is reasonable, which means that our reproducing computation, in particular (3.1), is correct to the first approximation.

If we put $v_1 = 90^\circ$ for $t = 830$ (al-Ma'mūn) ~ 1630 (Kepler) [$v_1 (= 180 - A) = 93 \sim 83^\circ$], we obtain a very appropriate orientation for explaining the historical records; the error in the most fundamental observation of the equinoxes is directly transformed into that in the eccentricity:

$$\Delta k \approx \Delta\gamma, \tag{3.3}$$

hence

$$k \approx 2e + \Delta\gamma, \quad (v_1 \approx 90^\circ). \tag{3.4}$$

According to this, all historical values of the Sun's eccentricity should deviate from the double modern eccentricity $2e$ by the quantity of the observational errors of the equinoxes $\Delta\gamma (> < 0; \text{cf. chs. 4, 5})$.

Likewise we have

$$\Delta\theta \approx \frac{-\Delta\gamma \cos K + \Delta\delta}{2e \sin K} + \frac{3}{4}e \cos K, \quad (v_1 = 90^\circ). \quad (3.5)$$

Putting further $K = \pm 90^\circ$ we have

$$\Delta\theta \approx \pm \frac{\Delta\delta}{2e}, \quad (v_1 \approx 90^\circ, K = \pm 90^\circ). \quad (3.6)$$

Putting now only $K = \pm 90^\circ$ in (3.1) we have

$$\begin{aligned} \Delta k &= \pm \Delta\gamma \sin v_1 + \Delta\delta \cos v_1 \pm \frac{3}{2}e^2 \sin 2v_1 \cos v_1 \\ \Delta\theta &= -\frac{\Delta\gamma \cos v_1 \mp \Delta\delta \sin v_1}{2e} \pm \frac{3}{2}e \sin^2 v_1 \cos v_1 \quad (K = \pm 90^\circ). \end{aligned} \quad (3.7)$$

As we shall see below, all historical records of the solar elements reveal their very characteristic errors in accordance with the tendencies shown above.

Due to Hipparchus and Ptolemy the above Eqs. (3.7) deserve special attention. With $K = +90^\circ$ (Hipparchus, Ptolemy, al-Battānī) we put (3.7) as consisting of two terms relating to observations $(\Delta\gamma, \Delta\delta)$ and the hypothesis (e^2, e) respectively:

$$\begin{aligned} \Delta k &= (\Delta\text{obs})_k + (\Delta\text{hyp})_k \\ \Delta\theta &= \Delta A = (\Delta\text{obs})_A + (\Delta\text{hyp})_A \end{aligned} \quad (3.8)$$

and for simplicity we put $|\Delta\gamma| = |\Delta\delta|$. We then obtain

$$\begin{aligned} (\sin v_1 \pm \cos v_1)_{\max, \min} &= \pm\sqrt{2} && \text{at } v_1 = \pm\pi/4 + \pi \\ (\sin 2v_1 \cos v_1)_{\max, \min} &= \pm 1/\sqrt{2} && \text{at } v_1 = \pm\pi/4 + \pi \\ (\sin^2 v_1 \cos v_1)_{\max, \min} &= \pm 0.385 && \text{at } v_1 = \pm 54.7^\circ + \pi. \end{aligned} \quad (3.9)$$

With (3.9) we obtain small maximum errors arising from the hypothesis:

$$\begin{aligned} (\Delta\text{hyp})_{k, \max, \min} &= \pm \frac{3}{2\sqrt{2}}e^2 = \pm 0.0003 \\ (\Delta\text{hyp})_{A, \max, \min} &\approx \pm 0.6^\circ \end{aligned} \quad (3.10)$$

Neglecting these small error sources and putting simply $|\Delta\gamma| = |\Delta\delta| = \Delta$ we may describe

Table 3.1. Computational deduction of the historical solar elements - eccentricity and apogee - by means of the modern parameters.

a	b	c	d	e	f	g	h	i	
Length of the seasons									
			recorded		modern		recorded – modern		
	t	K (°)	$t_3 - t_1$ d h m d	$t_2 - t_1$ d h m d	$t_3 - t_1$ d	$t_2 - t_1$ d	$\Delta(t_3 - t_1)$ d h	$\Delta(t_2 - t_1)$ d h	
1	Hipparchus Alm.III, 4	c. –130	90	187			+0.6339	+0.4846	
				94 1/2	186.3661	94.0154	+15.213	+11.630	
2	Ptolemy Alm. III, 4	c.+140	90	187			+0.5378	+0.5961	
				94 1/2	186.4622	93.9039	+12.908	+14.307	
3	al-Battānī Op. ast. XXVIII	882/883	90	186 14 45 186.6146	93 14 93.5833	186.5899	93.5166	+0.0247 +0.593	+0.0668 +1.603
4	Copernicus De rev. III,16	1515	225	186 8 36 186.3583	231 15 0 231.6250	186.5407	231.7199	–0.1823 –4.376	–0.0949 –2.277
5	Tycho Op. om. II, 19–24	1583	45	186 18 41 186.7785	46 2 48 46.1167	186.5269	46.0727	+0.2516 +6.038	+0.0440 +1.056
		1588	45	186 18 30 186.7708	46 2 55 46.1215	186.5258	46.0704	+0.2450 +5.881	+0.0511 +1.226
		1588	135	„ 140 8 50 140.3681	„ 140.1665	„	„	„	+0.2015 +4.837
6	Remarks		3-d, -e: ca.		Maeyama (1988), Figs. 1.1–1.7				
1–, 2–d, –e: = Geminus, Manitius, 8, cf. also 210ff. col. c: solar longitudes at t_2 . 4-d: $365\frac{1}{4}$ (III, 13) –178;53,30 (III, 16). For the notorious value $186;5\frac{1}{2}$ cf. e.g. Swerdlow-Neugebauer, 172. 4-e: (4-d) +45;16. cols. j, k : $\Delta(t_3 - t_1)\omega/2$, $\Delta\gamma - \Delta(t_2 - t_1)\omega$.									

Note a generally good agreement between the historical records and our reproduced values (cols. o/p, r/s).

$$\Delta k_{\max, \min} \approx \pm \sqrt{2} \Delta, \quad \Delta (\text{rad.})$$

$$\Delta A_{\max, \min} \approx \pm \frac{\Delta}{\sqrt{2}e} \approx \pm 40 \Delta, \quad \Delta (^\circ). \quad (3.11)$$

Both extreme errors are therefore proportional to the underlying observational errors, Δ . As we shall see below, both observational quantities, $\Delta\gamma$ and $\Delta\delta$, roughly correspond to individual observational errors:

$$\Delta\gamma \approx -\Delta t_1 \omega \approx \Delta t_3 \omega \approx -\Delta l_1 \approx \Delta l_3$$

$$\Delta\delta \approx -\Delta t_2 \omega \approx -\Delta l_2 \quad [\text{cf. (4.1), (4.2)}].$$

Putting therefore $\Delta \approx |\Delta t_{1,2,3}| \omega \approx |\Delta l_{1,2,3}|$ for simplification, we obtain approximate numerical connections between observational errors and the resulting errors of the

Table 3.1. (Continued)

j	k	l	m	n	o	p	q	r	s
Eccentricity						Apogee (°)			
modern			computed		recorded	computed		recorded	
$\Delta\gamma$ (°)	$\Delta\delta$ (°)	A (°)	e	Δk	k	k p	$\Delta\theta$	A'	A'
+0.312	-0.165	66.527	0.017558	+0.006284	0.041400	2;30 0.041667	-1.278	65.249	65;30 65.5
+0.265	-0.323	71.124	0.017457	+0.006289	0.041203	2;29,30 0.041528	-6.719	64.405	„ 82;17
+0.012	-0.054	83.782	0.017169	+0.000323	0.034661	2;4,45 0.034653	-1.672	82.109	82.283
-0.090	+0.004	94.615	0.016912	-0.001472	0.032351	0.0323	+2.122	96.737	96 2/3 96.667
+0.1240	+0.0806	95.780	0.016883	+0.002174	0.035941	0.0359194	-0.114	95.667	95;44 95.733
+0.1208	+0.0704	95.866	0.016881	+0.002094	0.035857	0.0358416	-0.351	95.515	95;30 95.5
„	-0.0769	„	„	+0.002090	0.035853	0.0358388	-0.414	95.453	95;27 95.45
Conn. d. temps Expl. suppl.				(3.1)	$= 2e + \Delta k$	1-p: ca.	(3.1) = A + $\Delta\theta$		

dependent parameters. Through a combination of observational errors of $\pm 1/4^d$ and $\pm 1^h$ e.g. the error in the location of the apogee at the time of Hipparchus and Ptolemy can attain extreme values of $\pm 9^\circ$ and $\pm 1.5^\circ$ (Tab. 3.2, 2-h, -i, 1-h, -i).

Only the maximum attainable errors are shown above. As we can easily see in (3.7), the actual errors can vary greatly within these limits depending on the combination of and variation in observational errors.^{3,1}

4. Errors in the underlying observations and the orbital elements of the Sun

In the preceding section our computation reproduced the historical records of the solar elements to a sufficient order of accuracy by taking into account only the errors in three observations necessary for their deduction. It follows that all other error sources such as year length and the motion of the perigee are of minor magnitudes and the three solar longitudes recorded as observed are solely responsible for those attested orbital

Table 3.2. Observational errors and the resulting maximum attainable errors in the dependent parameters (eccentricity and apogee)

a	b	c	d	e	f	g	h	i	j	
$\Delta \approx \Delta t_{1,2,3} ^{(d)}$ $\approx \Delta I_{1,2,3} ^{(\circ)}$		$ \Delta k _{\max}$	$ \Delta A _{\max}$ ($^{\circ}$)	$ \Delta k _{\max}$ at t =			$ \Delta A _{\max} (^{\circ})$ at t =			
				-130	+140	+880	-130	+140	+880	
1	$1/24 = 1^h$	0.04	0.001	1.7	0.001	0.0009	0.0008	1.6	1.5	1.3
2	$1/4$	0.25	0.006	10	0.006	0.006	0.005	9	9	8
3	1	1	0.025	40	0.023	0.022	0.019	38	36	31
4	$v_1(= 180^{\circ} - A)$ in (3.7) with $K = +90^{\circ}$		$\pm \pi/4 + \pi$		113.5°	108.9°	96.2°	113.5°	108.9°	96.2°

elements of the Sun. Our final problem is thus to search for the origins of the true solar longitudes as recorded by Ptolemy, al-Battānī etc.

Since we may now consider the two terms $\Delta\gamma$ and $\Delta\delta$ in (3.1) as resulting only from the errors in the observed times, $\Delta t_n = t'_n - t_n$ ($n = 1, 2, 3$), we obtain from (2.10)

$$\begin{aligned}
\Delta\gamma &= \frac{1}{2}(\Delta t_3 - \Delta t_1)\omega = \frac{1}{2}(\Delta L_3 - \Delta L_1) \\
&= \frac{1}{2}[\Delta l_3 - 2e\Delta v_3 \cos v_3 + \dots - (\Delta l_1 - 2e\Delta v_1 \cos v_1 + \dots)] \\
&\approx \frac{1}{2}(\Delta l_3 - \Delta l_1) \approx \Delta l_3 \approx -\Delta l_1,
\end{aligned} \tag{4.1}$$

$$\Delta\delta = \Delta\gamma - (\Delta t_2 - \Delta t_1)\omega \approx \frac{1}{2}(\Delta l_3 + \Delta l_1) - \Delta l_2 \approx -\Delta l_2. \tag{4.2}$$

Here the only error sources $\Delta\gamma$ and $\Delta\delta$ are directly connected with the longitudinal errors, Δl_{1-3} .^{4.1}

We shall analyse our problem by following Tycho's rigorous computation and clear demonstration. Our main concern below is with the deduction of Tycho's longitudinal errors—the deviation of the Sun's positions in the orbit l' , which he actually observed, from the longitudes l which he sought to determine:

$$\Delta l_n = l'_n - l_n; \quad n = 1, 2, 3.$$

Tycho determined his three solar longitudes l_n first by measuring the Sun's meridional altitudes, which was virtually the only observation to be made, and then by converting them into longitudes with his parameters (Fig. 4.1).

His longitude l is thus given by his parameters as

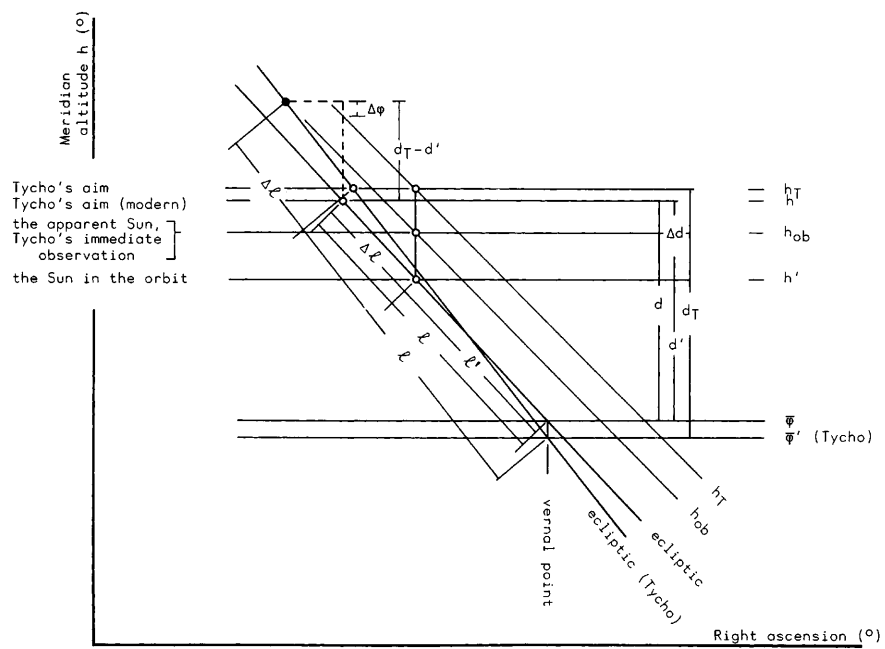
$$\sin l = \frac{\sin d_T}{\sin \epsilon'}, \tag{4.3}$$

where he deduced the declination d_T from his immediately observed solar altitude at the meridian h_{ob} corrected for refraction and solar parallax:

$$d_T = h_{\text{ob}} + p' \cos h - r' - \bar{\varphi}'. \tag{4.4}$$

The modern theoretical declination at this instant is given by

$$d' = h_{\text{ob}} + p \cos h - r \cot h - \bar{\varphi}. \tag{4.5}$$



Putting this into (2) we derive Tycho's longitudinal error Δl as a function of his erroneous parameters. However, it is convenient to consider the total error Δl as consisting of two terms, the Δd - and the $\Delta \varepsilon$ -term [cf. (4.6)].

