

A Neolithic Oral Tradition for the van der Waerden/Seidenberg Origin of Mathematics

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Communicated by A. SEIDENBERG

Introduction

In his recent book [19] VAN DER WAERDEN "ventured a tentative reconstruction of a mathematical science which must have existed in the Neolithic Age, say between 3000 and 2500 B.C., and spread from Central Europe to Great Britain, to the Near East, to India, and to China." He argues that a common origin* is highly probable. These views are consistent with those of SEIDENBERG [12], who says, "What was this older, common [to Pythagorean and Old-Babylonian mathematics] source like? I think its mathematics was very much like what we see in the Sulvasutras." SEIDENBERG does not mean the common source was necessarily the Sulvasutras, rather that it "is to be sought either in the Vedic mathematics or in an older mathematics very much like it." VAN DER WAERDEN bases much of his reconstruction on the Chiu Chang Suan Shu (Nine Chapters on the Mathematical Art), a Chinese collection of mathematical problems, written during the Han-period (200 B.C. to 220 A.D.). The Chiu Chang is, according to VAN DER WAERDEN, a more systematic and richer source than the Babylonian texts. SEIDENBERG's primary source is the Sulvasutras, ancient Indian sacred works on altar constructions, in addition to EUCLID's Elements and the Babylonian mathematical tablets. Both VAN DER WAERDEN and SEIDENBERG discuss the important and difficult questions relating to the dating of their primary source materials and the likely dependencies among them. Basing themselves on their answers to these questions and their analysis of much of the oldest known mathematical thought, both men argue for a common source for the Indian, Babylonian, Greek, and Chinese mathematics.

Here I shall give a small, coherent, and basic core of geometry concerning

* On page 10 of [19] the discussion of a common origin occurs in the context of Pythagorean triples and the Theorem of PYTHAGORAS. Later, on page 33, VAN DER WAERDEN speaks of "a common mathematical doctrine from which these ideas were derived." The ideas in question are "the mathematical and religious ideas current in England in the Neolithic Age, in Greece, in India, and in China ..."

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MATHEWS
1985

rectangles and their parts, including the right triangles on their diagonals and the gnomons in their corners, which may serve as what VAN DER WAERDEN has called an "oral tradition current in the Neolithic age." I hope to give this hypothesized ancient core some credence through its relation to the Chiu Chang and its explanatory power.

My work owes much to SEIDENBERG's papers [9–13], VAN DER WAERDEN's earlier book *Science Awakening* [17], and NEUGEBAUER & SACHS' work [8] on Babylonian mathematics. Its present form arose out of a study of VOGEL's translation [16] of the Chiu Chang.

I have tried to avoid both the use of modern algebraic symbolism, which I feel is misleading, and any suggestion that, for example, those ancients who spoke of the difference of two geometric squares may be credited to some extent with knowledge of the identity $a^2 - b^2 = (a - b)(a + b)$. I see no necessity for taking any such position. If, however, by algebra one means certain recurrent patterns of thought or, even, motor activity (I am here thinking of patterns intrinsic to mental or written arithmetic calculations or kinesthetically experienced patterns in the moving of stones, beads, rods, or dust in a computing device), then one may credit the Indians, Babylonians, Greeks, and Chinese with some knowledge of algebra. One may conjecture that an algebraic tradition, including viewpoint, technique, and notation, arose out of attempts to codify the active parts of those numerical algorithms which began as geometric relations.

I. Gnomons

The idea of a gnomon or gnomon-construction may have arisen from dualisms or equivalences occurring among the rituals of the Vedic religion. In [13] SEIDENBERG notes that the circle and square were regarded as dual figures, and further, circle : square = Gārhapatya : Āhāvaniya (circular fire altar : square fire altar) = earth : sky = human : divine. The principal expression of the equivalence of two geometric figures or two altars is the equality of their areas. In addition to the circle/square equivalence, the pair oblong/square was also considered. SEIDENBERG finds in the needs of ritual a motivation for squaring the oblong. He notes that in the Sulvasutras are found the following directions:

If you wish to turn an oblong [ABCD—see Figure 1] into a square take the tiryānmāni, i.e., the shorter side of the oblong, for the side of a square [AEFD], divide the remainder [EBCF] into two parts and inverting join these two parts to the sides of the square. Fill the empty space by adding a small piece. It has been taught how to deduct it.

The last sentence refers to use of the Theorem of PYTHAGORAS, given elsewhere in the Sulvasutras, for finding a square equal to the difference of two squares.

It is clear from the Figs. 1(2) or 1(3) that the Vedic literature includes clear references to the gnomon figure.* SEIDENBERG observes that the desire to square

* The carpenter's square along the right-side and bottom of Fig. 1(3).

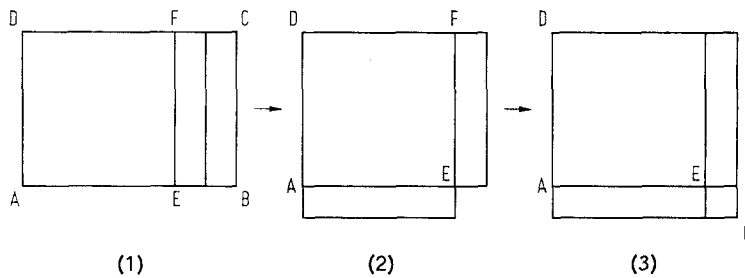


Fig. 1

an oblong provides a context in which the idea of a gnomon may have originated as well as the need to subtract one square from another. I shall return to this last very interesting point in Section III on the Theorem of PYTHAGORAS.

I shall propose Proposition I, Book II, of EUCLID's Elements* (EII.1, hereafter) as the first result in the neolithic core.

I.1 In Figure 2(1), if the side BC of the rectangle BCHG is cut into any number of segments, the rectangle with sides BG and BC is equal (in area) to the rectangles with sides BG and each of the segments.

In Figure 2(2) is given a rectangle ABCD with diagonal AC. If any point I on AC is chosen, two rectangles** AI and IC are determined. Each of these rectangles determines a gnomon: AI is the gnomon-corner and EF and HG are the gnomon-arms of gnomon ABFIGDA, while IC is the gnomon-corner and (again) EF and HG are the gnomon-arms of gnomon CDHIEBC. The rectangles EF and HG are often called gnomon-complements or complements (of one another). The rectangles AF and AG (HC and EC) are called the gnomon-bases of gnomon ABFIGDA (CDHIEBC).

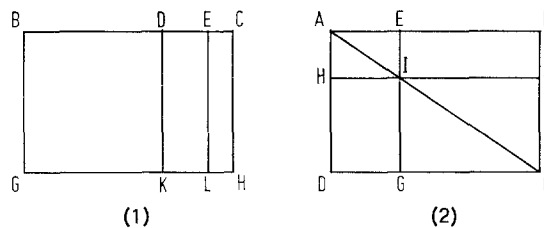


Fig. 2

I.2 The gnomon-complements are equal (in area).

This result may be interpreted as a result on similar triangles ($\triangle AEI \sim \triangle ICG$ and so $AE : EI = GC : GI$). I shall avoid this interpretation on the grounds that the equality of the complements is a less sophisticated result. A simple argument

* All references to EUCLID's Elements are to HEATH's translation [2].

** I shall refer to rectangles, such as EBFI, by their diagonals, here EF.

for I.2 may be based on the assumption that the diagonal of a rectangle divides it into two equal pieces.* Since one-half of rectangle AI and rectangle HG and one-half of rectangle IC is equal to one-half of rectangle AI and rectangle EF and one-half of rectangle IC, we see that rectangle HG is equal to rectangle EF.

From the equality of the complements it follows that the gnomon-bases AF and AG are equal. I list this as

I.3 The gnomon-bases are equal (in area).

II. Equivalent to Oblongs and Sums and Differences of Squares Using Gnomons

Closely connected to the discovery and subsequent use of the gnomon is the problem of forming the difference of two squares. There are at least two ways in which the difference of squares may be considered. One way has already been mentioned, to find (the side of) a square equal to the difference. A second is to arrange the two squares so that their difference may be seen as the area inside the larger and outside the smaller. I shall discuss the latter first because it is simpler and leads to a solution of the former, that is, to the Theorem of PYTHAGORAS.