Note: By our description above Tycho's solar observations can be followed accurately, an indication that Tycho strived for obtaining his solar position, l , with a high precision (cf. Tab. 4.1., 5-q, -r).

The same quantity Δl is also obtainable to the first approximation by simply adding $(d_T - d')$ to the modern declination d as indicated by „- -●- -“ [cf. Maeyama (1974), 39, (15)].

For a general application it is convenient to express longitudinal error Δl approximately as a function of two error sources, Δd and $\Delta \varepsilon$. Since the former consists of three error sources, Tycho's errors in the determination of the solar orbit all originate from the following four sources ($\Delta \varphi - \Delta \varepsilon$):^{4.2}

declination (Δd)

geographical latitude ($\Delta \varphi$)

solar parallax (Δp)

refraction ($r' - r \cot h$)

obliquity of the ecliptic ($\Delta \varepsilon$)

$$\Delta l = l' - l \cong - \frac{\Delta d \cos d - \Delta \varepsilon \sin d \cot \varepsilon}{\cos l \sin \varepsilon}$$

$$= f(\Delta d, \Delta \varepsilon),$$

$$\Delta d = d' - d = \Delta p \cos h - r'_h + r \cot h - \Delta \bar{\varphi},$$

$$h_{\text{ob}} \cong h \cong h_T \cong h' \text{ for } \cos h, \cot h; r' \text{ at } h; r \cong 60'' \text{ at } h > 10^\circ,$$

$$\Delta p = 3' - 9'' = 0.048^\circ,$$

$$\Delta l_n = l'_n - l_n; n = 1, 2, 3; \text{ (cf. Fig. 4.1).} \quad (4.6)$$

Putting the longitudinal errors Δl_{1-3} obtainable from (4.6) into (4.1)–(4.2) we can derive from (3.1) the deviations Δk and $\Delta \theta (= \Delta A)$ and, by means of the modern parameters at the epoch, we finally arrive at the solar orbital elements corresponding to the historical records. Our computed values agree closely with Tycho's parameters (Tab. 4.1., 5-q, -r). It follows that the above series of Eqs (3.1) can be assumed to be an adequate mathematical expression which transmits, simply and accurately, Tycho's actual procedure of deducing the solar orbital elements.

Three of the above four error sources, namely the three contributions to the error in declination, are in principle those astronomical quantities which should be, but were not necessarily, determined as independent parameters. This is a long complex problem.^{4.3}

According to the above equations all these three error sources of declination involve similar contributions to the longitudinal error Δl :

$$\Delta l_d \cong -2.5 \Delta d \frac{\cos d}{\cos l}. \quad (4.7)$$

Hence, their individual contributions at the equinoxes are of the same order in absolute values:

$$\Delta l \cong \mp 2.5 \Delta d \cong \mp 2.5 \times [\Delta p \cos h; -(r'_h - r \cot h); -\Delta \bar{\varphi}]; \quad l_{1,3} = 0, 180^\circ, \quad d = 0. \quad (4.8)$$

The obliquity of the ecliptic can be determined by observation virtually in three ways. From the two observable, extreme altitudes of the Sun, $h_{\max, \min}$

$$\begin{aligned} h_{\max} &= \bar{\varphi} + \varepsilon - p \cos h_{\max} + r \cot h_{\max}, \\ h_{\min} &= \bar{\varphi} - \varepsilon - p \cos h_{\min} + r \cot h_{\min}, \end{aligned} \quad (4.9)$$

we obtain

$$\begin{aligned} \varepsilon'_1 &= \frac{1}{2}[(h_{\max} + p' \cos h_{\max} - r'_{h \max}) - (h_{\min} + p' \cos h_{\min} - r'_{h \min})], \\ \varepsilon'_2 &= (h_{\max} + p' \cos h_{\max} - r'_{h \max}) - \bar{\varphi}, \\ \varepsilon'_3 &= \bar{\varphi} - (h_{\min} + p' \cos h_{\min} - r'_{h \min}), \end{aligned}$$

hence, the originating errors

$$\begin{aligned} \Delta \varepsilon_1 &= \frac{1}{2}[\Delta p(\cos h_{\max} - \cos h_{\min}) - r'_{h \max} + r'_{h \min} + r(\cot h_{\max} - \cot h_{\min})], \\ \Delta \varepsilon_2 &= \Delta p \cos h_{\max} - r'_{h \max} + r \cot h_{\max} - \Delta \bar{\varphi}, \\ \Delta \varepsilon_3 &= \Delta \bar{\varphi} - \Delta p \cos h_{\min} + r'_{h \min} - r \cot h_{\min}. \end{aligned} \quad (4.10)$$

Likewise, if one determines the colatitude $\bar{\varphi}$ by means of the Sun's observation as

$$\begin{aligned} \bar{\varphi}'_1 &= \frac{1}{2}[h_{\max} + h_{\min} + p'(\cos h_{\max} + \cos h_{\min}) - r'_{h \max} - r'_{h \min}], \\ \bar{\varphi}'_2 &= h_{\max} - \varepsilon' + p' \cos h_{\max} - r'_{h \max}, \\ \bar{\varphi}'_3 &= h_{\min} + \varepsilon' + p' \cos h_{\min} - r'_{h \min}, \end{aligned}$$

we obtain its errors, similarly to the derivation above, as dependent on that body:

$$\begin{aligned} \Delta \bar{\varphi}_1 &= \frac{1}{2} \Delta p(\cos h_{\max} + \cos h_{\min}) - r'_{h \max} - r'_{h \min} + r(\cot h_{\max} + \cot h_{\min}), \\ \Delta \bar{\varphi}_2 &= -\Delta \varepsilon + \Delta p \cos h_{\max} - r'_{h \max} + r \cot h_{\max}, \\ \Delta \bar{\varphi}_3 &= +\Delta \varepsilon + \Delta p \cos h_{\min} - r'_{h \min} + r \cot h_{\min}. \end{aligned} \quad (4.11)$$

Although refraction as an atmospheric phenomenon was known in antiquity and Islam and later in the west, it seems that its values for astronomical use, as a function of altitude, were first determined systematically by Tycho.^{4.4} Another particularly important parameter was the solar parallax. Ptolemy's original mean value of this parallax,

Table 4.1. Computational deduction of the historically recorded lengths of the seasons by means of the underlying parameters, historically recorded and modern (cf. Tabs. 3.1., 5.1., 5.2., Figs. 4.1, 4.2)

a	b	c	d	e	f	g	h	i	
		Colatitude (°)			Solar altitude (°)		Refraction		
		recorded	modern		$t_{1,3}$	t_2			
		K (°)	$\bar{\varphi}'$	$\bar{\varphi}$	$\Delta\bar{\varphi}$	$h_{1,3}$	h_2	$r'_{1,3}$	r'_2
1	Hipparchus	90	c. 54						
2	Ptolemy	90	59;2						
			59.033	58.800	+0.233	58.800	82.478	0	0
3	al-Battānī	90	53;59						
			53.983	54.074	−0.090	54.074	77.658	0	0
4	Copernicus	225	35;40,30						
			35.675	35.641	+0.034	35.641	19.261	0	0
5	Tycho	45	34;5,30					45''	
			34.092	34.093	−0.001	34.093	50.464	0.013°	0
		45				''	''	''	''
		135				''	''	''	''

6 Remarks

3-c: Nallino II, 12, n.4.

cols. j, k: mean horizontal parallax, $p = 8.80''$, Expl. Suppl., 490.

1. 1: Hipparchus, not deducible due to the lack of his data.

2-p, -r, 3-p, -r: Ptolemy/al-Battānī not deducible from (4.6) because $l_2 = K = 90^{\circ}$.

Is. 1-4, cols. o-r: $p' = 0$ for all but Tycho; If $p' = 2;51 - 3'$ applied, then $\Delta(t_3 - t_1)^{(d)} = -1.010$ (Ptolemy), $+0.5759$ (al-Battānī), $+0.1394$ (Copernicus);

$\Delta(t_2 - t_1) = +0.2066$ (Copernicus), thus the deviations larger than the results given above for $p' = 0$; cf. Tycho $+2.888$ and $+2.210^h$ (5-q, -r) for $p' = 0$.

Note the historical records [(), cols. q, r], reproducible roughly (Copernicus), accurately (Tycho with $p' = 3'$) and not reproducible (Ptolemy and al-Battānī). In the latter case their recorded parameters contain problems (see text). Hence, Tycho's observational behaviour can accurately be known.

Table 4.1. (Continued)

j	k	l	m	n	o	p	q	r
	Solar parallax		Obliquity of the ecliptic (°)			Longitude (°)	Lengths of the seasons (recorded - modern)	
$p'(^{\circ})$	$\Delta p (^{\circ})$	ε'	ε	$\Delta\varepsilon$	$\Delta l_{1,3}$	Δl_2	$\Delta(t_3 - t_1)$ d h (h)	$\Delta(t_2 - t_1)$ d h (h)
2;51		23;51,20					-1.134	
	+0.045	23.856	23.678	+0.177	± 0.559		-27.2 (+12.9)	
c. 3		23;35					+0.427	
	+0.048	23.583	23.584	-0.001	∓ 0.211		+10.251 (+0.593)	
3		23;28,30					-0.067	-0.060
	„	23.475	23.502	-0.027	± 0.033	-0.026	-1.617 (-4.38)	-1.437 (-2.28)
3		23;31,30					+0.2676	+0.0558
	„	23.525	23.492	+0.033	∓ 0.132	-0.077	+6.422 (+6.038)	+1.339 (+1.056)
							„	„
					„	„	(+5.881)	(+1.226)
							„	+0.2118
					„	+0.077	+5.084	+5.084
							(,,)	(+4.837)
					(4.6)		col. q := $2(-\Delta l_1)/\omega$	
							col. r := $(\Delta l_2 - \Delta l_1)/\omega$	

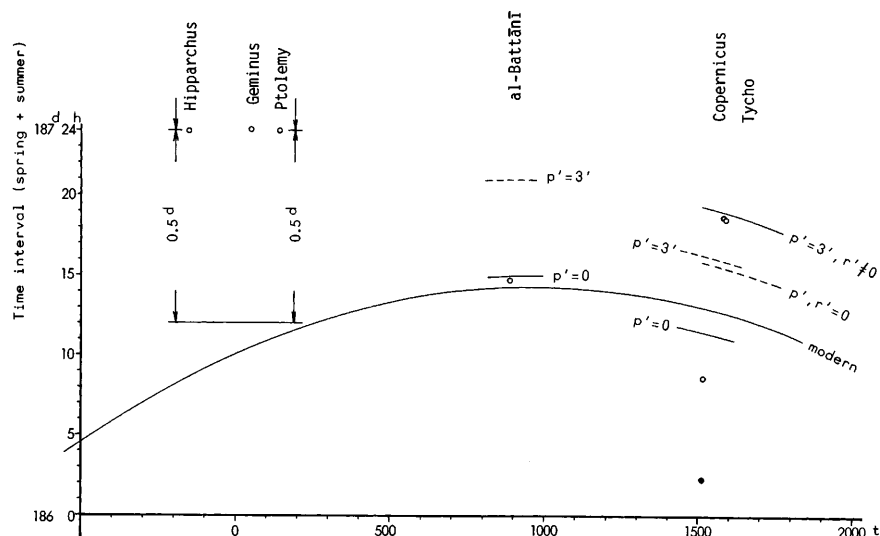


Fig. 4.2. The time interval between two equinoxes ($l_3 - l_1 = 180^\circ$); historical measurements and modern values as functions of time

$t = -500 \sim +2000$ (Tabs. 3.1., col. h; 4.1, col. q; 5.1; Figs. 4.3–4.5, 5.4–5.6)

○	historical records and our reproducing computations.
—	our different reproductions – $\Delta = +13.82^h$ (al-Battānī), $+3.35^h$ (Copernicus), $+2.89^h$ (Tycho), cf. Tab. 4.1., 6 Remarks – they do not agree with the records, hence they are not valid.
----	computed with Copernicus' data, $186;5\frac{1}{2}$, not valid (Tab. 3.1., 4-d, 6).
●	the historical record (187^d) with the error-allowance $\pm 0.5^d$, cf. text.
○	our tentative computation with $\bar{\varphi}'_2 = 54.077^\circ$ [(4.11)], $p' = 0$;
—	$\Delta(t_3 - t_1) = +0.034^d = +0.81^h$, but see the most plausible explanation in ch. 5.2, Fig. 5.3.
—	al-Battānī
—	modern values
	Maeyama (1988), Fig. 1.4.

Note al-Battānī's and Tycho's high accuracy of the immediate observations and their accurate and inaccurate values due to the neglect and application of the erroneous solar parallax of $3'$ respectively (cf. ch. 5).

$2'51''$ (Alm. V, 18; H 442–3), remained virtually unaltered up to the time of Kepler. All these parameters are directly connected with the determination of the lengths of the seasons, as shown in Fig. 4.2. Their great influence on the determination of the solar elements will be treated separately in the next chapter.