In Figure 3 are given several configurations showing the difference in area of two squares.**

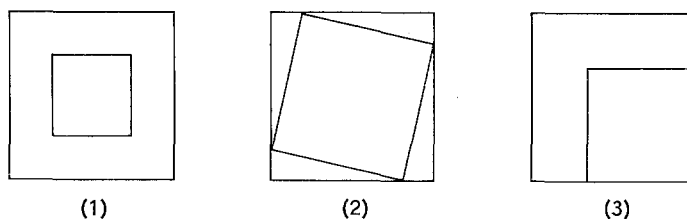


Fig. 3

A figure closely related to that in Figure 3(2) is found in an ancient Chinese work, the Chou Pei, and the gnomon in Figure 3(3) is found in both China and India (for China, see LAM & SHEN [5]; for India the idea of a gnomon is clearly present in the Sulvasutras, as noted on page 194, above). I start with the gnomon-figure.

In Figure 4(1) there is a square ABCD and a gnomon-construction. I shall refer to each of the gnomon-corner squares AI and IC as the complement of the other.

The first result is evident.

II.1 In Figure 4(1), the difference between (the areas of) the full square (AC) and either of the gnomon-corners is equal to each of

* This is one of a number of more elementary results which must have been part of the ancient core.

** The kind of configuration shown in Figure 3(2) limits the size of the inner square.

A (Sum Form): the sum of the (areas of) complementary gnomon-corner and the gnomon-complements;

B (Product Form): the (area of) rectangle formed by aligning the complementary gnomon-corner and the gnomon-complements.

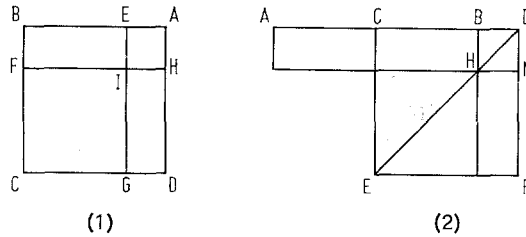


Fig. 4

The Product Form occurs in EUCLID II.6. EUCLID's figure is shown in Figure 4(2). In Figure 4(2), in which BC is equal to CA , EII.6 asserts that rectangle AM together with square HE is equal to square DE . Rectangle AM will be called the aligned gnomon.

A second configuration of a square within a square is given in Figure 3(1), and repeated in Figure 5(1) and 5(2) with two possible dissections. In Figure 5(3) there is a rearrangement of Figure 5(2).

It will be convenient to use the following conventions. In Figure 6, if AB and CD are given lengths, there are several elementary, associated lengths: the sum EH of AB and CD , their difference GF , their arithmetic mean EM , and FM , which is half their difference. I shall denote the last two of these by $HS(AB, CD)$, and $HD(AB, CD)$. It is evident that the longer of AB and CD is the sum of HS and HD while the shorter is the difference.

II.2 In Figure 5(1), the difference of the squares with sides AB and CD is four times the rectangle with sides $HS(AB, CD)$ and $HD(AB, CD)$.

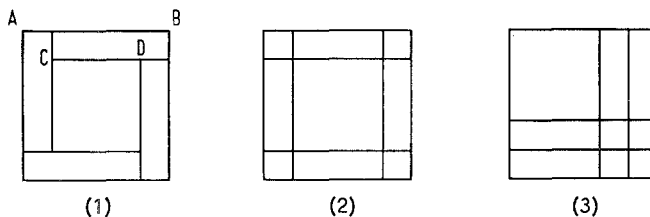


Fig. 5

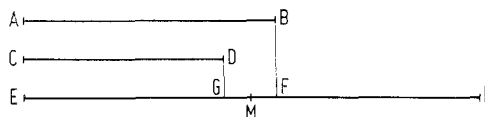


Fig. 6

This result suggests that HS and HD are important in the ancient tradition. This will become ever clearer in the remainder of this paper.

Although there is implicit in Figure 5(2) yet another way of expressing the difference of two squares, I have given this figure for the purpose of showing that a further dissection of Figure 5(1) followed by a rearrangement gives Figure 5(3). This will be useful later.

In Figure 7 are given two arrangements for adding two squares. Although other arrangements are possible—one surely would think of a figure like that in Figure 3(1)—a search has not yielded any other arrangements that are attested in any ancient tradition (to my knowledge), are closely related to the ancient tradition proposed here, or appear to lead to useful equivalences. Nonetheless, and because it appears to me that the sum of squares lies a bit deeper than the difference of squares, I may have overlooked other possibilities.

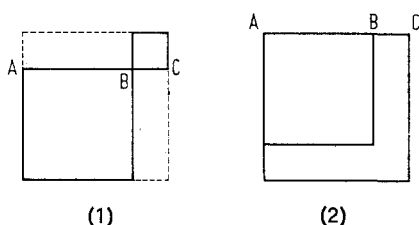


Fig. 7

My discussion of the sum of two squares using the arrangement in Figure 7(2) is motivated by Problem 11 of Chapter IX of the Chiu Chang Suan Shu, which I will discuss later. An analysis of this problem suggests that in adding the squares with sides AB and AC, one should consider the square with side HS(AB, AC). This leads to the dissection in Figure 8, wherein $AD = HS(AB, AC)$.

Since adding the squares on AB and AC together counts all rectangles of the dissection in Figure 8 once excepting AG, which is counted twice, while doubling the square on AD may be tallied by the check marks shown—note that instead of checking the rectangle beneath BD twice, a check was put into its equal, just adjacent, and likewise elsewhere in the dissection—the following result is evident:

II.3 In Figure 8, the sum of the squares on AB and AC is the sum of twice the square on HS(AB, AC) and twice the square on HD(AB, AC).

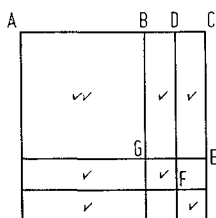


Fig. 8

The counting argument of II.3 may be used in the case where D does not divide BC equally. It follows, then, that

$$\begin{aligned} \text{square on AB and square on AC} &= \text{square on (AB and BD joined) and} \\ &\quad \text{square on (AB and DC joined) and} \\ &\quad \text{twice the rectangle EF.} \end{aligned}$$

This result appears to explain Problem 12 in Chapter IX of the Chiu Chang. We shall come back to this later.

There is one result remaining to be discussed in this section, the relation between oblongs and differences of squares. Although I have put it last, this result is part of a procedure given at the beginning of my discussion of the ancient tradition, in and just prior to Figure 1, namely, how to square an oblong. One first transforms the oblong into a gnomon, recognizes the gnomon as the difference of two squares, and then uses the Theorem of PYTHAGORAS to get the side of the desired square. The result at hand comes from this procedure by omitting the last step. The procedure which transforms an oblong into a gnomon-construction contains within its dynamics, structures, and results a significant fraction of the entire ancient tradition and, in embryo, the greater part of this tradition.

II.4 The area of the rectangle with sides AD and DC (in Figure 1) is the difference of the squares with sides HS(DC, AD) and HD(DC, AD).

This seminal result is intimately related to II.1A and II.1B. First, II.4 as stated is equivalent to II.1B. Secondly, from noting that the rectangle with sides AD and DC is equal to the square on AD and the rectangle with sides AD and the difference between sides DC and AD, II.4 may be seen as equivalent to II.1A.

The proposed ancient core will be completed in Section III, wherein the Theorem of PYTHAGORAS will be added to the eight results given thus far. Since the discussion of the Theorem of PYTHAGORAS will lead us away from these eight results it is appropriate to comment now upon some connections between them and EUCLID's Elements. The eight results I.1–II.4 arose from my attempts to understand the received arithmetic in Chapter IX of the Chiu Chang. Given this, it is of interest to note that the proposed ancient core is very nearly all of EUCLID's Book II together with one or two propositions from Book I.

The correspondence between the proposed ancient core and EUCLID's Elements is shown below.

	Ancient Core	EUCLID
i)	I.1	EII.1
ii)	I.2	EI.43
iii)	I.3	EI.43
iv)	II.1A	EII.4, EII.7
v)	II.1B	EII.6
vi)	II.2	EII.8
vii)	II.3	EII.9, EII.10
viii)	II.4	EII.6, EII.14
ix)	III.1	EI.47

The ancient core, with the Theorem of PYTHAGORAS (III.1 here; EI.47 in EUCLID), comprises EI.43 and EI.47 of Book I together with all but three (Propositions 11–13) of the fourteen propositions in Book II.