For a great time span, $t = 1250 \pm 450 \cong 800 \sim 1700$, during which the position of apogee remained at a nearly right angle to the vernal equinox, we approximate the true anomaly of our first longitude ($l_1 = 0$) by $v_1 \cong 90^\circ$ and obtain from (3.1), (4.1) and (4.8)^{4.5}

$$\begin{aligned}
\Delta k &\cong \frac{1}{\sin K} [\Delta \gamma (\sin K - \frac{3}{4} e \cos^2 K) + \frac{3}{4} \Delta \delta e \cos K] + \Delta k_h \\
&\cong \Delta \gamma \cong \left(\frac{\Delta t_3 - \Delta t_1}{2} \right) \omega \cong \frac{\Delta l_3 - \Delta l_1}{2} = -\Delta l_1 = \Delta l_3 \cong 2.5 \Delta d, \\
&\quad (v_1 = 90^\circ), \tag{4.12}
\end{aligned}$$

$$\begin{aligned}
\Delta \theta &\cong \frac{1}{2e \sin K} (-\Delta \gamma \cos K + \Delta \delta) + \frac{3}{4} e \cos K, \quad (v_1 = 90^\circ), \\
&= \frac{\Delta \delta}{2e} = -\frac{\Delta l_2}{2e} \quad (v_1, K = 90^\circ). \tag{4.13}
\end{aligned}$$

(4.12) shows a remarkably simple relationship, consisting of proportionalities, between observation [time (t) and position (l)] and dependent parameter (eccentricity, k). This only becomes visible if their quantities are evaluated as deviating from the modern values at the epoch, as shown above. The deviation in time and position (Δt , Δl) comes from the one in declination (Δd) which ultimately originates from the errors in a few independent parameters (r' , p' , φ') [cf. (4.6)–(4.8)]. Figs. 4.3–4.5 illustrate the simple proportionalities just mentioned.

If we now put in (4.6) roughly $\Delta \bar{\varphi} = 0$, $\Delta p \cong p'$, we obtain

$$\Delta l_1 = -\Delta l_3 \cong -\frac{-\Delta r_{1,3} + p' \cos \bar{\varphi}}{\sin \varepsilon} (l_1 = 0, l_3 = 180^\circ; \Delta r_1 = \Delta r_3 = r' - r \cot \bar{\varphi}), \tag{4.14}$$

hence, we roughly have from (3.1)

$$\begin{aligned}
\Delta k = k - 2e &\cong \Delta \gamma \cong -\Delta l_1 \cong \Delta l_3 \cong \frac{-\Delta r_{1,3} + p' \cos \bar{\varphi}}{\sin \varepsilon} \cong \frac{p' \cos \bar{\varphi}}{\sin \varepsilon} \cong 2p', \\
&\quad (\text{roughly for } \bar{\varphi} \cong 35^\circ; \text{ e.g. Copernicus and Tycho}). \tag{4.15}
\end{aligned}$$

Consequently, those who took the solar parallax seriously into account had an inevitable tendency to obtain too great an eccentricity:

$$\begin{aligned}
k &\cong 2e + 2p' \\
&\cong 2e + 0.00175 \cong 0.0369 (t = -130) \sim 0.0361 (+880) \sim 0.0355 (+1600) \\
&> 2e, \quad (p' \cong 3'). \tag{4.16}
\end{aligned}$$

or, bisecting the distance between the central body and the reference point for the uniform angular motion and denoting it as e'

$$\begin{aligned}
e' &\cong e + p' \\
&\cong e + 0.0009 \cong 0.0178 (t \cong 1600) \\
&> e. \tag{4.17}
\end{aligned}$$

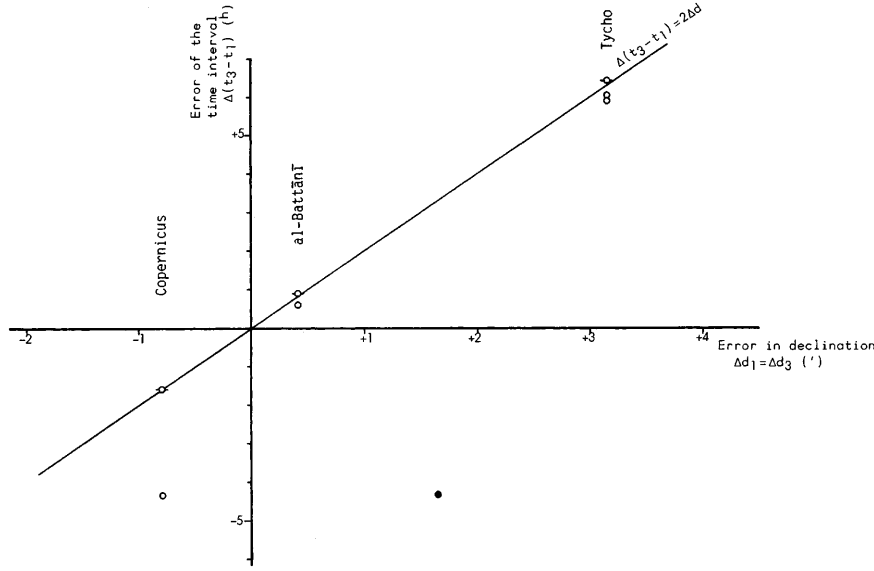


Fig. 4.3. The error of the time interval between two equinoxes as a function of the error of declination (cf. Figs. 4.2, 4.4, 4.5)

The error of the time interval between two equinoxes changes proportionally to the error in declination and their relationship can simply be expressed as

$$\Delta(t_3 - t_1) = 2(-\Delta l_1)/\omega \approx 5\Delta d_1/\omega^{(d)}, \quad (1)$$

hence roughly, e.g., for $\Delta d(^{\circ})$

$$\approx 2\Delta d_{1,3}^{(h)}, \text{ cf. (4.8).} \quad (2)$$

Therefore each minute of an arc in declination yields an error of 2 hours in the length between the equinoxes.

- historical records - modern values relating to our computed values of Δd_1 [(4.6)]; +0.40' (al-Battānī, our tentative computation, cf. Fig. 4.2, esp. ch. 5.2), -0.79' (Copernicus), +3.16' (Tycho).
- our reproductions [(4.6)–(4.8) and (1), (2) above].
- Copernicus' value computed with $p' = 3'$; $\Delta d = +1.64'$ (not valid).

Note the high validity of the simple correspondence, $1'(\Delta d) \sim 2^h[\Delta(t_3 - t_1)]$.

(4.15)–(4.17) show that the eccentricity of the solar orbit to be determined from two equinoctial observations at roughly $t = 1250 \pm 450 \cong 800 \sim 1700$ exceeds the modern value by the value adopted for the solar parallax. Thus, the above eccentricities correspond precisely to e.g. the well-known value of Tycho (0.03584) and to its rounded and bisected one of Kepler (0.018). This has already been shown elsewhere.^{4.6}

Since the values adopted for the solar parallax decreased because of its new determinations as an independent parameter, the determined eccentricity necessarily decreased

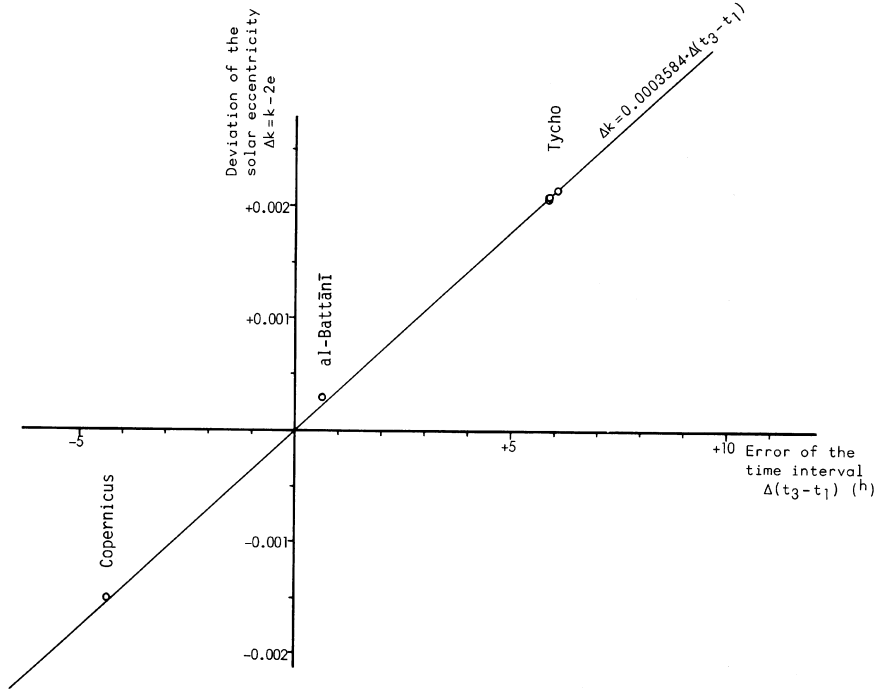


Fig. 4.4. Deviation of the solar eccentricities (dependent parameter) from the modern values as a function of the error in the time interval between two equinoxes (observations); cf. Figs. 4.2, 4.3, 4.5

- historical records - modern double eccentricity ($2e$)
- $\Delta k \approx -\Delta l_1 = \Delta l_3 \approx \Delta(t_3 - t_1)\omega/2$ [cf. (4.12), $A, v_1 = 90^\circ$],
e.g. here $\Delta k = 0.0003584 \cdot \Delta(t_3 - t_1)$.

Hence, an error of $\pm 1^h$ in the time interval between two equinoxes (observations) changes the solar eccentricity k (dependent parameter) by ± 0.00036 .

Note: At $A, v \approx 90^\circ$ [cf. (4.12)] the deviation Δk is proportional to the error of the underlying time interval between two equinoxes, hence roughly applicable for $t = 1250 \pm 450 = 800 \sim 1700$.

too [(4.15)–(4.17)]. This actually began to occur at around Kepler's death (1630) and, as evident from the equations, the solar eccentricity then converged rapidly towards the modern value.^{4.7} Contrary to this later historical process, those astronomers who neglected the solar parallax in their computation were – independently of the epochs due to the actual small amount $p = 8.8'' \approx 0$ – close to obtaining an accurate eccentricity corresponding to the modern value. An outstanding example was al-Battānī (cf. Figs. 2.2, 4.2–4.5; ch. 5.2).

It may be of interest to present our greatly simplified computation. Once again we put $v_1 = 90^\circ (= 180 - A)$ in (3.1) for the whole time interval between Hipparchus

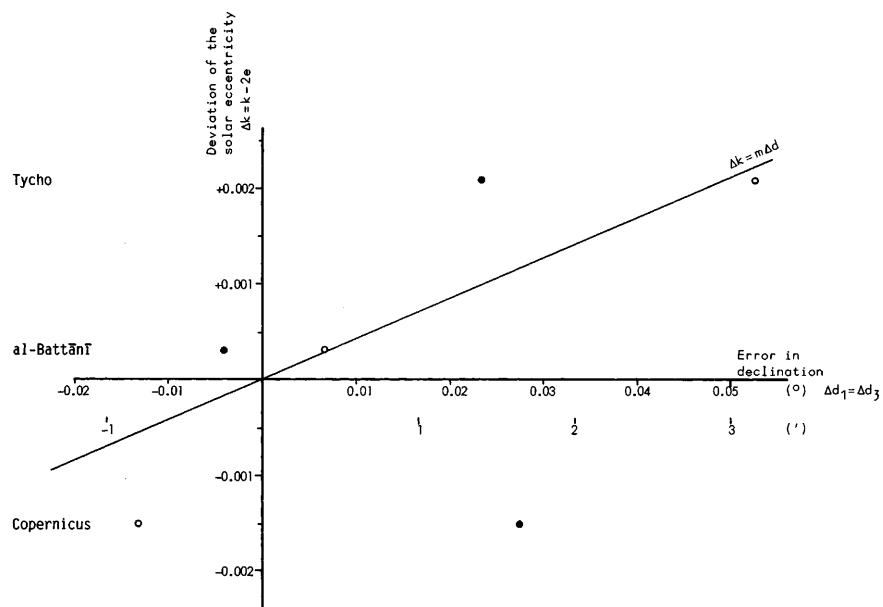


Table 4.2. Approximate solar elements computed from (4.18); cf. Table 3.1.

	$K(^{\circ})$	Δk	k'	$\Delta A = \Delta\theta(^{\circ})$	$A''(^{\circ})$
Hipparchus	90	+0.005445 [+0.006284]	0.04056 (0.04167)	-4.70 [-1.28]	61.83 (65.5)
Ptolemy	90	+0.004625 [+0.006289]	0.03954 (0.04153)	-9.25 [-6.72]	61.87 (65.5)
al-Battānī	90	+0.000209 [+0.000323]	0.03455 (0.03465)	-1.57 [-1.67]	82.21 (82.28)
Copernicus	225	-0.001571 [-0.001472]	0.03225 (0.0323)	+1.98 [+2.12]	96.59 (96.67)
Tycho	45	+0.002108 [+0.002094]	0.03587 (0.03584)	-0.116 [-0.351]	95.75 (95.5)
	135	+0.002108 [+0.002090]	0.03587 (0.03584)	-0.156 [-0.414]	95.71 (95.45)
Remarks	[]: accurately computed from (3.1), Table 3.1., cols. n, q (): recorded values, Table 3.1., cols. p, s.				

$v_1 = 90^{\circ}$ in (3.1); cf. (3.3) – (3.6), (4.12), (4.13)

$$\Delta\gamma = \frac{\Delta(t_3 - t_1)}{2} \omega; \text{ (4.1), Tab. 3.1., cols. h, j}$$

$$\Delta\delta = \left[\frac{\Delta(t_3 - t_1)}{2} - \Delta(t_2 - t_1) \right] \omega \approx -\Delta t_2 \omega \approx -\Delta l_2; \text{ (4.2),}$$

Tab. 3.1., cols. h, i, k

$$e_t, A_t : \text{modern values at the epoch } t. ^{4.8} \quad (4.18)$$

Tab. 4.2. shows the high accuracy of our much simplified Eqs. (4.18) in the case of al-Battānī, Copernicus and Tycho; this accuracy is due to the good approximation of $v_1 \approx A \approx 90^{\circ}$.

If we, once again, simplify our equations above and put $v_1 \approx A \approx 90^{\circ}$, $K = \pm 90^{\circ}$ in (4.12), (4.13), (4.18), we obtain numerically simple expressions:

$$\begin{aligned} \Delta k &\approx \frac{\Delta(t_3 - t_1)}{2} \cdot \frac{\pi}{180} \\ \Delta A &\approx \mp \frac{\Delta t_2}{2e} \approx \mp 30 \cdot \Delta t_2(^{\circ}). \end{aligned} \quad (4.19)$$

In the case of al-Battānī we then derive

$$\begin{aligned} \Delta k &\approx \frac{0.0247}{2} \cdot \frac{\pi}{180} = +0.00022 \quad (+0.000323) \\ k' &\approx 0.034554 \quad (0.034661) \end{aligned}$$

$$\Delta A \approx -30[0.0668 + (-0.0247/2)] \approx -1.63^\circ \quad (-1.672^\circ)$$

$$A' \approx 82.15^\circ \quad (82.109^\circ), \quad (\text{cf. Tab. 3.1., } l.3),$$

agreeing with the accurately computed values in parentheses.

Consequently, the deviation of the eccentricity from $2e$ and the error of the apogee are approximately equal to the error in the time-determination of the equinoxes, $\Delta t_1 \approx -\Delta t_3$, and to 30 times the error of the solstices respectively.

5. Explanatory remarks on the underlying observations of the Sun

Our computation has shown that the solar elements of all astronomers under consideration can be reproduced accurately by means of the modern values if we take the underlying observations, the lengths of the seasons, into account (Tab. 3.1., cols. o/p, r/s). There are, however, inconsistencies between observations and some independent parameters, in particular, in the case of Ptolemy and al-Battānī. The outstanding example is the geographic latitude, which is supposed to be one of the most basic parameters in any observation, and yet the recorded geographic latitudes do not agree with the actual observations of the lengths of the seasons.

We claim, however, that this problem of apparent inconsistencies can be solved to a certain extent. For this purpose we shall first make our preliminary study using Tycho's observations. In the years 1584–1588 Tycho rigorously observed the instants of the equinoxes, vernal and autumnal, and obtained a constant interval between the two equinoxes (Op. om. II, 15). Our comparison of his observations with Tuckerman's Tables (Tab. 5.1.) clearly shows, as expected, that the total error of Tycho's time interval between the two equinoxes consists of two observational errors of equal magnitude (cols. d, g, note), and that therefore his observations at the two equinoxes were made under the same condition. Assuming that this argument is applicable in the case of other astronomers as well as in Tycho's case, we can deduce approximate errors at the equinoxes in time and position from the lengths of the seasons recorded by these astronomers:

$$\Delta t_{1,3} = \mp \frac{\Delta(t_3 - t_1)}{2},$$

$$\Delta l_{1,3} \approx \mp \frac{\Delta(t_3 - t_1)}{2} \omega,$$

and hence:

$$\begin{aligned} \Delta t_2 &= \Delta(t_2 - t_1) + \Delta t_1^{(d)}, \\ \Delta l_2 &\approx [\Delta(t_2 - t_1) + \Delta t_1] \omega(^{\circ}). \end{aligned} \tag{5.1}$$

Table 5.1. Tycho's determination of the vernal and autumnal equinoxes, 1584–1588
(Op. om., II, 15).

a	b	c	d	e	f	g	h
		Vernal equinox		Autumnal equinox			
		t_1	Δt_1	t_3	Δt_3		
		Tycho	modern	Tycho	Tycho	modern	Tycho
				- modern			- modern
		d h m	d	d h m	d	d h	d h
1	1584	9 21 30		−0.1318	12 16 0		+0.1262
		9.8958	10.0276	−3.162	12.6667	12.5404	+3.029
2	1585	10 3 19		−0.1281	12 21 49		+0.1275
		10.1382	10.2663	−3.074	12.9090	12.7815	+3.060
3	1586	10 9 8		−0.1329	13 3 38		+0.1212
		10.3806	10.5134	−3.189	13.1514	13.0302	+2.908
4	1587	10 14 56		−0.1299	13 9 26		+0.1334
		10.6222	10.7521	−3.117	13.3931	13.2597	+3.202
5	1588	9 20 45		−0.1308	12 15 15		+0.1271
		9.8646	9.9954	−3.140	12.6354	12.5083	+3.051
6	Remarks	$\bar{\Delta}t_1 = -0.1307^d = -3.136^h$		$\bar{\Delta}t_3 = +0.1271^d = +3.050^h$			
		$\sigma = 0.0016 = 0.039$		$\sigma = 0.0039 = 0.094$			
		$ \bar{\Delta}t_1 + \bar{\Delta}t_3 = +0.2578^d = +6.186^h$		$ \bar{\Delta}t_1 - \bar{\Delta}t_3 = +0.086^h = +5.2^m$			

Tycho's equinoctial determinations (the immediate observations, corrected for parallax, refraction and equation of time and interpolated) are accurately reproducible by means of his astronomical data (φ' , p' , r') and the modern parameters [cf. (4.6)].

Unlike his length of the tropical year his time interval from the vernal to the autumnal equinox is thus systematically too long by ca. 6^h (Figs. 4.2–4.5, 5.4–5.6):

Tycho–modern values:

$$|\bar{\Delta}t_1| + \bar{\Delta}t_3 = +6.19^h \text{ (av., 6 Remarks):Tuckerman}$$

$$\Delta(t_3 - t_1) = +5.88 \text{ (Tab. 3.1., 5-h): Maeyama (1988), Fig. 1.4}$$

$$2(\mp \Delta l_{1,3})/\omega = +6.42 \text{ (Tab. 4.1., 5-q): Computed from (4.6)}$$

The small differences in the above values are primarily due to the neglect of perturbations in the apparent solar motion [ca. 30^m max(= 15^m × 2); cf. Maeyama (1988), 5]. Tycho's error is therefore mainly caused by his adoption of the traditional solar parallax too great by $\Delta p' \approx 3'$ [cf. Maeyama (1974), passim, esp. 51]. The corrections to Tuckerman's tables at Tycho's time are small and were here omitted [cf. Britton's study (1992), 17, 153–178].

Note: Tycho's determination of two equinoxes shows systematic errors of the same amount [$|\bar{\Delta}t_1| \approx \bar{\Delta}t_3 \approx \Delta(t_3 - t_1)/2 \approx 3.1^h$ (cols. d, g, h)] indicating that his solar observations were equally conditioned at those two instants.

Putting Δl into (4.6) we obtain the error of the colatitude $\Delta \bar{\varphi}$, on which the whole observational operations are supposed to have been based. With $\Delta l_{1,3}$ we may deduce a simple equation giving the relevant value, namely the error of the meridian altitude at which the equinoctial observations of the Sun were actually made:

Table 5.2. The colatitudes ($= 90^\circ$ -geographical latitudes), practically used and recorded (cf. Tab. 5.1.)

a	b			c		d	e
	Errors of colatitude ($\Delta\tilde{\varphi}$)					p', r'	Remarks
	practically used			recorded			
1 Hipparchus	-0.115°	-6.90'		uncertain		0	
2 Ptolemy	-0.098	-5.86		+0.233°	+14'	0	
3 al-Battānī	+0.006	+0.35		-0.074	-4.42	0	$\varphi' = 36^\circ$
4				-0.090	-5.42		$\varphi' = 36^\circ 1'$; Op. astr. II, 12
5 Copernicus	+0.057	+3.43					
6	+0.026	+1.55				0	$K = 225^\circ$
7	av +0.042	+2.49		+0.034	+2.07		
8 Tycho	+0.0034	+0.205	+12.3''			$p' = 3'$	
9 1588	+0.0026	+0.154	+9.2			$r'_{1,3} = 45''$	$K = 45^\circ$
10	-0.0014	-0.084	-5.0			$r'_2 = 0$	$K = 135^\circ$
11	av +0.0015	+0.091	+5.5	-0.0011	-0.067	-4''	
12 Remarks	Deduced from the recorded lengths of the seasons [from (5.2); <i>ls.</i> 6,9, 10 from (4.6) and (5.1)]. 1–, 2–b: A striking agreement with the systematic error in declination, ca. -6' ~ -7', as found by Fortheringham, 412 ff. and Britton (1992), 18–24.			1-c: $\varphi, \varphi' \approx 36^\circ$		<i>ls.</i> 1–7: $p' = 2'51'' \sim 3'$ not applied, cf. text. For r' (<i>ls.</i> 8–11) cf. Op. om. II, 64.	

Note the great discrepancies between the observing conditions (col. b) and the recorded values (col. c) in the case of Ptolemy and al-Battānī (*ls.* 2-4, cols. b-c) and Tycho's excellent agreement; $-5 \sim +12''$ versus $-4''$ (*ls.* 8-11, cols. b-c); cf. text.

$$\Delta\bar{\varphi} = -\frac{\Delta(t_3 - t_1)}{2}\omega \sin \varepsilon + \Delta p \cos \bar{\varphi} - r' + r \cot \bar{\varphi}. \quad (5.2)$$

Our computation shows Tycho's marvellous agreement between the descriptions ($-4''$) and the practice of observations ($-5 \sim +12''$; Tab. 5.2., *ls.* 8-11, cols. b-c).

Based on the results derived above we shall now analyse the individual observational conditions.

5.1. Hipparchus-Ptolemy (ca. -130/+140)

Since we have more general data about Ptolemy than about Hipparchus, we shall first try to clarify the problem in Ptolemy's case. In principle we presuppose that Ptolemy actually did what he said with regard to his observations.

His fundamental parameters for φ and ε were deduced from his observations of two extreme zenith-distances of the Sun, $z'_{\max, \min}$:

$$\varphi' = \frac{1}{2}(z'_{\max} + z'_{\min}) = 30^\circ 58', \quad (\text{Alm V, 13; H 407, 409}),$$

$$2\varepsilon' = z'_{\max} - z'_{\min} = 47^\circ 40' \sim 47^\circ 45', \quad (\text{Alm I, 12; H 68}),$$

from which we have

$$z'_{\max} = 54^{\circ}48' \sim 54^{\circ}50.5',$$

$$z'_{\min} = 7^{\circ}8' \sim 7^{\circ}5.5'.$$

Now, from (5.1) and the lengths of the seasons measured by Hipparchus which Ptolemy claimed to have reconfirmed by his own observations,^{5.1} we may deduce

$$\Delta t_{1,3} = \mp 0.2689^{\text{d}} = \mp 6.45^{\text{h}}$$

$$\Delta t_2 = +0.3272 = +7.85, \quad (\text{Tab. 3.1., 2-h,-i}),$$

hence, from (5.2) we obtain the colatitude, the meridian altitude of the Sun which he observed at the equinoxes

$$\Delta \bar{\varphi} = -5.86' \approx -6', \quad (\text{Tab. 5.2., 2-b}).$$

Since the zenith-distance of his equator gave him his geographic latitude

$$\varphi' = 30^{\circ}58' (\varphi = 31^{\circ}12'; \Delta \varphi = -14'),$$

and we can put

$$\varphi' = \varphi - \Delta z - \Delta \bar{\varphi},$$

his zenith must have been inclined to the south by the amount (which we will call Ptolemy's error of zenith):

$$\begin{aligned} \Delta z &= +14'(-\Delta \varphi; \text{his error}) + 6'(-\Delta \varphi; \text{deduced from his lengths of the seasons}) \\ &= +20'. \end{aligned} \quad (5.3)$$

His extreme zenith-distances of the Sun corrected for this error will then be

$$z''_{\min} = 7^{\circ}8' \sim 7^{\circ}5.5' + 20' = 7^{\circ}28' \sim 7^{\circ}25.5'$$

$$z''_{\max} = 54^{\circ}48' \sim 54^{\circ}50.5' + 20' = 55^{\circ}8' \sim 55^{\circ}10.5'.$$

Comparing these values to the modern ones deducible from (4.9)

$$z_{\min} = 7^{\circ}31.2'$$

$$z_{\max} = 54^{\circ}51.4',$$

two observations at the solstices and the resulting parameters:

$$\begin{aligned}
 z'_{\min} &= 7^{\circ}8' [+20'(\text{error of zenith}) = 7^{\circ}28'] ; \Delta z_{\min} \approx -3' \\
 h'_{\min} &= 35^{\circ}12' ; \Delta h_{\min} \approx +3' \\
 2\varepsilon' &= (90^{\circ} - h'_{\min}) - z'_{\min} = 47^{\circ}40' ; \Delta \approx +20' \\
 \varphi' &= [(90 - h'_{\min}) + z'_{\min}]/2 = 30^{\circ}58' ; \Delta \approx -14'.
 \end{aligned} \tag{5.4}$$

None of the above values contradicts his allegedly reconfirmed lengths of the seasons – correct to 0.6^d (Tab. 3.1., 2-h, -i) – or his observational conditions which we deduced by means of these lengths (the meridian altitude of his equator, $\Delta\bar{\varphi} = -6'$; Tab. 5.2., 2-b). His error of the zenith as deduced by us, $20'$, is large indeed, however, so is the error of the maximum difference of the solar altitudes 2ε (c. $20'$) as well as the error of the geographic latitude ($14'$). In fact, our computation shows that if he had related his geographic latitude to the correct zenith he would have obtained entirely different lengths of the seasons, $\Delta(t_3 - t_1) = -1.13^d$ (Tab. 4.1., 2-q) instead of $+0.54^d$ (Tab. 3.1., 2-h). In that case he should also have found the solar elements to be entirely different from his records.