The propositions EII.2 and EII.3 are special cases of EII.1, which corresponds exactly to I.1. The correspondence ii) is clear—I note only that EUCLID's proposition is stated for parallelograms instead of rectangles. As to iii), I.3 follows easily from I.2.

The results EII.4 and EII.7 are slightly different ways of relating the pieces of the dissection of a square, shown in Figure 4(1). EII.4 states that the full square AC is equal to the two gnomon-corners and the gnomon-complements, while EII.7 states that the full square AC and either gnomon-corner is equal to the other gnomon-corner and twice the gnomon-base. The correspondence iv) notes yet another view of the result underlying all of EII.4, EII.7, and II.1A. It is reasonable to expect both redundancy and the separate statement of special cases in the development of mathematics. It is common knowledge that EUCLID's Elements has instances of redundancy as well as separately stated special cases. Accordingly, I have not tried to refine the ancient core proposed here, but have left it as it emerged from the Chiu Chang.

I noted correspondence v) just after II.1B, calling attention also to EUCLID's figure (see Figure 4(2)), which explicitly displays what I have called the aligned gnomon.

I have included II.2 among the results in the ancient core even though it does not appear to be explicitly related to the problems of Chapter IX of the Chiu Chang. It was included because of its intimate relationships to II.3 and to the Theorem of PYTHAGORAS. The result II.2 and the corresponding EII.8 given in EUCLID are the same. The figure used here (Figure 5(1)) is not in the Elements and allows a somewhat shorter proof. HEATH [2, page 391, vol. 1] also gives a proof using Figure 5(1). EUCLID's figure for EII.8 is that shown in Figure 8, which is used here in II.3. I have noted earlier that each of Figures 8 and 5(2)—the latter is very nearly the same as Figure 5(1)—is a rearrangement of the other.

In his commentary on EII.9 and EII.10, HEATH [2, pp. 394–398, vol. 1] notes that in the proof of EII.9 EUCLID has used the Theorem of PYTHAGORAS for the first time in Book II, that his proof and accompanying diagram are not in the style of the first eight propositions in this book, and that each of EII.9 and EII.10 may be proved independently of the Theorem of PYTHAGORAS, entirely within the methods and style of EII.1–8. The counting argument for II.3 sketched here is well within the spirit of Book II; indeed the diagram used here for the proof occurs in EII.8.

The corresponding viii) is clear. The two figures—EUCLID's (given in Figure 4(2)) and Figure 1—are strongly related. The two constructions are the same, step by step. As to EII.14, which is “to construct a square equal to a given rectilineal figure,” EUCLID proves it using EII.5 and EI.47 (Theorem of PYTHAGORAS). The purpose and the construction procedure given in the Sulvasutras for squaring an oblong, given here on page 194, are the same as in the Elements.

Finally, III.1 and EI.47 are the same result. I will give a detailed discussion III.1 in the next section.

III. The Diagonal Square

I do not suppose the term “diagonal square” which I shall use in this section to denote the square whose side is the diagonal of a given rectangle or the hypotenuse of a right triangle will have wide appeal. Nonetheless, I shall use it for brevity and, most of all, to avoid using the inappropriate but universally used “Theorem of PYTHAGORAS” or “Pythagorean Theorem.” I shall talk about the diagonal square and the Diagonal Square Theorem.

If one writes out a conjectured ancient tradition there is in the lineal nature of writing a first result, a second, and so on. The very procedure one uses to prepare a piece of writing has within it a powerful impetus which orders the output. Those who write pieces on the history of mathematics are subject to a second ordering impetus, that which arises out of our view of proper mathematics, wherein one deduces one result from another so that there always is an ordering of results leading from axioms to the result at hand.

These remarks form a background to this third section, which is third for reasons I believe have some validity. Nonetheless, insights and new results which ultimately form a mature field of mathematics are not necessarily discovered in an order which turns out at maturity to be acceptable.

The religion-based motivation for squaring an oblong leads to the problem of finding a square equal (in area) to the difference of two squares, that is, to the Diagonal Square Theorem. SEIDENBERG [13] gives the diagram shown in Figure 9(1) and makes the comment, “In trying to subtract a square from a square, one would place the smaller square into the larger and look at the difference. This could well lead one to the contemplation of something like [Fig. 9(1)].”

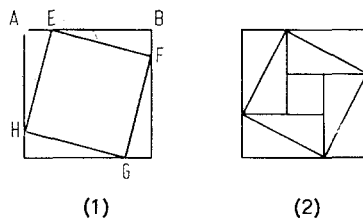


Fig. 9

SEIDENBERG goes on to say, “This figure unquestionably was contemplated in ancient times. The Chou Pei, an ancient Chinese work, has ... [Fig. 9(2)] ...”

SEIDENBERG notes that a proof of the Diagonal Square Theorem (DST) follows easily from Figure 9(1). (He sketches a proof; see also the argument below.) Thus he can argue that “the geometry of the Sulvasutras stems from the philosophy of equivalence through area.”

There are many paths to the Chou Pei figure. One may consider adding or subtracting squares using one or the other of Figures 3(1), 7(1), or 7(2). In each of these arrangements the sides of the two squares are the *given quantities*. Through dissection, each of Figures 3(1) and 7(2) can be made to have the form shown in Figures 5(1) or 5(2). (It is useful to bear in mind that Figure 5(3) is a rearrangement

of Figure 5(2).) Now, in Figure 10 I have added the four diagonals of the rectangles on the outside of Figure 5(1) and included the dotted line segments which take Figure 5(1) to 5(2).

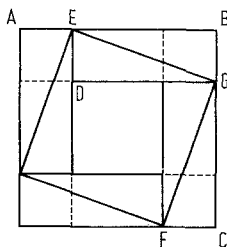


Fig. 10

Independently of how Figure 10 came about, it is clear that the large square (AC) can be dissected in two ways: either

(1) square on DG plus square on ED plus twice rectangle EG
or

(2) square on EG plus twice rectangle EG.
The DST follows immediately, that is,

III.1 In Figure 10, the square on DG plus the square on ED is equal to the diagonal square EG.

It is easy to check that the “given quantities”—call them XY and WZ in Figures 3(1) and 7(2)—do not appear in the DST arising from Figure 10, which came from a “natural” dissection or rearrangement of these figures; rather it is the quantities HD(XY, WZ) and HS(XY, WZ) which appear.

The part of Chou Pei in which the figure like that in Figures 9(2) or 11(2) appear has a passage relating to the figure. I quote it in full, from NEEDHAM [7].

Thus, let us cut as rectangle, and make the width 3 wide, and the length 4 long. The diagonal between the corners will then be 5 long. [Rectangle G*E* in Figure 11(2) may be taken as the rectangle in question, though the relative sizes are not the same as in the Chou Pei. Next follows a procedure for completing the figure, from rectangle G*E*.] Now after drawing a square on this diagonal [square E*F*], circumscribe it by half-rectangles like that which has been left outside, so as to form a plate [the square on side A*B*]. Thus the outer half-rectangles of width 3, length 4, and diagonal 5, together make two rectangles; then the remainder is of area 25. [Since A*B* would have length 4 plus 3, the square on A*B* would have area 49; this square minus two rectangles (which lie outside of square E*F*, in the form of 4 triangles) gives the area of E*F* as 49 minus 24, or 25.]

The last square-bracket insert is just (2) of the proof of the DST, appearing just before III.1.

This discussion has left out Figure 7(1), which perhaps is the most natural arrangement for adding two squares. The segments AB and BC are the given quantities. If we complete the figure by adding lines as shown in Figure 11(1), where the square DB is equal to square BE, then a rearrangement (suggested by the small numbers) leads to Figure 11(2), which is labeled as in Figure 10. It is clear that E^*D^* is BC and D^*G^* is AB. The given quantities are, in this case, part of the DST.

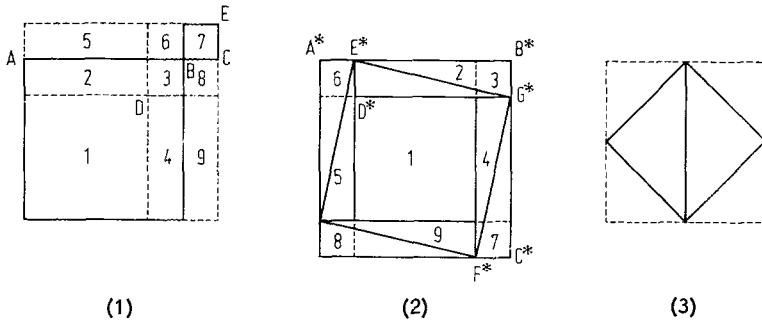


Fig. 11

On the basis of the natural arrangement of two squares in Figure 7(1) and the direct occurrence of the given quantities in the DST, one might conjecture that the DST was found through adding squares using 7(1), a rearrangement into the Chou Pei figure, and an argument similar to that given just prior to III.1. SEIDENBERG has pointed out (private communication), however, that this conjecture has several flaws. There is in the Vedic rituals a motive for seeking a means of constructing the side of a square whose area is the difference of two squares, and there is a plausible argument for how one might then come to Figure 9(1) and thence to the Chou Pei figure in Figure 9(2). Neither SEIDENBERG nor I can give a similar motive for adding two squares. Even if there were such a motive, there appears to be no textually-based reason for adding the diagonals in the constructions leading to Figure 10, other than, of course, that such leads to the Chou Pei figure.

Among the Vedic rules for altar construction is "The cord which is stretched across a square produces an area of double the size." (See THIBAUT [15, page 233].) No proof is given in the Sulvasutra. An immediately suggested figure, see Figure 11(3), provides a transparent and compelling argument for this rule. The diagram in Figure 11(3) is a special case of the Chou Pei diagram in Figure 9(2).

IV. The Problems of the Ninth Chapter of the Chiu Chang Suan Shu

I have now described the hypothesized "small, coherent, and ancient geometric tradition." I used the ninth chapter of the Chiu Chang as a source of ideas for the nine components of this ancient tradition. In the balance of this paper I shall discuss the problems of Chapter IX in terms of these nine components. Because to

a considerable extent I constructed/conjectured an ancient tradition using the Chiu Chang, one would naturally expect explanations of the Chiu Chang based on such a hypothesized tradition to be reasonably coherent and well fitted. Until I can thoroughly test this conjecture on, say, the Babylonian corpus, I can argue for the merits of my conjecture only on such grounds as the simplicity of explanation it allows, or its congruence with received results or figures. I shall use my translation into English of VOGEL's [16] translation. I shall compare my reconstruction with those of VOGEL, VAN DER WAERDEN [19], WANG & NEEDHAM [20], and SWETZ & KAO [14]. VOGEL notes that there are no diagrams in Chapter IX of the Chiu Chang.

Problems 1, 2, and 3. These three problems deal with the (3, 4, 5) right triangle. Problem 1 will serve to illustrate all three. "Now we have a horizontal side of 3 feet and a vertical side of 4 feet. Question: How long is the hypotenuse? Answer: 5 feet. Rule: Multiply each of the horizontal and vertical sides by itself, add these products, and take the square root of the sum."

Problems 4 and 5. These two problems are mathematically the same as the first three but are two steps harder. First, the problems are stated in non-mathematical terms, so that the reader is required to understand that the problem can be formulated as one concerning a right triangle. Secondly, the given numerical quantities are no longer whole numbers; however, the quantities are always chosen (here, as well as elsewhere in Chapter IX) so that all necessary square roots are rational. In addition to III.1, the solution of Problem 4 uses the result that an angle inscribed in a semicircle is a right angle and that of Problem 5 uses the result that a cord wrapped around a vertical cylinder and rising uniformly may be viewed as the hypotenuse of a right triangle wrapped around the cylinder, with one side of the triangle remaining horizontal and equal in length to the circumference of the cylinder.

Problems 6–10. The received solutions of this group of problems depend upon the result II.1, which gives alternative expressions to the difference of two squares. I shall give several of these problems in detail.

Problem 6. "We now have a square reservoir, with side 1 chang [1 chang = 10 ch'ih; these are length units]. A reed is growing in the center and its top is 1 ch'ih out of the water. If the reed is inclined towards the shore, the top just reaches the water of the shore. Question: What are the depth of the water and the length of the reed? Answer: The water is 1 chang, 2 ch'ih and the reed is 1 chang, 3 ch'ih.

Rule: Multiply half of the side of the reservoir by itself; decrease this by the product of the length of the reed above the water with itself; divide the difference by twice the length of the reed above the water. This gives the depth of the water. Add to this the length of the reed above the water. This gives the length of the reed."

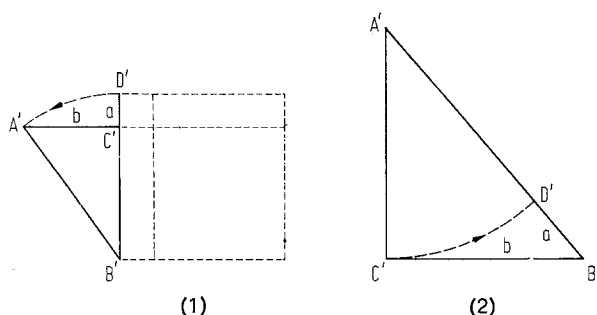


Fig. 12

Neither the hypotenuse nor the vertical side of the right triangle $A'B'C'$ in Figure 12(1) is known; their difference $D'C'$ and the horizontal side of the triangle are known. Now referring to Figures 4(1) and 12(1), identifying AD with $A'B'$ (or its equal $B'D'$) and IG with $C'B'$, and using III.1 and II.1A gives directly that

$$\begin{aligned} \text{square on } A'C' &= \text{square on } A'B' \text{ minus square on } B'C' \\ &= \text{square on } D'C' \text{ and twice the rectangle} \\ &\quad \text{with sides } D'C' \text{ and } C'B'. \end{aligned}$$

From this it is clear that the unknown side $C'B'$ is given by

$$\begin{aligned} (b \cdot b - a \cdot a) \div (2 \cdot a) &= (5 \cdot 5 - 1 \cdot 1) \div (2 \cdot 1) \\ &= 12 \text{ ch'ih} \\ &= 1 \text{ chang, } 2 \text{ ch'ih}. \end{aligned}$$

VOGEL's comment on this problem is that the initial relation is $(x + a)^2 = x^2 + b^2$ —where $x = B'C'$ —from which $x = (b^2 - a^2)/(2a)$, as in the received arithmetic. This leaves the transition from the initial relation to the recipe unexplained.

Problem 7. “We now have an upright pole and a cord attached to the top of the pole. If the rope hangs naturally, 3 ch'ih of its length lies upon the ground. If the end of the cord is placed 8 ch'ih from the pole, the [taut] rope is in a straight line from the top. Question: What is the length of the rope? Answer: 1 chang, $2\frac{1}{21}$ ch'ih.*

Rule: Multiply the distance from the foot of the pole to the end of the rope by itself and divide the result by the amount of rope lying on the ground. Add to this result the amount of rope lying on the ground and halve this result. This is the length of the rope.”

* It should be $2\frac{1}{6}$ ch'ih.

In Figure 12(2) is given a diagram for Problem 7. It is obvious from Figures 12(1) and 12(2) that Problems 6 and 7 are similar. If we apply III.1 and II.1B, and identify $A'B'$ with AD and $A'C'$ with IG , it follows directly that

square on $B'C'$ = the rectangle with sides $D'B'$ and twice
 $A'B'$ subsequently decreased by $D'B'$.

Thus $A'B'$ is

$$\begin{aligned} ((b \cdot b) \div a) + a) \div 2 &= (((8 \cdot 8) \div 3) + 3) \div 2 \\ &= (21\frac{1}{3} + 3) \div 2 \\ &= 24\frac{1}{3} \div 2 \\ &= 12\frac{1}{6} = 1 \text{ chang, } 2\frac{1}{6} \text{ ch'ih.} \end{aligned}$$

I shall omit any discussion of Problems 8–10; although their settings vary, the mathematical model and solution techniques are the same as for Problem 7.

Problems 11 and 12. These problems appear to be slightly more advanced than the first ten problems and their solutions require a result using the sum of squares rather than the difference.

Problem 11. “We now have a door whose height is 6 ch’ih, 8 ts’un [10 ts’un = 1 ch’ih] more than its width. The distance between the [diagonal] corners is 1 chang. Question: What are the height and width of the door? Answer: The width is 2 ch’ih, 8 ts’un and the height is 9 ch’ih, 6 ts’un.

Rule: Multiply 1 chang by itself; this is the beginning amount. Multiply half of the difference by itself, double the result and subtract this from the beginning amount. Halve the remainder and take the square root. Decrease this result by half of the difference—this is the width of the door. Increase the result by half the difference—this is the height of the door.”