By means of the modern values and his errors Ptolemy's obliquity of the ecliptic is approximately given as

$$\varepsilon' = \frac{1}{2}[2\varepsilon - \Delta z_{\min}(-3') - \Delta h_{\min}(+3') + \text{error of zenith}(20')] = 23^{\circ}50'; \Delta\varepsilon \approx +10',$$

which is too large simply by half the error of zenith. Also his geographic latitude of Alexandria,

$$\varphi' = \varphi(31^{\circ}12') - \text{error of zenith}(20') - \Delta\bar{\varphi}(-6') = 30^{\circ}58', \quad \Delta\varphi = -14', \tag{5.5}$$

is too small by his error of zenith and of the meridian altitude of the instrumental equator [$-20' - (-6') = -14'$].

It is particularly interesting that we find the same phenomenon in al-Battānī's observations (ch. 5.2). Our description above may be offered in addition to several other possibilities hitherto claimed.^{5.2}

The fact that Ptolemy, ca. 300 years later, still found no need to correct Hipparchus' solar data has frequently been called into question.^{5.3}

Due to the fraction $1/4^d$ in the year's length, of which no one was unaware in their time, Hipparchus and Ptolemy sought to attain at least that order of accuracy in their solar theories. The time interval, $(t_3 - t_1)$ and $(t_2 - t_1)$, can therefore contain roughly the double order of accuracy, $\pm 0.5^d$. Within that range Ptolemy seems to have found that Hipparchus' values agree with his own measurements (0.5, 0.6^d resp. at his time; Tab. 3.1., 2-h, -i).

Though both show very similar deviations from the modern values (Fig. 5.2, 1), Ptolemy's error in the equation of the center $\Delta\delta$ is twice as much as Hipparchus' (2), so that his final error turns out to be much greater than Hipparchus' (3).

This can also be shown by a simple computation. If we put $K = 90^\circ$ and (just for a simple comparison) $v_1 = 90^\circ$ for 113° and 109° (Tab. 3.1., 1-, 2-l), we obtain roughly from (3.1)

$$\Delta\theta = \Delta A \approx +\frac{\Delta\delta}{2e} \approx 30 \cdot \Delta\delta \quad [\text{cf. (3.7)}]. \quad (5.6)$$

Putting the difference

$$\Delta(\Delta\delta) = \Delta\delta_{\text{pt}} - \Delta\delta_{\text{Hi}} \approx -0.16^\circ, \quad (\text{Tab. 3.1., 1-, 2-k}),$$

we obtain the difference in their errors as

$$\Delta(\Delta A) = \Delta A_{\text{pt}} - \Delta A_{\text{Hi}} \approx 30 \cdot \Delta(\Delta\delta) \approx -4.8^\circ, \quad (\text{Tab. 3.1., 1-, 2-q}), \quad (5.7)$$

which roughly agrees with the actual longitudinal motion of the apogee in 270 years, 4.6° , which virtually corresponds to Ptolemy's error, -5.6° (Tab. 3.1., 1-, 2-l, -q, -s; Fig. 5.2).

Contrary to the case of the eccentricity, Ptolemy's solar apogee shows a much greater error than Hipparchus' because his underlying errors in time determination, though very similar to Hipparchus', tended to enlarge the error of the equation of the center $\Delta\delta$, which plays an important role in determining the solar apogee [(4.13), Fig. 5.2].

The seemingly complex problem of the apogee is in fact simple, as appears from the expression (3.7). The accuracy of the determined location of the apogee corresponds precisely to the errors of the underlying observations. If these errors are given within a certain amount, say $\pm 1/4^d$, the accuracy can vary (neglecting the small amount of $\pm 0.6^\circ$ due to the hypothesis [(3.10)]) within the corresponding maximum range, $\pm 9^\circ$ (Tab. 3.2, 2-h, -i), including, of course, the possibility of becoming „0“. This was the case for Hipparchus and Ptolemy.^{5.4}

Because of the low order of accuracy in general, the question of whether the solar parallax was taken into account cannot be answered, nor is it of importance.

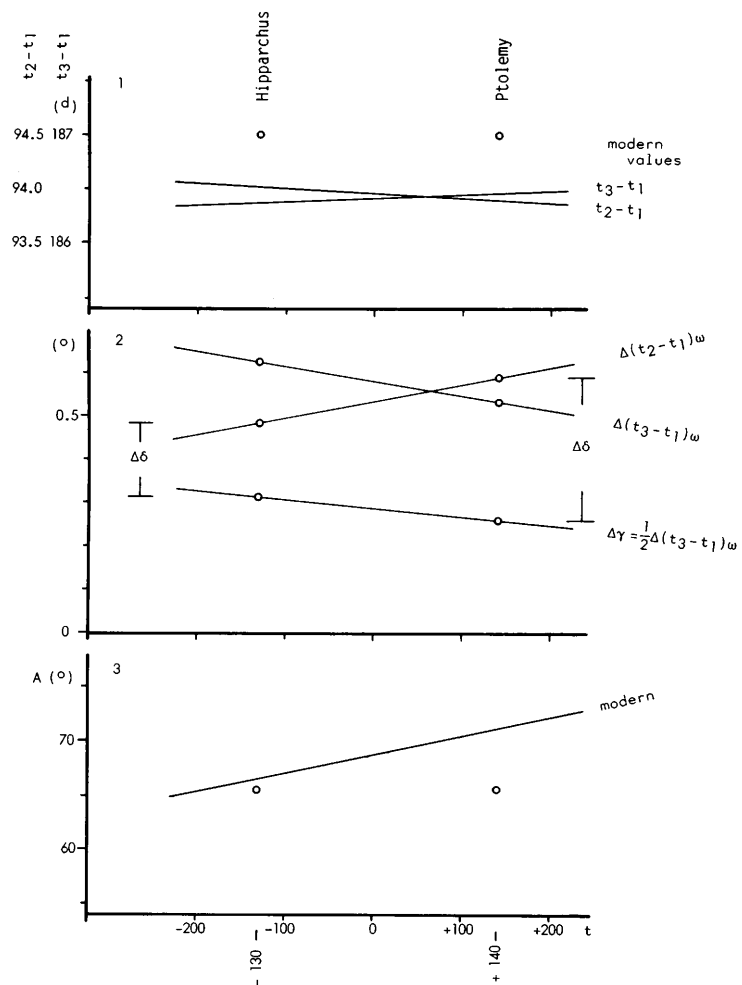
5.2. *Al-Battānī*

As in the case of Ptolemy, al-Battānī's lengths of the seasons show no agreement with his still more basic observational data such as the geographic latitude.

The error in his geographic latitude $\Delta\varphi = 0.09^\circ = 5.4'$ (Tab. 4.1., 3-e) would cause an error of more than 10^h (Tab. 4.1., 3-q) in the length of the half-year, substantially greater than his actual error of 0.6^h (Tab. 3.1., 3-h).

Unlike Ptolemy, however, al-Battānī gives us his observations of two extreme zenith-distances of the Sun (Op. astr. IV, Nallino I, 12):

$$\begin{aligned} z'_{\min} &= 12^\circ 26' \\ z'_{\max} &= 59^\circ 36', \end{aligned} \quad (5.8)$$



This quantity eventually turned out to be the error of Ptolemy at t_2 , $l_2 = 90^\circ$, which was added to the error of Hipparchus.

3. Hence, the same solar apogee resulting from the same underlying observations shows two entirely different deviations from the actual position at the epochs. Ptolemy's result therefore shows an additional error of

$$\Delta(\Delta A_{\text{pt}}) \approx \Delta(\Delta \delta)/2e \approx -4.6^\circ,$$

corresponding roughly to the motion of the apogee in that time interval, $0.0172 \times 270 \text{ years} = 4.6^\circ$, connected to the phenomenon that the length of the seasons changes as a function of the tropical longitude of the solar apogee.

This can be shown by means of our much simplified equation. The deviation of a time interval from its mean value is given as a function of the initial mean anomaly α_0 by

$$(s\Delta S) \approx -\frac{S}{\pi} 2e \cos\left(\alpha_0 + \frac{M}{2}\right) \sin \frac{M}{2} \dots = f(\alpha_0), \text{ (d), [Maeyama (1988), eq. (3.7)],}$$

where S is the length of the mean tropical year in days. Putting $\alpha_0 \approx v_1$ and $v_1 + \Delta v$ for Hipparchus and Ptolemy respectively at $l_1 = 0$ and further for the mean anomalistic motion of the Sun for the two time intervals $M \approx l_2 - l_1 = 90^\circ$ and $\approx l_3 - l_1 = 180^\circ$, we obtain the error which Ptolemy had to derive additionally to that of Hipparchus according to the equations (4.13), (4.18), (5.1) above as

$$\Delta A_{\text{pt}} - \Delta A_{\text{Hi}} \approx \frac{\Delta(\Delta \delta)}{2e} \approx \frac{360}{\pi} \Delta v [\sin(v_1 + 45^\circ) \sin 45^\circ - \frac{1}{2} \cos v_1].$$

Putting $v_1 = 90^\circ$ for simplification, roughly corresponding to the decrease of the true anomaly v_1 , the increase of the longitude of the apogee, in that time lapse becomes

$$\approx \Delta v \approx 109 - 113.5^\circ = -4.5^\circ.$$

Note: From the same underlying data, satisfying approximately the presupposed condition of both astronomers, one logically obtains the same orbital elements which, however, show entirely different errors at two different epochs, since the lengths of the two seasons concerned decreased and increased respectively in relation to the tropical longitude of the apogee. The resulting additional error of Ptolemy therefore corresponds to the longitudinal advance of the solar apogee since Hipparchus' time. The case of the eccentricity is entirely different due to its slow change.

where the modern values are

$$z_{\min} = 90 - h_{\max} = 12.339^\circ = 12^\circ 20' 22''$$

$$z_{\max} = 90 - h_{\min} = 59.484^\circ = 59^\circ 29' 2''; \quad [\text{cf. (4.9)}],$$

hence

$$\Delta z_{\min} = +5' 38''$$

$$\Delta z_{\max} = +6' 58''. \quad (5.9)$$

Since the deviations would have been still greater if he had corrected his observed values for the solar parallax ($\Delta z_{\min} \approx +6.3'$, $\Delta z_{\max} \approx +9.5'$), we are certain that al-Battānī, like most astronomers up to Tycho's time, neglected the solar parallax for his solar determination.

Knowing this, we can now determine the colatitude, at which his instrument is supposed to have been fixed:

$$\Delta\bar{\varphi} = +0.006^\circ = +21'' \quad (\text{Tab. 5.2., 3-b}),$$

hence

$$\varphi'' = 90 - (\bar{\varphi} + \Delta\bar{\varphi}) = 35^\circ 55' 14'' = 35.921^\circ,$$

which deviates considerably from his own determination:

$$\varphi' = (z'_{\min} + z'_{\max})/2 = 36^\circ 1' \quad (\text{Nallino I,12, n.4}),$$

$$\varphi' - \varphi'' = +5.8'.$$

Notably, all three deviations above [(5.9)] are of the same magnitude, ca. $+6'$, which would indicate that there is only one systematic error-source, the zenith. Fig. 5.3 shows our approximate picture of how al-Battānī is supposed to have made his solar determinations.

Although the error in his assumed zenith-direction is very great ($0.1^\circ = 6'$), we are able to reproduce all numerical determinations of al-Battānī's solar elements to $\pm 40'' = \pm 0.01^\circ$, if we give to his immediate observations this amount as a limit of uncertainty. Due to the remarkable accuracy of his immediate observations we can see no other feasible way of explanation. As to the fact that al-Battānī gives all solar observational values only to a minute of arc, we have to consider a rounding error of $\pm 30''$, which is close to our above allowance of $\pm 40''$. Only in this way, it seems, can we clarify his extremely accurate measurements of the equinoxes and the summer solstice with small errors of $\Delta t_{1,3} = \mp 0.3^h$ and $\Delta t_2 = +1.3^h$ [(5.1); Tab. 3.1., 3-h, -i; Fig. 5.3].

With respect to the immediate observations of the Sun, al-Battānī's accuracy is comparable with that of Tycho.

5.3. Copernicus

As is well known and as we mentioned above,^{5.5} Copernicus' numerical data frequently contain ambiguities and sometimes notorious inconsistencies. Unlike in the case of Tycho and Kepler, this was certainly a consequence of his huge task to propagate a new astronomical system which he hoped would function quantitatively for many centuries – replacing Ptolemy's traditional world system which was based on observations of which the earliest, Babylonian, ones dated from the year -720 .

With regard to the geographical latitude of Frauenburg $-54^\circ 19' 1/2'$ given in the “De revolutionibus” (III, 2) – we note that Copernicus seems to have employed this value consistently, because he gives there $-8^\circ 40'$ for the declination of Spica which is too small ($+1.9'$) precisely by that amount of his error in the colatitude $\Delta\bar{\varphi} = +2.07'$.^{5.6} This quantity in fact agrees precisely with what we deduced from his observations on the lengths of the seasons, $+1.55'$ and $+3.43'$ (Tab. 5.2., 5 ~ 7-b, -c).