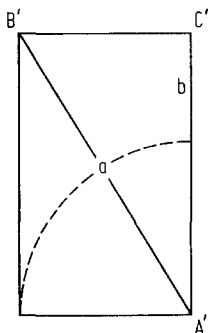


Fig. 13

From Figure 13 it is seen that the new feature here is that the diagonal is given, not a side as in Problems 6–10. When we identify $B'C'$ of Fig. 13 with AB of Fig. 8 and $A'C'$ with AC and use II.3 and III.1, it follows directly that

$$\begin{aligned} \text{square on } B'A' &= \text{the sum of twice the square on} \\ &\quad \text{HS}(B'C', A'C') \text{ and twice the square} \\ &\quad \text{on } HD(B'C', A'C'). \end{aligned}$$

The side $HD(B'C', A'C')$ is just $b \div 2$. The side $HS(B'C', A'C')$ corresponds to AD in Figure 8; AD is a kind of “mean-square” between the squares on AB and AC of Figure 8. If the latter are added it is natural to suppose the sum is related to twice the mean-square. Perhaps this is one way of viewing II.3. In any case the above result allows the calculation of the mean-square:

$$2 \cdot \text{square on } AD = a \cdot a - 2((b \div 2) \cdot (b \div 2)),$$

and now “halve the remainder [difference] and take the square root.” The side AD is now known:

$$AD = \sqrt{(10 \cdot 10 - 2(3.4 \cdot 3.4)) \div 2} = 6 \text{ ch'ih}, 2 \text{ ts'un}.$$

From Figure 8 it is clear that

$$\text{width} = AD - BD = 6.2 - 3.4 = 2 \text{ ch'ih}, 8 \text{ ts'un},$$

$$\text{height} = AD + BD = 6.2 + 3.4 = 9 \text{ ch'ih}, 6 \text{ ts'un}.$$

These calculations fit the Rule very well indeed.

Now turning to Problem 12, we have some indication that it was given as a generalization of Problem 11. For one thing, it is the immediately following problem; also, if II.4 was in fact used in Problem 11, a straight-forward adaptation of this result fits the received arithmetic of Problem 12.

Problem 12. “We now have a door whose height and width are not known; both are shorter than a bamboo pole whose length is unknown. If the bamboo is held horizontally, it comes out 4 ch'ih longer than the width; if held vertically it comes out 2 ch'ih longer; if laid in the diagonal it comes out even. Question: What are the height, the width, and the diagonal of the door? Answer: The width is 6 ch'ih, the height is 8 ch'ih, and the diagonal is 1 chang.

Rule: Multiply the deficiencies in the vertical and horizontal together, double the product and take the square root of the result. Add to this the deficiency in the vertical to give the width of the door; add to it the deficiency in the horizontal to give the height of the door; add to it both deficiencies to give the diagonal of the door.”

In Figure 14(1) is given a sketch of the context of the problem. Each of $A'B'$, $B'C'$, and $C'A'$ is unknown. In Figure 14(2) is given a figure similar to that of

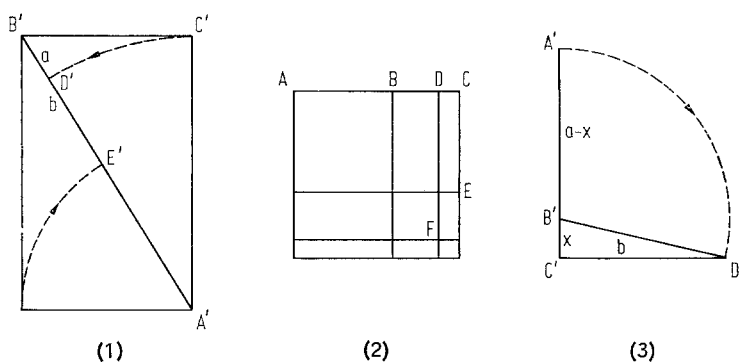


Fig. 14

Figure 8 (which accompanies the result II.3 used in Problem 11). Make the identifications: $A'B'$ is AC , BD is deficiency b , and DC is deficiency a . The argument used to justify II.3 and the analogy between Figures 8 and 14(2) suggest that the side AB , which is neither the width, height, nor diagonal of the door, will play a role in the calculations. From Figure 14(2) it follows, using the argument for II.3 as a guide, that

square on AB and square on AC = square on $(AB \text{ and } BD \text{ joined})$ and
square on $(AB \text{ and } DC \text{ joined})$ and
twice the rectangle EF .

Since $(AB \text{ and } BD \text{ joined})$ is the height of the door, $(AB \text{ and } DC \text{ joined})$ its width, and AC its diagonal, it follows, using III.1, that

$$\text{square on } AB = \text{twice rectangle } EF.$$

Now, using the given numerical data, we have

$$\text{square on } AB = 2 \cdot 4 \cdot 2 = 16,$$

$$AB = 4,$$

$$AB \text{ and } DC \text{ joined} = 4 + 2 = 6 \text{ ch'ih} = \text{width},$$

$$AB \text{ and } BD \text{ joined} = 4 + 4 = 8 \text{ ch'ih} = \text{height},$$

$$AB \text{ and } BC \text{ joined} = 4 + (2 + 4) = 10 \text{ ch'ih} = 1 \text{ chang} = \text{diagonal}.$$

VOGEL's comments on this problem [16, pp. 96, 121] are as follows (my translation): From the right triangle [see Figure 14(1) and denote $A'B'$ by z] we have $z^2 = 2z^2 - 2 \cdot (a + b) \cdot z + a^2 + b^2$. Adding $2ab$ to both sides gives $2ab = [z - (a + b)]^2$ or $z - (a + b) = \sqrt{2ab}$. Hence $z - a = b + \sqrt{2ab}$ and $z - b = a + \sqrt{2ab}$. VOGEL notes that the intermediate calculations are not given by the Chiu Chang and suggests that the first equation was completed through the formula $a^2 + 2ab + b^2 = (a + b)^2$, which VOGEL views as geometrically based.

Problem 13. “Now we have a bamboo pole with a height of 1 chang. The end was broken and touched the earth 3 ch’ih from the root. Question: How high is the break? Answer: $4\frac{11}{20}$ ch’ih.

Rule: Square the distance from the root; the result should then be divided by the height. From the height of the pole subtract the result and take half of the remainder. This is the height of the break.”

In Figure 14(3) is given a figure for this problem. We use II.4 and III.1. Through considering the first and third diagrams in Figure 1 and noting from Figure 14(3) that it is a that is given, we let AD be A'B' decreased by B'C' and DC be A'C'. From II.4 and III.1 we have

$$\begin{aligned}\text{square on } C'D' &= \text{the difference of the squares on } B'D' \text{ and } B'C' \\ &= \text{the rectangle with sides DC and AD.}\end{aligned}$$

This gives the arithmetic

$$(3 \cdot 3) \div 10 = 10 - 2 \cdot B'C'$$

or

$$B'C' = 4\frac{11}{20} \text{ ch'ih.}$$

Problem 14. In this problem (and in Problem 21) a notion of speed is introduced. In this context “Pythagorean triples” are calculated, perhaps in a manner similar to that proposed by VAN DER WAERDEN [17] for the triples in Plimpton 322. I shall propose a way of calculating these triples which is based on the ancient core, fits the received arithmetic, and agrees closely with VAN DER WAERDEN’S original idea.

Problem 14. “Now we have two people starting from the same place. The speed of A is 7 and the speed of B is 3. B goes east and A first goes 10 pu south and then towards the north-east until he meets B. Question: How far did each of A and B travel? Answer: B went $10\frac{1}{2}$ pu east. A went $14\frac{1}{2}$ pu in the diagonal direction until he met B.

Rule: Multiply each of 3 and 7 by itself, add the products and halve the sum. Take this result as the coefficient of the diagonal way of A. Subtract the coefficient of the diagonal way from the product of 7 with itself. The difference is the coefficient of the southern way. Multiply 7 by 3; the product is the coefficient of the eastern way of B. Lay out the 10 pu towards the south and multiply it by the coefficient of the diagonal path of A. Again lay out the 10 pu and multiply it by the coefficient of the eastern way of B. Each product is a dividend. Divide the dividends by the coefficient of the southern way. Each time we get the amount of the way.”