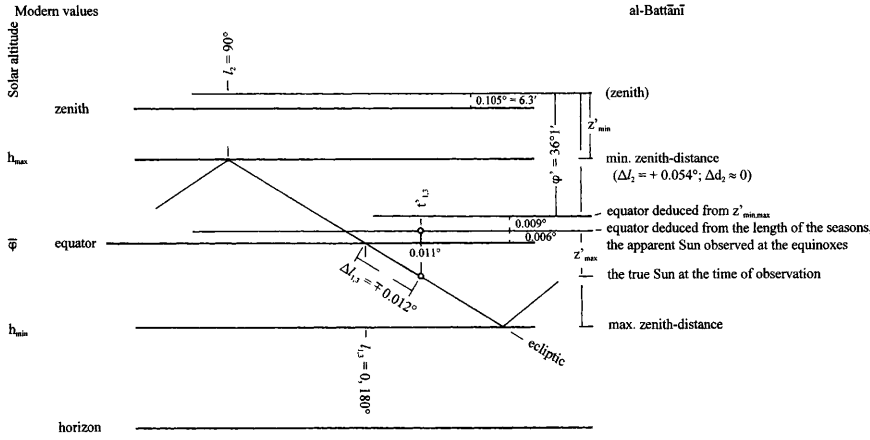


Fig. 5.3. Al-Battānī's meridional observation of the apparent Sun (approximately reproduced)

We put al-Battānī's zenith as deviating from the true one by the quantity deducible from his two immediate observations of the solar zenith-distance [(5.9)]:

$$\Delta z_{\text{zenith}} = (\Delta z_{\text{min}} + \Delta z_{\text{max}})/2 = 6.3' = 0.105^\circ.$$

We then obtain the two extreme zenith-distances as measured from his zenith by means of the modern values:

$$z'_{\text{min}} = \varphi - \varepsilon - r \tan z_{\text{min}} + p \cos(90 - z_{\text{min}}) + \Delta z_{\text{zenith}} = 12.444^\circ = 12^\circ 26' 40''; \Delta = -40''$$

$$z'_{\text{max}} = \varphi + \varepsilon - r \tan z_{\text{max}} + p \cos(90 - z_{\text{max}}) + \Delta z_{\text{zenith}} = 59.589^\circ = 59^\circ 35' 20''; \Delta = +40''$$

$$\varepsilon' = (z'_{\text{max}} - z'_{\text{min}})/2 = 23.572^\circ = 23^\circ 34' 20''; \Delta = +40''$$

$$\varphi' = (z'_{\text{max}} + z'_{\text{min}})/2 = 36.017^\circ = 36^\circ 1'; \Delta = 0,$$

where al-Battānī's values are reproduced within a range of $\pm 40''$.

With a deviation of only $0.009^\circ \approx 30''$ from his geographic latitude [$\varphi' = 36^\circ$ (recorded); $= 36^\circ 1'$ (computed); Tab. 5.2., 3-, 4-e], the equator of his instrument is situated as deduced by us (Tab. 5.2., 3-b). Thus he was able to make most accurate observations of the Sun at the equinoxes and summer solstice:

$$\Delta l_{1,3} = \mp 0.012^\circ, \quad \Delta t_{1,3} = \mp 0.3^h$$

$$\Delta l_2 = +0.054^\circ, \quad \Delta t_2 = +1.3^h \text{ [according to (5.1); Tab. 3.1., 3-h, -i].}$$

Note: The coincidence of the two locations of the equator ($\Delta \approx 0.009^\circ \approx 30''$; cf. e.g. Tycho, $-5 \sim +12''$ versus $-4''$, Tab. 5.2., 8 $\sim$ 11-b, -c) – the one computed from his extreme zenith-distances of the Sun [(5.9)] and the other deduced from his lengths of the seasons (Tab. 5.2., 3-b) – indicates a high plausibility of our reproduction above.

With his colatitude and his other data such as the obliquity of the ecliptic, but discarding his notoriously erroneous year length ($186;5 \frac{1}{2} + 178;53 \frac{1}{2} = 364;59$: III,16), we could reproduce reasonably well his errors in the time intervals, taking also accidental errors into account (Tab. 4.1., 4-q, -r).

Copernicus most probably determined the colatitude and the obliquity of the ecliptic by means of the two extreme altitudes of the Sun:

$$h_{\max} = 59.152^\circ = 59^\circ 9.1'$$

$$h_{\min} = 12.213^\circ = 12^\circ 12.8' \quad [(4.9)];$$

hence

$$\bar{\varphi}'_1 = 35.682^\circ = 35^\circ 40.9' \quad [(4.11); p', r' = 0]$$

$$\varepsilon'_1 = 23.469^\circ = 23^\circ 28.1' \quad [(4.10); p', r' = 0].$$

These two values are close to Copernicus', the deviations being only ca. $20''$:
Copernicus:

$$\bar{\varphi} = 35^\circ 40' 30'' \quad (\text{Tab. 4.1., 4-c})$$

$$\varepsilon = 23^\circ 28' 30'' \quad (\text{Tab. 4.1., 4-l}).$$

Copernicus' actual deviations from the modern theoretical values, $\Delta\bar{\varphi} = +2.1'$ (Tab. 4.1., 4-e) and $\Delta\varepsilon = -1.6'$ (4-n), therefore correspond to

$$\Delta\bar{\varphi}'_1 = +0.042^\circ = +2.5' \quad [(4.11); p', r' = 0]$$

$$\Delta\varepsilon'_1 = -0.033^\circ = -2.0' \quad [(4.10); p', r' = 0], \quad (5.10)$$

and are attributable primarily to the neglect of refraction.

The above correspondence shows that on determining his solar elements Copernicus also neglected the traditional solar parallax $3'$ (Tab. 4.1., Remarks), despite having mentioned it (IV, 21).

As we have seen above, both al-Battānī and Copernicus followed almost the same procedures based on similar observations in order to determine the solar orbit. The question that arises is why their results differ strongly from one another and why the former was more accurate than the latter. Neglecting all minor factors we can reduce the question to the problem of refraction, particularly because both al-Battānī and Copernicus neglected the solar parallax of $3'$.

Designating for simplification the deviation between their errors by " Δ ", we obtain roughly

$$\Delta(\Delta\bar{\varphi}) \approx \frac{1}{2}r[(\cot h_{\min})_{\text{Cop}} - (\cot h_{\min})_{\text{al-B}}] \approx +1.4' [p', r' = 0 \text{ in } (4.11)];$$

hence from (4.8)

$$\Delta(\Delta l_1) \approx +3.5' \approx +0.06^\circ,$$

thus the deviation in the time interval

$$\Delta[\Delta(t_3 - t_1)] \approx \frac{2[-\Delta(\Delta l_1)]}{\omega} \approx -0.12^d \approx -2.8^h,$$

agrees roughly with

$$\text{Copernicus} - \text{al-Battānī} = -4.4 - (+0.6) = -5.0^h \text{ (Tab. 3.1., 3-, 4-h),}$$

or more precisely with

$$= -1.6 - (+0.6) = -2.2^h \text{ (Tab. 4.1., 3-, 4-q).} \quad (5.11)$$

The above value $\Delta(\Delta l)$ is introduced into the equation of the center and further into the eccentricity as

$$\Delta(\Delta k) \approx \Delta(\Delta \gamma) \approx -\Delta(\Delta l_1) \approx -0.06^\circ \approx -0.001 \text{ rad,}$$

corresponding to

$$(\Delta k)_{\text{Cop}} - (\Delta k)_{\text{al-B}} \approx -0.0015 - 0.0003 = -0.0018$$

or, neglecting the decrease of the value in the time interval of ca. 630 years ($\Delta k_t \approx 2\Delta e_t \approx 0.0005$), roughly to

$$k_{\text{Cop}} - k_{\text{al-B}} \approx 0.0323 - 0.0346 = -0.0023. \quad (5.12)$$

The different orders of accuracy of the solar determinations of two astronomers can therefore be reduced essentially to the effect of refraction at two different minimum solar altitudes, or simply to two different geographical latitudes:

$$\text{Copernicus: } h_{\min} \approx 12.1^\circ; r \cot h_{\min} \approx 4.6'$$

$$\varphi = 54.359^\circ \text{ (Frauenburg)}$$

$$\text{al-Battānī: } h_{\min} \approx 30.5^\circ; r \cot h_{\min} \approx 1.7'$$

$$\varphi = 35.926^\circ \text{ (al-Raqqā)}$$

$$\Delta \text{ refraction} = 3'$$

$$\Delta \varphi = 18.4^\circ; \text{ Tab. 4.1., 3-, 4-d.} \quad (5.13)$$

It is strange that in determining the solar orbit Copernicus – unlike many other astronomers, especially Tycho – used the low solar altitude at $l_2 = 225^\circ$ as one of the three fundamental observations for his determination.

The values of the solar eccentricities which would result from accurate observations, following their procedures described above ($p', r' = 0$), are equal to the values given by the two astronomers to the third place after the comma:

Table 5.3.

	$\Delta I_{1,3}$	$\Delta(t_3 - t_1)$	Δk	k'	records
Copernicus	$\pm 0.051^\circ$	$-0.104^d = -2.49^h$	-0.00089	0.0329	0.0323
al-Battānī	± 0.010	$-0.020 = -0.49$	-0.00017	0.0342	0.03465
			(3.1), (4.6)	Tab. 3.1., 3–, 4–o, –p	

In short, Copernicus' ignorance of refraction caused too great a colatitude [$\tilde{\varphi}' > \tilde{\varphi}$, (5.10)], forcing him to measure too low declinations of the celestial bodies [$d' < d$ (e.g., $= 0$ for the Sun at the equinoxes)], thus further to determine, e.g. too short a time interval between two equinoxes (Tab. 4.1., 4-q). This gave him too small an eccentricity of the solar orbit (Tab. 3.1., 4-n).^{5.7}

Hence Copernicus' enigmatically small eccentricity of the Sun was simply caused by the effect of refraction. This actually occurred, 1) comparing to al-Battānī, both of them ignorant of refraction, by the high geographical latitude of Copernicus' observing place, and 2) comparing to Tycho, both having equally high latitudes, by Copernicus' determination of his colatitude not as an independent but as a dependent parameter by means of the solar altitudes, as shown above [cf. (5.10)]. All this can be clearly explained by means of few equations, (4.6) –(4.8), (4.12) (Figs. 4.2–4.5).

It is now clear why numerous Muslim astronomers succeeded in determining the solar orbit accurately:^{5.8}

1. They generally had high observational accuracy,
2. They neglected the (traditional) solar parallax,
3. They observed high solar altitudes at the meridian due to low geographical latitudes thus their observations were scarcely affected by refractions.

Al-Battānī's case is simply an example of our argument above.

Two fundamental parameters of the Sun, the obliquity of the ecliptic and the eccentricity, which had tacitly been considered as astronomical constants, decreased from the time of Hipparchus and Ptolemy over al-Battānī's down to Copernicus' epoch much in the same manner. Copernicus' diligent and accurate determination of these parameters [cf. (5.10), Tab. 5.3] must have been for him clear evidence for the necessity of a new theory asserting that both parameters changed as a function of time. Together with those different values for the precessional constant of the equinoxes as historically recorded, he was thus led, as if by an *invitatio divina*, to his theory of the 1717 year-cycle.^{5.9} From our above statements it follows that all this can be reduced to the problem of his determination of the colatitude by means of the solar altitudes. The colatitude was too

high by 2 minutes of arc, caused by the effect of refraction, of which Copernicus, like all other astronomers of that time, had no quantitative knowledge. And this is eventually attributable to the high geographical latitude where he had to observe the minimum solar altitude (ca. 12°).

My early conjecture that Copernicus had favoured his own theory of the 1717 year-cycle by deliberately choosing a very low solar altitude, 19° , at $l_2 = 225^\circ$ is in reality ungrounded. This choice gave no numerical contribution to that theory, since the $\Delta\delta$ -term for the determination of the eccentricity in (3.1) is negligibly small. [cf. (4.2)].

Thus, the neglect of the traditional solar parallax of $3'$, which favoured the accuracy of the solar determination by al-Battānī and many other Muslim astronomers, was spoiled in the case of Copernicus by his use of the Sun's low position $h_{\min} \approx 12^\circ$ due to the effect of refraction.

5.4. Tycho Brahe

Tycho's astronomical procedures, unlike those of his predecessors, can precisely be followed with no ambiguities (a high degree of testability), a fact which is certainly more significant than the question of observational accuracy. We can in fact for the first time give our deep insight into an astronomer's daily activity.

By simply applying his data to his procedures as he describes them we could reproduce his underlying time determinations at t_1 , t_2 and t_3 , and his two time intervals $(t_2 - t_1)$ and $(t_3 - t_1)$, to fractions of an hour between $0.13^h = 8^m$ and $0.55^h = 33^m$ (Tab. 4.1., 5-q, -r).

The neglect of solar parallax and refraction ($p', r' = 0$) changes the time interval between the two equinoxes approximately by

$$\begin{aligned} \Delta(t_3 - t_1)_{p', r'} &\approx 2 \times 2.5 \times (-p' \cos h_{1,3}; +r') \\ &\approx (-0.21^d)_{p'}, (+0.06^d)_{r'} \text{ respect. [(4.8)],} \end{aligned}$$

hence his error of $+0.27^d$ would improve to $+0.12^d$:

$$\begin{aligned} \Delta(t_3 - t_1) &= +0.27 \text{ (Tab. 4.1., 5-q, or } +0.25, \text{ Tab. 3.1., 5-h)} - 0.21 + 0.06 \\ &\approx +0.12^d \approx +2.9^h \text{ (Tab. 4.1., Remarks).} \end{aligned} \tag{5.14}$$

Figs. 5.4–5.5 show Tycho's solar observations again in relation to the two fundamental, independent parameters, solar parallax and refraction, and Fig. 5.6 illustrates the origin of his error, +6 hours, in the length of the half-year in contrast to the error in the tropical year, –3 seconds.

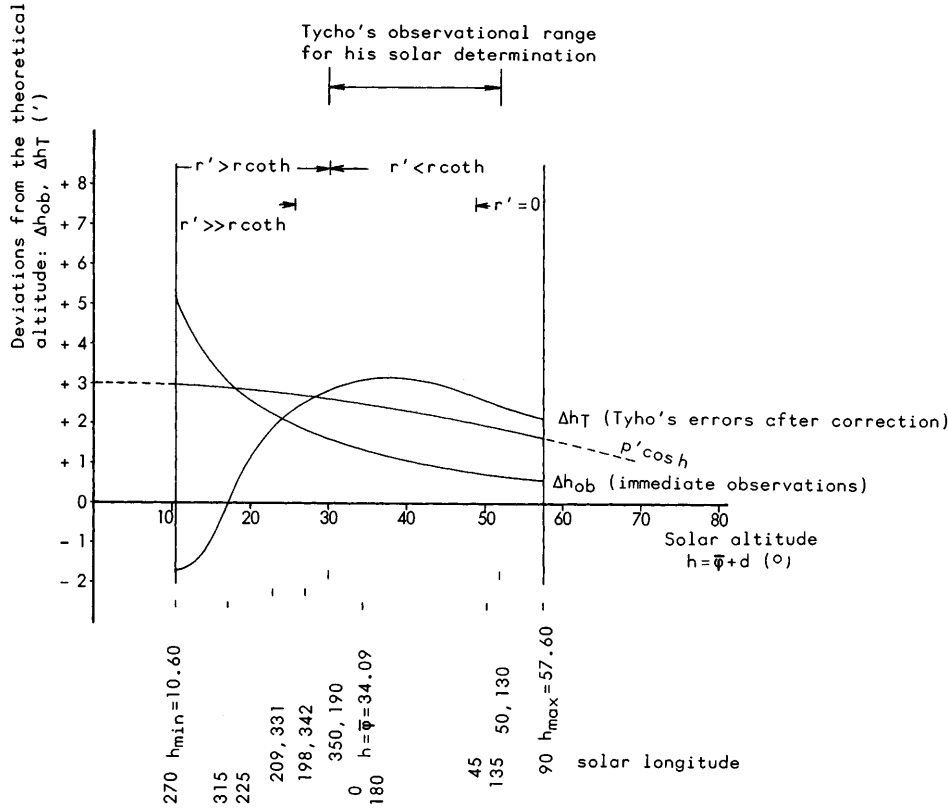


Fig. 5.4. Deviations of Tycho's solar altitudes over the horizon, immediately observed (h_{ob}) and corrected for solar parallax and refraction (h_T), from the theoretical values (h): cf. Figs. 4.1, 5.5, 5.6.

Tycho's immediate observations:

$$\Delta h_{ob} = h_{ob} - h = -p \cos h + r \cot h,$$

Tycho's errors after correction for p' , r' :

$$\Delta h_T = h_T - h = \Delta p \cos h + r \cot h - r'.$$

Note: In Tycho's observational range for his solar determination his errors are primarily covered by his correction for the solar parallax, $p' \cos h$. The irregular curve of Tycho's errors after correction originates from his erroneous values for refraction in which in turn the erroneous value for the solar parallax $3'$ is fully involved.

From our statements above it follows that, unlike the case of his predecessors, there are no ambiguities in Tycho's last stages. In particular he corrected all his solar observations for solar parallax and refraction, and criticized his predecessors for their neglect of these two problems: With all these facts he himself did not determine the solar parallax but always accepted and employed the traditional value $3'$ for the mean horizontal solar parallax.^{5.10} As for the refraction he determined it not as an independent but as a dependent parameter,

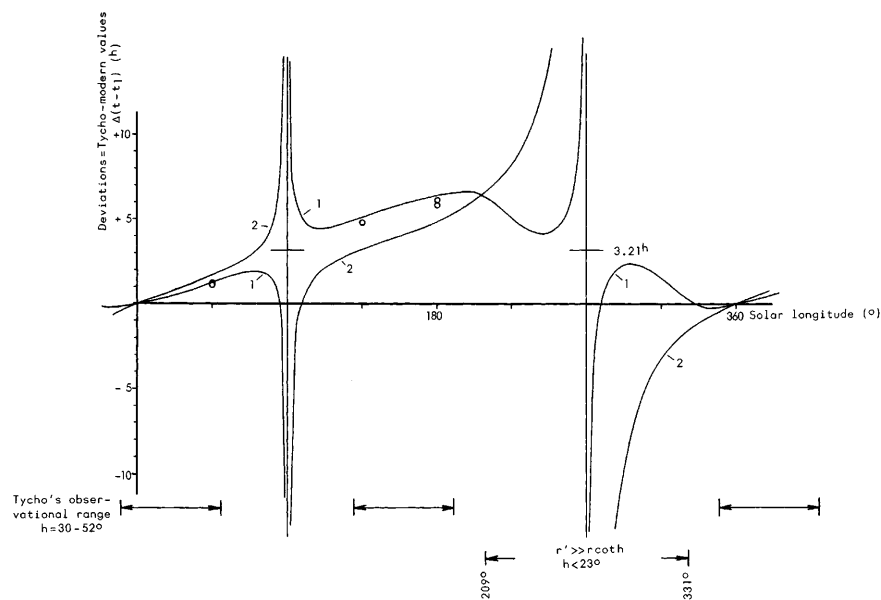


Fig. 5.5. Deviations of Tycho's time intervals from the modern values as a function of the solar longitude:

$$\Delta(t - t_1)^{(h)}; t_1 \text{ at } l_1 = 0, t \text{ at } l = 0 - 360^\circ \text{ (cf. Figs. 4.2, 4.5, 5.6)}$$

Tycho's errors:

1. the total error

$$\Delta(t - t_1) = \frac{24(\Delta l - \Delta l_1)}{\omega} = \frac{24\Delta l}{\omega} + 3.21^h$$

$$\Delta l_1 = -0.132^\circ \text{ (Tab. 4.1., 5-o; cf. also Tab. 5.1.), } \Delta l \text{ according to (4.6).}$$

2. the error caused by $p' = 3'$

$$\Delta(t - t_1)_p, \text{ computed from (4.6), (4.8) only with } \Delta p.$$

° : Tycho's records (Op. om. II, 15, 19-24) - modern values [Maeyama (1988)]: cf. curves 1.

Note: Tycho's errors are accurately reproduced by the curves 1. In the observational range for his solar determination Tycho's errors of the time intervals are primarily caused by his erroneous solar parallax, $p' = 3'$ (curve 2). Irregular parts of the curves are directly connected, as indicated, with Tycho's erroneous values for refraction caused again by those for the solar parallax.

that is, he deduced the deviations of the observed solar altitudes from those given by his solar theory without the term for refraction, and then identified the deviations as the solar refractions. Like other parameters, such as the obliquity of the ecliptic [(4.10)], his solar refractions are therefore distorted by the erroneous value of the traditional solar parallax.^{5.11}

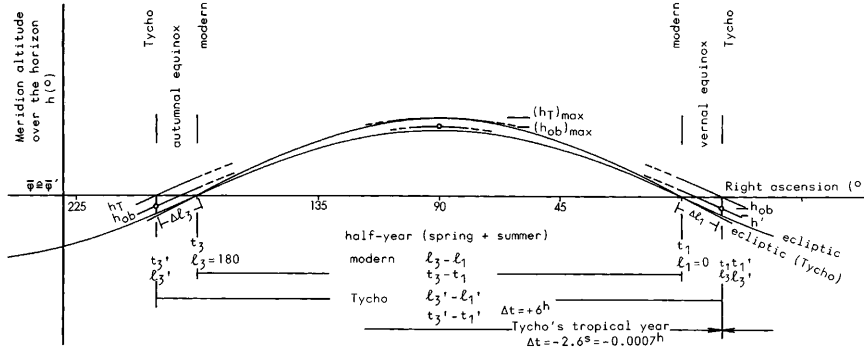


Fig. 5.6. Tycho's determination of the equinoxes (Tab. 5.1., Figs. 4.1, 4.2, 4.5, 5.5)

Tycho determined the equinoxes by observing the meridional altitude of the apparent Sun (h_{ob}), corrected for the solar parallax and refraction (h_T) such that this altitude is equal to his colatitude:

$$h_T = h_{ob} + p' \cos h - r' = \bar{\varphi}'.$$

Tycho's resulting time interval between the two equinoxes relating to the theoretical altitude of the true Sun h' in the ecliptic is therefore too long by $\Delta t = \Delta t_3 - \Delta t_1 \approx +6^h$, corresponding to $\Delta l = \Delta l_3 - \Delta l_1 \approx +0.26^\circ$ (Tab. 3.1., 5-h; 4.1, 5-o). His tropical year, measured at the same position in the ecliptic, is accurate to $\Delta t = -2.6^s = -0.0007^h$, an accuracy which he obtained by comparing his observations with the old ones made 100 years earlier by B. Walther (Op. om. II, 42–44).

In order to minimize the effects of the parallax and of refraction Tycho determined the geographical latitude by the fixed stars (Op. om. X, 297). Using the thus independently determined (and therefore accurate) colatitude and the maximum observable solar altitude $(h_{ob})_{\max}$ corrected for the solar parallax ($r'_h = 0$), he then determined his final obliquity of the ecliptic:

$$\varepsilon' = (h_{ob})_{\max} + p' \cos h_{\max} - \bar{\varphi}' = (h_T)_{\max} - \bar{\varphi}'; \Delta\varepsilon \approx p' \cos h_{\max} = 1.61' [\approx \Delta\varepsilon_2 = 2.16' (4.10)];$$

$$\text{Tycho} = 23;31,30; \text{X, 293}].$$

This result was therefore too large, the difference being nearly equal to the deviation caused by his erroneous solar parallax [Maeyama (1974), 40–42].

Note: Tycho's determinational error of the length of the half-year, 6^h , and that of a year, 0.0007^h , caused by his solar parallax, $3'$, are typical examples of his rigor; his observational accuracy was constant, but he strictly kept to the – erroneous, traditional – solar parallax, the only astronomical quantity which he accepted without his own determination. In his later stage all solar observations were made under the same conditions (cf. Fig. 4.1).

Here is where Kepler begins to struggle for his *new astronomy*, using Tycho's most accurate immediate observations of the Sun but corrected for the traditional solar parallax and the refraction according to Tycho.^{5.12}

Summary

From Hipparchus down to the time of Kepler's *Astronomia nova* (1609) the Sun's orbit was determined by means of the solar hypothesis according to Hipparchus and the length of two seasons measured by three observations of the Sun. In the so determined solar elements we therefore have two error sources:

1. the error of the solar hypothesis,
2. the error of the observations.

We have first deduced some simple formulae showing the theoretical deviations of the solar elements determined by Hipparchus' hypothesis from the modern ones and shown that these deviations are negligibly small compared with our very varying historical records of the solar elements (ch. 2). We have then deduced other formulae capable of reproducing the historical records of the solar elements as a function of the underlying observational errors (ch. 3).

Since all this functioned precisely enough, we then proceeded to clarify the basic conditions under which the recorded observations were actually made. This could be done sufficiently well by means of our analysis of Tycho's observations into which we were able, thanks to his consistency of description and conduct, to give our deep insight (chs. 4, 5). An outstanding example thereof is the marvelous agreement between his independently determined geographical latitude and his instrumentally fixed colatitude as deduced by us from his various lengths of the seasons to ca. ± 10 seconds of arc, which means an agreement of his three independent operations - determination of the latitude, installation of the instrument and observation of the Sun.

Being convinced that, apart from the problem of accuracy, Tycho's observational procedures were in principle the same as those of earlier astronomers, we then applied our discovery about Tycho's observations to our other astronomers. In the case of Ptolemy and al-Battānī we thus came to find a systematic error, each in different magnitudes, which seems to clear up all their apparent inconsistencies and uncertainties, such as the obliquity of the ecliptic (Ptolemy) and the geographical latitude (Ptolemy, al-Battānī), to a permissible minimum.

Thus we have eventually arrived at our aim of connecting the historically determined Sun's orbits with their immediate observations. The results obtained strongly vary from case to case but are associated with individual circumstances and characteristics of their times in history. Those who neglected the erroneous traditional solar parallax were capable of determining the Sun's orbit – depending on the order of accuracy of the immediate observations and on their frequency – accurately within the effect of refraction. A contrast is therefore clearly visible between al-Battānī and Copernicus due to the greatly different latitudes of their observing places (refraction). Contrary to all his predecessors Tycho consistently took the solar parallax into account and therefore shows us ubiquitously his errors arising from the traditional quantity of $3'$.

Tycho's length of the half-year between two equinoxes was erroneous by 6 hours, but that of the tropical year was accurate to 3 seconds of time. These facts can now

be simply explained. In each determination the same rigorous observation of the Sun occurred twice, equally displaced by the solar parallax; twice inversely in the former and twice the same in the latter. As a result the effect of Tycho's erroneous value for the solar parallax upon the measured intervals in time and position turned out to be double in the former, whereas it disappeared without trace in the latter.

This most remarkable phenomenon was later easily recognized by those astronomers, who took the solar parallax as a fundamental independent parameter, such as Horrocks (ca. 1638), Flamsteed (ca. 1675) and Cassini (ca. 1684) [Maeyama (1974), 51, n.6].

However dramatic our conclusion may sound, much corroborating evidence can be found. One example may suffice here. Tycho's solar tables, computed with his solar eccentricity 0.03584 (too large by the error of the half-year, i.e. the double of the solar parallax) were the corner stone for Kepler's *Astronomia nova* (1609). Moreover, Kepler introduced the corresponding rounded bisected eccentricity 0.018 (too large by the erroneous solar parallax of 3') into his final *Tabulae Rudolphinae* (1627).

Exactly on this basis – the most extensive solar observations ever made in number and accuracy but corrected for the solar parallax and refraction all according to Tycho – Kepler had to begin with his struggle for a *new astronomy* of the solar system. As we have frequently shown, many questions of astronomy at that time can therefore ultimately be reduced to the problem of the solar parallax and refraction.