So as to not interrupt the discussion of the rule itself I shall first discuss the use of III.1 and II.4. If we refer to Figure 15(1) and use III.1, the product

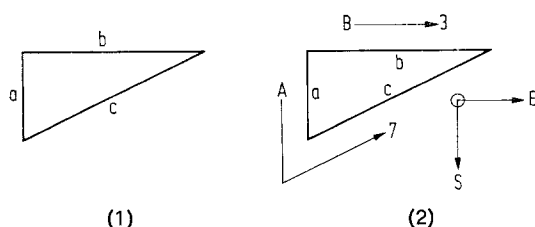


Fig. 15

of b with itself is the difference of the products of c and a with themselves. From II. 4, which uses Figure 1, this difference of squares may be thought of as a product of sides AD and DC . Moreover, c and a are seen to be $HS(AD, DC)$ and $HD(AD, DC)$. It is also clear from II.4 and Figure 1 that $c + a = DC$ and $c - a = AD$. Let DC and AD be measured by numbers p and q , respectively; these results may be summarized in

$$b \cdot b = c \cdot c - a \cdot a = p \cdot q,$$

$$c = (p + q) \div 2,$$

$$a = (p - q) \div 2,$$

$$c + a = p,$$

$$c - a = q.$$

Problem 14 is that of two persons A and B moving at different rates during the same time period, that is, by reference to Figure 15(2), the distances $c + a$ and b are completed in equal times. Hence $(c + a) \div 7 = b \div 3$. From $b \cdot b = (c - a) \cdot (c + a)$ and $(c + a) \div 7 = b \div 3$ it follows that $(c + a) \div (c - a) = 7 \cdot 7 \div 3 \cdot 3$.^{*} In a triangle similar to the desired triangle one will have $c + a = 7 \cdot 7 = p$ and $c - a = 3 \cdot 3 = q$. It then follows that—here begins the received arithmetic—

$c = (7 \cdot 7 + 3 \cdot 3) \div 2 = 29$, $a = p - c = 7 \cdot 7 - 29 = 20$, and $b = 7 \cdot 3 = 21$. Since these numbers are proportional to the lengths of the sides of the triangle, it remains to use the given information that the southern way has length 10 pu. Thus, multiply all sides by 10 and divide by a . This gives the diagonal and eastern ways, $(10 \cdot 29) \div 20 = 14\frac{1}{2}$ pu and $(10 \cdot 21) \div 20 = 10\frac{1}{2}$ pu.

Problem 15. This problem is the first of two problems on inscribed figures in right triangles.

^{*} Once the geometry of the problem is translated into arithmetic equalities such as these, one may infer the statement (*) as follows: 1) From $b \cdot b = (c - a) \cdot (c + a)$ it follows that $((c - a) \div b) \cdot ((c + a) \div b) = 1$; hence the two factors are reciprocals. 2) From $(c + a) \div 7 = b \div 3$ it follows that $(c + a) \div b = 7 \div 3$. Putting 1) and 2) together shows that $(c - a) \div b = 3 \div 7$. Hence $(c + a) \div (c - a) = 7 \cdot 7 \div 3 \cdot 3$. This argument is due in part to SEIDENBERG (private communication).

Problem 15. “Now we have a horizontal side of 5 pu and a vertical side of 12 pu. Question: How large is the square inscribed in the right triangle? Answer: The side of the square is $3\frac{9}{17}$ pu.

Rule: Add the horizontal and vertical sides; this is the divisor. Multiply the horizontal and vertical sides; this is the dividend. Divide the dividend by the divisor. The result is the side of the square.”

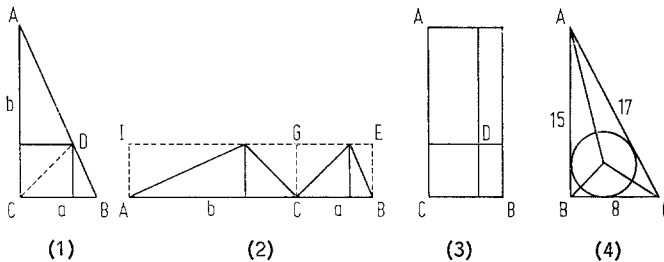


Fig. 16

A figure is given in Figure 16(1). The side of the square CD is to be found. If the figure is cut along the dotted line from D to C, leaving an uncut hinge at C the figure may be opened as in Figure 16(2). The dotted lines make four rectangles, each of which is bisected by a diagonal. It is clear from I.1 that the area of AE is the sum of the areas of the two rectangles IC and GB. Arithmetically,

$$(b + a) \cdot \text{side of square} = b \cdot a.$$

This is the received arithmetic.

The idea of the dissection along CD is due to VAN DER WAERDEN [19, p. 54]. VOGEL [16, p. 97] states that the formula $b \cdot a / (b + a)$ follows from similarity or the equality of the gnomon complements (I.2). The gnomon VOGEL has in mind must be that in Figure 16(3), obtained from Figure 16(1) by drawing through A and B lines parallel to CB and CA, respectively, and extending the sides of the inscribed square. Since $b \cdot a = \text{sum of the areas of rectangles AD, DE, DB, and CD}$, and the areas of DE and CD are equal, it follows that $b \cdot a = \text{areas of rectangles AF and CG}$, that is, $b \cdot a = b \cdot \text{side of square} + a \cdot \text{side of square}$.

Each of these explanations of Problem 15 lies within the proposed ancient core. I have come to prefer VAN DER WAERDEN's idea because it is applicable to the next problem, while it appears that VOGEL's approach is not.

Problem 16. This problem asks for the diameter of the circle inscribed in an (8, 15, 17) right triangle. See Figure 16(4). VOGEL [16] suggests that the given rule may be based on a dissection of the figure. Although VOGEL gives few details it would appear that he had in mind something quite similar to VAN DER WAERDEN's solution for Problem 15. If in Figure 16(4) cuts are made from the center of the inscribed circle to the three vertices, leaving uncut hinges at B and C, the figure may be opened out on either side of BC, so that A, B, and C lie in a straight line. From this figure, which is analogous to Figure 16(2), the received arithmetic $((15 + 8 + 7) \cdot (\text{diameter} \div 2) = 15 \cdot 8)$ follows directly.

Problems 17–21. Each of these problems may be called a “city problem”. The first three may be seen as variations on I.2, the equality of gnomon-complements. The fourth is more difficult, appearing to require two results from the ancient core, I.3 and II.4. Problem 21 is yet another variation, combining a “city problem” with a “speed problem.” This is consistent with the trend of the problems, from simple to complex.

Problem 18. “Now we have a city with a rectangular boundary. From east to west it measures 7 li and from north to south 9 li. In the middle of each side is an open door. If we go out the east door 15 li we come to a tree. Question: How many steps out the south door must we go to see the tree? Answer: 315 pu.

Rule: Multiply the number of steps from the east door towards the south, just up to the corner, by the number of steps from the south door east, just up to the corner. The product is the dividend. Take the number of steps the tree is from the door as divisor. Divide the dividend by the divisor.”

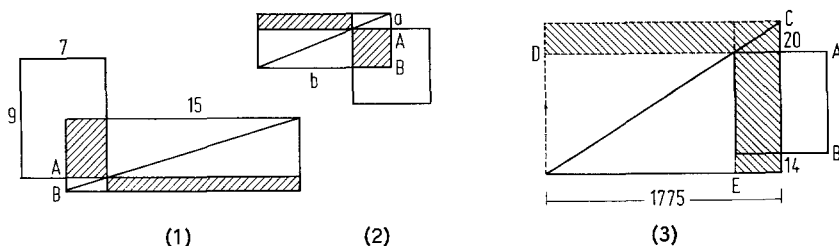


Fig. 17

Using I.2 and Figure 17(1)—and converting li to pu (1 li = 300 pu = 300 “steps”)—gives immediately that

$$AB \cdot 4500 = 1050 \cdot 1350 \text{ and so } AB = 315 \text{ pu.}$$

The solution to Problem 19—see Figure 17(2)—is a variation on that of Problem 18 in that one of the gnomon-complements is a square with unknown side AB and the other has known sides.

Problem 20. “Now we have a city with a square boundary. We do not know the length of the side. In the middle of each side is an open door. A person goes out of the north door 20 steps and finds a tree. If a person goes out the south door 14 steps and then turns towards the west 1775 steps, then he can see the tree. Question: What is the length of the side of the city? Answer: 250 pu.

Rule: Multiply the number of steps taken from the north door by the number of steps towards the west. Double this and let it be the *shih*. Add the number

of steps taken from the south and north doors and let it be the *tsung-fa*. Extract the square root to give the desired side.”*

Referring to Figure 17(3) and using I.3 we see clearly that

$$20 \cdot 1775 = (\text{half of unknown side}) \cdot (\text{unknown side and } 20 \text{ and } 14)$$

or

$$2 \cdot 20 \cdot 1775 = (\text{unknown side}) \cdot (\text{unknown side and } 20 \text{ and } 14).$$

The Rule contains instructions on computing $2 \cdot 20 \cdot 1775$ and $20 + 14$ and then either invokes a procedure related to square roots or, if one assumes the Rule means one is to compute $\sqrt{(2 \cdot 20 \cdot 1775)/34}$, is completely wrong. The answer of 250 pu is correct. I shall give an explanation for a solution within the ancient tradition.

By use of II.4 the last equation may be written as

$$2 \cdot 20 \cdot 1775 = \text{difference of the squares with sides HS (unknown side and } 20 \text{ and } 14, \text{ unknown side) and HD (unknown side and } 20 \text{ and } 14, \text{ unknown side)}.$$

This leads to the arithmetic (note $17 = (20 + 14) \div 2$)

$$2 \cdot 20 \cdot 1775 + 17 \cdot 17 = \text{HS} \cdot \text{HS}$$

or

$$\sqrt{2 \cdot 20 \cdot 1775 + 17 \cdot 17} - 17 = 250 \text{ pu}.$$

This solution is *not* described in the Rule. Certainly one may observe in both places the partial results $2 \cdot 20 \cdot 1775$ and $20 + 14$ and note that both refer to a square root operation. I note that this problem is the only one in Chapter IX for which the arithmetic recipe is not elementary and complete within the problem. I will comment further upon this problem in Section V.

Problem 21. This is another “city problem,” but one having features of two earlier problems. One part of the problem involves velocities; the handling of this in the solution is the same as in Problem 14. The geometry is similar to that in Problem 18; I.3 is used instead of I.2.

Problems 22–24. Each of these problems describes a situation involving a remote object. In the first problem the distance to a tree is wanted; given are data relating the tree to a known, nearby figure. In Problem 23 the height of a mountain is wanted; given are data on its relationship to a nearby, vertical pole. One problem is based on I.2, the other I.3. Problem 24 involves a well of unknown depth; given are data obtained at ground level. Each of these three problems uses either I.2 or I.3 in a straightforward manner.

* *Shih* and *tsung-fa* are technical terms occurring in the square root algorithm. See Section V.

I now give a table summarizing the usage of the proposed ancient core in the 24 problems of Chapter IX of the Chiu Chang. I note that the problems are grouped and, within a group, tend to become more complex, sometimes combining in one problem features of several problems. In Chapter IX as a whole the problems tend from very simple applications of the DST to quite difficult, multi-stage problems. The last three problems are relatively simple compared to Problem 21.

VOGEL [16, p. 120] argues that the Chiu Chang is not a textbook, rather a collection of problems, since the methods of calculation are only explained in detail in a few problems. WANG & NEEDHAM [20, p. 351] note—though on a different Chapter of the Chiu Chang—that “The extreme brevity of the text is presumably due to the custom of oral teaching by qualified mathematicians, and to the literary desire of avoiding undue repetition.”

It seems to me that Chapter IX represents a deliberate and fully considered attempt to order a set of problems, with features of repetition, generalization, and the blending of problem contexts and solution techniques from several problems into one. This more nearly characterizes a textbook than a problem collection.

Problem	Ancient Core	Category
1	III.1 (DST) }	Purely geometric/arithmetic
2	III.1 }	
3	III.1 }	
4	III.1	Sawing plank from a log
5	III.1	Vine-helix problem
6	III.1 & II.1A	Reed in reservoir
7	III.1 & II.1B	Cord on pole
8	III.1 & II.1B	Beam leaning on wall
9	III.1 & II.1B	Cutting a log
10	III.1 & II.1B	Door problem
11	III.1 & II.3	Door problem
12	III.1 & II.3 generalized	Door problem
13	III.1 & II.4	Broken bamboo
14	III.1 & II.4	Velocities
15	(III.1)* & I.1	Inscription problem
16	(III.1)* & I.1	Inscription problem
17	I.2	City problem
18	I.2	City problem
19	I.2	City problem
20	I.3 & II.4	City problem
21	III.1 & I.3 & II.4	City problem & velocities
22	I.2	Remote object problem
23	I.3	Remote object problem
24	I.2	Remote object problem

* DST not explicit.

V. Square Roots, Problem IX.20, and Horner's Method

I particularly wish to make an argument supporting my reconstruction of Problem 20 since my use of II.4 is not consistent with the views of WANG & NEEDHAM [20] and D. B. WAGNER (private communication). These distinguished scholars have an extraordinarily impressive knowledge of both the ancient Chinese language and Chinese mathematics. Given this, I am very hesitant in advancing explanations at variance with their opinions.

WANG & NEEDHAM's paper is "Horner's Method in Chinese Mathematics." Their main "... new contribution ... is the attempt to show that the mathematicians of the 1st century B.C. in China understood the essentials [of HORNER's method]." They also say, "It is now generally accepted by historians of mathematics that the method of HORNER (1819) for solving higher numerical equations appeared in a substantially equivalent form in the works of the Sung algebraists such as Ch'in Chiu-shao (1247). This method took its origin much earlier in the procedures for root extraction of the Han dynasty. These are to be found in the 'Nine Chapters of Mathematics' (Chiu Chang Suan Shu) but the text has long been very obscure, and it is the purpose of the present paper to attempt an explanation of its meaning." Their reconstruction/exposition of the Chiu Chang method of extracting the square root of number is meticulously done and very clear. Some fifteen years later LAM LAY YONG [3] wrote a paper, based on a work by YANG HUI in 1261, supporting WANG & NEEDHAM and also presenting an argument for a geometric basis for the arithmetic root extraction techniques given in the Chiu Chang. My own explanation, arrived at independently of LAM LAY YONG but influenced by certain Babylonian procedures and a somewhat obscure and, in my opinion, questionable paper by GANGULI [1] on Vedic mathematics, parallels that of LAM LAY YONG.

The surviving Chinese procedures on square root extraction are described in terms of operations on a counting board or calculating device. Although it happens to be a base 10 device, the underlying explanation does not depend upon the choice of base. * In 1968 VOGEL stated his opinion that the Chiu Chang method was based upon an algebraic identity. I believe, however, with SEIDENBERG and VAN DER WAERDEN, that the basis of much of Babylonian, Chinese, and Vedic mathematics is at bottom geometric. The computational algorithms resting on the underlying geometry are, however, by and large what have survived and are the raw material with which we work.

The result II.1B is sufficient to explain the received arithmetic, allowing for subsequent adaptation to a computing device and an inevitable codification of the geometric procedure into an algorithm, complete with names for the various numbers appearing in the computation.

I assume that a numerical measure N of the side of a square with given area was sought and that the metrological system was sufficiently developed that N was thought of as, for example, A degrees, B minutes, and C seconds (Babylonian) or a chang, b ch'ih, and c ts'un. For simplicity I will continue as if there were

* The algorithm is, in fact, the same as the well known "divide and average" algorithm, which is the same as "NEWTON's method".

only three digits in N . The method of the algorithm is not restricted to a fixed number of places.

Using WANG & NEEDHAM's example, we seek the side N of a square with given area, say 55,225. We therefore seek digits a , b , and c for which $(a \cdot 100 + b \cdot 10 + c) \cdot (a \cdot 100 + b \cdot 10 + c) = 55,225$. By trial— $(3 \cdot 100) \cdot (3 \cdot 100) = 90,000 > 55,225$; $(2 \cdot 100) \cdot (2 \cdot 100) = 40,000 < 55,225$ —we find $a = 2$. From II.1B, the difference of the squares on TZ (55,225) and TU (40,000) in Figure 18 is the aligned gnomon. If we neglect the very thin outer gnomon and the small square with diagonal UV, we have

$$55,225 - 40,000 > 2 \cdot (100 \cdot a) \cdot (10 \cdot b)$$

or

$$15,225 > 4,000 \cdot b.$$

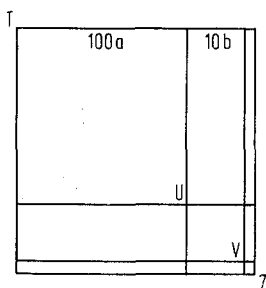


Fig. 18

This gives a good estimate of b ; in this case b is less than 4. We may check if $b = 3$ is correct (is $(2 \cdot 100 + 3 \cdot 10) \cdot (2 \cdot 100 + 3 \cdot 10) < 55,225$?). The algorithm now repeats the same reasoning for the remaining digits:

$$55,225 - 40,000 - 30 \cdot (2 \cdot (200) + 30) > 2 \cdot 230 \cdot c,$$

$$2,325 > 460 \cdot c.$$

This shows that $c < 6$. It follows, as before, that $c = 5$ is correct. Indeed, $235 \cdot 235 = 55,225$.