Notes

1. Cf. Tab. 3.1., cols. a, p and ch. 5.2 below.
2. E.g. Delambre, *Hist. Astr. ancienne*, vol. 2, 117ff.; *Hist. Astr. moyen âge*, 35f.; *Hist. Astr. mod.*, 113f, 152 ff.; Neugebauer 57f.; Pedersen 144ff.; Petersen-Schmidt; Rome (1943); Swerdlow-Neugebauer 150ff.; Thoren 220ff. Those who analysed the observational errors will be dealt with below.
3. Although Ptolemy does not say that Hipparchus invented the solar model with the anomaly (Alm. III, 4), we call it below for simplification „Hipparchus' solar hypothesis“.
- 1.1. E.g. Kennedy (1963); Hartner-Schramm, 208f.
- 2.1. Cf. e.g. Brouwer-Clemence, 65; Maeyama (1974), 45.
- 3.1. Some statements of Petersen and Schmidt, e.g. $1/4^d$ versus $14\text{--}15^\circ$ error-interval, are therefore not incorrect (pp. 74, 80, 83, cf. also Toomer, 154) but the problem is more complex (cf. ch. 5.1 below).
- 4.1. For eqs. (4.1), (4.2) cf. eq. (5.1) below.
- 4.2. Cf. Maeyama (1974), 39f.
- 4.3. Thoren, e.g. 234f.; Maeyama (1974), e.g. 36f.
- 4.4. Tycho *Op. om.* II, 64, 136, 287; Cleomedes, *Czwalina* 78; Plinius, *Nat. hist.* II, 57. For Alhazen, Walther etc. cf. Thoren 227ff.
- 4.5. For eqs. (4.12), (4.13) cf. eq. (5.1) below.

- 4.6. Tab. 3.1., 5-p; Tycho Op. om. II 21–24; Kepler, Astr. nova, cap. 23–28, Tab. Rudol. 44ff; Maeyama (1974), 50–59.
- 4.7. Maeyama (1974), 58, Tab.8; (1990.1), 88, Tab. 5.1.; (1990.2), 136f., Figs. 1, 3.
- 4.8. For (4.18) cf. (5.1) below.
- 5.1. Ptolemy's allegedly more accurate values are in reality less accurate, cf. Fig. 5.2 below (cf. also e.g. Pedersen 148; Toomer 154, n. 47).
- 5.2. E.g. Britton (1969) has shown that the error of Ptolemy's obliquity of the ecliptic can be explained by the deviation of his observation from the meridian [also (1992), 1ff.]; cf. also Newton (1977), 96ff.; (1982), 31ff., supported by v. d. Waerden, 258. If our statement above is probable, Newtons probability such as „1 out of 10^{92} “ [(1977), 97] will be totally rejected.
- 5.3. That Hipparchus may have taken his $94\frac{1}{2}$ days for the spring from the Babylonian sources (Bowen-Goldstein, Jones 118) – and that, instead of $92\frac{3}{4}$ days for the summer or $187\frac{1}{4}$ days for the total from the same sources (cf. Kugler 84ff.), he took $92\frac{1}{2}$ and 187 days – and further that three centuries later Ptolemy reconfirmed, knowing all this or not, the correctness of Hipparchus' whole solar theory with those fundamental underlying lengths of the seasons – though he was in possession of his own, allegedly more accurate lengths (Alm III, 4; H 234; Toomer 154, n. 47) –, all this is beyond my understanding.
 In view of his observation of the stars and determination of the length of the year, etc. Hipparchus' determination of the Sun's orbit does not belong to his early stage soon after his Commentary (around $-140 \sim -150$) but undoubtedly to his later epoch (around -130) [Maeyama (1984), Figs. 7, 8, 301–305]. Britton (1992, 12ff.) tends to accept -134 for Hipparchus' determination of the length of the year. There is much evidence of Hipparchus' (early) dependence on the Babylonian sources (esp. Jones) and many questions can be raised about accuracy, precedents, and similarities with other sources. Yet his numerous observations of the Sun indicate that he tried to obtain all the solar data he needed and was confident about what he had. We accept Hipparchus' lengths of the seasons as his own; similarly we accept Ptolemy's claim to have measured the Sun's two extreme positions at the meridian (cf. ch. 5.1).
- 5.4. Cf. the ambiguous claim of Petersen and Schmidt about Hipparchus' accurate apogee-determination as a result of „coincidence“ (p. 74, 83). A good example may be given: If at $t \approx -1400$ ($v_1 = 135^\circ$) we take the observational errors of $\pm 1^\circ$ [$\Delta\gamma, \Delta\delta$ in (3.7)], ΔA will be $\pm 40^\circ$ (Tab. 3.2, 3-d) and negligibly small (≈ 0) respectively. Precisely based on their premise, $\Delta t_{1,2,3} = \Delta t$ (p. 82), but not $|\Delta t_{1,2,3}| = |\Delta t|$, so their claim only relates to the maximum possible errors (cf. also Pedersen 148f; Toomer 154, n. 47).
- 5.5. Tab. 3.1., l. 6.
- 5.6. Cf. e.g. Swerdlow-Neugebauer, 131.

- 5.7. The Commentariolus gives the eccentricity $1/25 = 0.04$, a conveniently rounded figure of Ptolemy's value $1/24 = 0.0416 \dots$, certainly for facilitating his numerical transformation of the Ptolemaic world-system.
- 5.8. Cf. e.g. Ibn al- A'lam (d. 985), max. equat. of center= $2;0,10^\circ$, Kennedy (1977), 21f.; al-Maghribi (d. 1283), $k = 2;5,57-2;5,59^p$, and Ibn al-Shāṭir (d. 1375), max. $eq. = 2;2,6^\circ$, Saliba (1985), 117–120, and (1987), 39ff.. Al-Bīrūnī mentions in his „Qānūn“ (Treatise 6, ch. 7) several similar solar elements, (I owe this information to Dr. Benno van Dalen). Cf. also van Dalen's contribution to *Encycl. of Islam*, al-Shams, the Sun.
- 5.9. Cf. Rheticus' Narratio prima; De rev. III, 6; also Moesgaard.
- 5.10. Cf. e.g. Thoren, 227.
- 5.11. Maeyama (1974), 46ff.
- 5.12. *Esp. Astronomia nova*, chs. 15–19; Wilson; Maeyama (1990.1).

References

- Al-Battānī: *Op. astr. Al-Battānī sive Albatēnii Opus astronomicum*. Ed. C. A. Nallino. 3 pts. Milano 1899–1907.
- Al-Bīrūnī: *Al-Qānūn al-Mas'ūdī, Mas'ūdī Canon*.
- Bowen, A. C. - Goldstein, B. R.: *Meton of Athens and astronomy in the late fifth century B.C. A Scientific Humanist: Studies in memory of Abraham Sachs*. E. Leichty, M. de J. Ellis, P. Gerardi (ed.), Philadelphia 1988, 39–81.
- Britton, J. P. (1969): *Ptolemy's determination of the obliquity of the ecliptic*. *Centaurus* 14, 29–41.
- (1992): *Models and Precision: The quality of Ptolemy's observations and parameters*. New York-London.
- Brouwer, D. - Clemence, G. M.: *Methods of celestial mechanics*. New York-London 1961.
- Caspar, M.: cf. Kepler.
- Clemence, G. M.: cf. Brouwer-Clemence, Woolard-Clemence.
- Cleomedes: *De motu circulari corporum caelestium*. A. Czwalińska: *Kleomedes, Die Kreisbewegung der Gestirne*. Leipzig 1927.
- Connaissance des temps, ou des mouvements célestes pour l'an ... Bureau des Longitudes, Paris 1953-.
- Copernicus, N.: *Commentariolus*.
- (1543): *De revolutionibus orbium coelestium, libri VI*.
- Czwalińska, A.: cf. Cleomedes.
- Dalen, B. v.: *Al-Shams, the Sun*. To appear in *Encycl. of Islam*.
- Delambre, J. B. J. (1817): *Histoire de l'astronomie ancienne*. 2 vols. Paris.
- (1819): *Histoire de l'astronomie du moyen âge*. Paris.
- (1821): *Histoire de l'astronomie moderne*. 2 vols. Paris.
- Explanatory supplement to the astronomical ephemeris and the American ephemeris and nautical almanac*. London 1961.
- Fotheringham, J. K.: *The secular acceleration of the Sun as determined from Hipparchus' equinox observations; with a note on Ptolemy's false equinox*. *Monthly Notices of the Royal Astr. Soc.* 78 (1918), 406–423.
- Geminus: *Isagoge. Gemini Elementa Astronomiae*. Ed. and German transl. C. Manitius, Leipzig 1898.
- Goldstein, B. R.: cf. Bowen-Goldstein.

- Hammer, F.: cf. Kepler.
- Hartner, W. - Schramm, M.: Al-Bīrūnī and the theory of the solar apogee: An example of originality in Arabic science. *Scientific Change*. A. C. Crombie (ed.), London 1963, 206–218.
- Heiberg: cf. Ptolemy.
- Hipparchus: Commentary. Hipparchi in Arati et Eudoxi phaenomena commentarium. Ed. and German transl., C. Manitius, Leipzig 1894.
- Jones, A.: Hipparchus's computations of solar longitudes. *J. Hist. of Astr.* 22 (1991), 101–125.
- Kennedy, E. S. (1963): Bīrūnī's methods on finding the solar parameters. *Belleten XXVII*, 31–36.
- (1977): The astronomical tables of Ibn al-Aʿlām. *J. Hist. of Arabic Science* 1 (1977), 13–23.
- Kepler, J. (1609): *Astronomia nova*. Pragae. *Gesammelte Werke* 3, ed. M. Caspar. München 1937.
- (1627): *Tabulae Rudolphinae*. Ulmae. *Gesammelte Werke* 10, ed. F. Hammer. München 1969.
- Kugler, F. X.: *Die Babylonische Mondrechnung*. Freiburg i. Br. 1900.
- Maeyama, Y. (1974): The historical development of solar theories in the late sixteenth and seventeenth centuries. *Vistas in Astronomy* vol. 16, 35–60.
- (1975): On the order of accuracy of Kepler's solar theory. *Vistas in Astronomy* vol. 18, 769–780.
- (1984): Ancient Stellar Observations. Timocharis, Aristyllus, Hipparchus, Ptolemy – the dates and accuracies. *Centaurus* 27, 280–310.
- (1988): The length of the seasons. *J. W. Goethe-Univ. IGN Preprint Series* 6, Frankfurt.
- (1990.1): Kepler's hypothesis vicaria. *Arch. Hist. Exact Sciences* 41, 53–92.
- (1990.2): Astronomy in the East and the West. 3 examples: parallax, precession, geometry. *Dall'Europa alla Cina: contributi per una storia dell'Astronomia*. Isaia Iannaccone e Adolfo Tambrello (ed.), Napoli, 129–139.
- Manitius, K.: cf. Geminus, Hipparchus, Ptolemy.
- Moesgaard, K. P.: The 1717 Egyptian Years and the Copernican Theory of Precession. *Centaurus* 13 (1968), 120–38.
- Nallino: cf. al-Battānī.
- Neugebauer, Otto: *A History of Ancient Mathematical Astronomy*. 3 pts. Berlin-Heidelberg-New York 1975.
- Newton, R. R. (1977): *The crime of Claudius Ptolemy*. Baltimore.
- (1982): The origins of Ptolemy's astronomical parameters.
- Pedersen, O.: *A Survey of the Almagest*. Odense 1974.
- Petersen, V. M.-Schmidt, O.: The determination of the longitude of the apogee of the orbit of the Sun according to Hipparchus and Ptolemy. *Centaurus* 12 (1967), 73–96.
- Plinius: *Naturalis Historia*. Ed. Jan-Mayhoff. 5 vols. Leipzig 1897–1933.
- Ptolemy (Claudius Ptolemaios): *Almagest*
- Heiberg, J. L. (ed.), *Syntaxis Mathematica*, Op. om. vol. I, 2 pts., Leipzig 1898, 1903.
- Manitius, K., Ptolemäus, *Handbuch der Astronomie*. Germ. trl. 2 vols., Leipzig 1912–13. Rpr. Leipzig 1963.
- Toomer, G. J., *Ptolemy's Almagest*. Engl. trl. London/New York 1984.
- Rheticus: Joachimus Georgius Rhaeticus, *Narratio prima*. Danzig 1540. Transl. in Rosen, 107–196, et al.
- Rome, A.: Les observations d'équinoxes de Ptolémée. *Ptolémée et le mouvement de l'apogée solaire*. *Ciel et Terre* 59 (1943), 1–15.
- Rosen, E.: *Three Copernican Treatises*. 3rd. ed. New York 1971.
- Saliba, G. (1985): Solar observations at the Maraghah Observatory before 1275: A new set of parameters. *J. Hist. Astr.* 16, 113–122.
- (1987): Theory and observation in Islamic astronomy: The work of Ibn al-Shāṭir of Damascus. *J. Hist. Astr.* 18, 35–43.
- Schmidt, O.: cf. Petersen-Schmidt.

- Schramm, M.: cf. Hartner-Schramm.
- Swerdlow, N. M.: Al-Battānī's determination of the solar distance. *Centaurus* 17 (1973), 97–105.
- Swerdlow, N. M.- Neugebauer, O.: *Mathematical Astronomy in Copernicus's De Revolutionibus*. 2pts. New York-Berlin-Heidelberg-Tokyo 1984.
- Thoren, V. E.: *The Lord of Uraniborg. A biography of Tycho Brahe*. Cambridge 1990.
- Toomer: cf. Ptolemy.
- Tuckerman, B.: *Planetary, lunar, and solar positions*. 2 vols. Philadelphia 1962, 1964.
- Van der Waerden, B. L.: *Die Astronomie der Griechen*. Darmstadt 1988.
- Vieta, F.: *Francisci Vietae Apollonius Gallus. Appendicula II*. Paris 1600. Repr. François Viète, *Opera mathematica*. Hildesheim-New York 1970, 343–346.
- Wilson, C.: The error in Kepler's acronychal data for Mars. *Centaurus* 13 (1969), 263–268.
- Woolard, E. W. - Clemence, G. M.: *Spherical astronomy*. New York-London 1966.

Institut für Geschichte der
Naturwissenschaften
Goethe Universität
Postfach 111932
60054 Frankfurt am Main
Germany

(Received April 2, 1996)