The arithmetic suggested by II.1B agrees in detail and order with that of the Chiu Chang, as clarified by WANG & NEEDHAM. LAM LAY YONG's arguments are the same and are based directly upon YANG HUT's commentary of 1261 on the Chiu Chang. YANG HUT's work contains a diagram identical to that in Figure 18—recall that the Chiu Chang as received has no diagrams.

If HORNER's method for polynomials is applied to finding a square root (so that the polynomial is taken to be $x^2 - n$, where n is the measure of the square whose side is wanted), it is easy to check (WANG & NEEDHAM give detailed examples) that the arithmetical work agrees with the Chiu Chang method. Recalling that HORNER's method for polynomials is attested in China in 1247, we naturally suppose that the thirteenth century Chinese based their method on the earlier Chiu Chang square root algorithm. Nobody disagrees with this. WANG & NEEDHAM

go further, however, and argue that in Problem 20 of the Chiu Chang we have “distinct evidence that at least towards the end of the pre-Christian era the Chinese applied their method [for calculating the square roots of numbers] to solve a quadratic equation of the type $x^2 + ax = b \dots$ ” The principal evidence is (1) the presence in Problem 20 of two technical words which appear in the Chiu Chang square root procedure and (2) the absence of a detailed recipe which is elementary and complete within the problem.

I can not account for either of these features of Problem 20 and am not qualified to discuss the linguistic evidence in the manuscript. I wish to observe, however, that Problem 20 is unique in that its Rule does not give a detailed, utterly explicit recipe for manipulating the given numbers to obtain the answer. Moreover, this problem is one of a cluster of several similar problems, the “city problems,” all others of which have explicit Rules, which, in my view, are based on the equality of the gnomon-complements. Finally, I have shown that II.4—which was also used in Problem 14—provides a straightforward solution to the problem.

VI. Summary

VAN DER WAERDEN’s and SEIDENBERG’s theories on the origin of mathematics continue work which at least for pre-Greek mathematics, began with NEUGEBAUER & SACHS [8], who hypothesized that the contents of the “geometrical algebra” in Book II of EUCLID’s *Elements* utilized results known in Mesopotamia. VAN DER WAERDEN and SEIDENBERG recently have hypothesized a development beginning well before 2000 B.C., initially centered upon an oral tradition of geometrical constructions and including the “Theorem of PYTHAGORAS”; this tradition subsequently divided into two great traditions—one geometric or constructive, the other algebraic or computational—which were transmitted separately to the Indians, Babylonians, Chinese, and Greeks. The first and last are thought to have received the geometrical tradition while the second and third received the algebraic tradition.

My own work was centered upon Chapter IX of the Chiu Chang, which I have used as a guide in my attempt to recover the oral tradition. I was also influenced by the work of NEUGEBAUER, SACHS, and VAN DER WAERDEN on Babylonian mathematics and by SEIDENBERG’s recent studies of Indian religious works on altar constructions.

The oral tradition I have proposed fits the Chiu Chang well in the sense that the core of nine geometric results leads one to the received arithmetic in the 24 problems. I have noted that this agreement is in part assured since the problems were used in choosing the results. Although I have not yet completed a systematic examination of the Babylonian materials, the oral tradition proposed here provides some explanation of them, including the generation of Pythagorean triples and the so-called normal problems.* In his recent book [19] VAN DER WAERDEN stresses

* NEUGEBAUER has referred to problems in which two numbers are sought and one is given their product and either their sum or difference as quadratic problems in normal form.

the many similarities between Babylonian and Chinese algebra, noting in both kinds of algebra the solution of quadratic equations; that sets of linear equations were solved by eliminating one unknown after the other; and the presence in both countries of numerical methods for calculating square and cube roots. I believe that the congruences with Books I and II of EUCLID, the Chou Pei diagram, and the results given in the Sulvasutras provide further evidence that the proposed Neolithic oral tradition has some explanatory power beyond the Chiu Chang.

If one were to use, say, II.4 repeatedly on problems including numerical measures of sides, the pattern imposed on the arithmetical reckoning by II.4 would be sufficiently strong to induce in the mind of the user an algorithm on the numbers themselves, an algorithm which would come to stand on its own, with the geometric result gradually assuming less importance. This is my view of the emergence of algorithmic/algebraic processes from the geometric. Without at all wishing to enter the geometric algebra controversy, I have tried in this paper to avoid using modern notation for algebraic identities implicit in geometric results. It seems to me that such use introduces an unnecessary and misleading bias into the discussion.

I have discussed some of VOGEL's comments on Chapter IX of the Chiu Chang, particularly when I felt his explanations of the problems were misleading or incomplete. In Problem 12, for example, VOGEL's explanation appears to me to be too algebraic for attribution to the authors or students of the Chiu Chang. If VOGEL intends to suggest through the algebra corresponding geometric results, then the suggested approach appears overly complex. For the most part our explanations are consistent. VOGEL often uses modern algebraic notation in his explanations and suggests the presence of such algebraic phenomena as factoring or completing the square. I find this contrary to my own reading. At the same time VOGEL also uses geometric figures and mentions in several places a geometric origin for the solution techniques.

My explanations are more complete (at the cost of being more speculative) than VOGEL's and in most problems exactly match the arithmetic given in the Chiu Chang. I have disagreed with VOGEL, WAGNER, and WANG & NEEDHAM on Problem 20; my discussion is given in Section V.

I believe my remark that the figure suggested by the Vedic result on doubling a square is a special case of the Chou Pei diagram—the figure also is given in PLATO's *Meno*, in the dialogue between SOCRATES and a boy—together with the interplay between Figures 7(1), 8, 11(1), 11(2), 11(3), and 9(2) discussed in Section III provide a useful summary of some bits and pieces related to the DST.

VAN DER WAERDEN (private communication) has raised the point that denoting line segments by two letters is not appropriate in describing an oral tradition. In an oral tradition one would use words such as hypotenuse, shorter leg, side, ..., or simply point to the line in question in a diagram or model. This had not occurred to me although I did work at the somewhat analogous matter of avoiding algebraic notation. I agree with VAN DER WAERDEN's view but have decided not to change the present paper.

To write down a proposed neolithic oral tradition (or ancient core) is, of course, speculative. Virtually by definition, written records were not produced in neolithic

times. What I have tried to do is to infer what was likely to have been the common oral heritage of the historical Chinese, Babylonian, Indian, and Greek mathematics. I have assumed that SEIDENBERG'S and VAN DER WAERDEN'S views on the origin of mathematics are more nearly correct than those who believe mathematics was independently discovered by several peoples. I have not tried to give a detailed discussion of their views here; I have tried to support their views by drawing together a reasonably complete oral tradition. I have argued that it explains the Chiu Chang well, matches Book II of EUCLID'S Elements closely, and, leaning principally on VAN DER WAERDEN'S papers and books, said that it appears to explain some of the Babylonian corpus. I am working on a paper which will attempt to go much further with the Babylonian materials.

Note added in proof. I have received a letter from Professor VOLKOV (Department of the History of Mathematics, Moscow) in which he discusses the meanings of the terms *shih* and *tsung-fa* associated with Problem 20 (see pp. 212–213). He argues that the received Rule for Problem 20 is not incomplete; rather, the two technical terms refer to a well known procedure for solving a quadratic equation numerically. I have stated that I am not competent in the Chinese language and therefore can not form an opinion on the merits of Professor VOLKOV'S (and others) arguments. Certainly they must be given some weight since they rest directly upon the received text. My own views are given in Section V. They rest upon the singularity of the Rule for Problem 20 and the observation that II.4 (which was used in Problem 14) provides a simple solution, one consistent with the remainder of the problems.

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(Received March 20, 1985)