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1968

CHAPTER 2

The Pythagorean Number Theory

The political events in Asia Minor during the sixth century B.C. caused a migration of Greeks towards Italy, Sicily, Thrace, and within Greece proper. But it was the intellectual reaction against the Ionian thinkers that gave birth to the new schools of thought in Croton, Elea, Abdera, and Athens. The first school of its kind was founded by Pythagoras of Samos (fl. 570 B.C.), whose intellectual interests were enriched by his knowledge of Oriental wisdom. Similarities and analogies with Eastern ideas appear in the doctrines and practices of the Brotherhood he organized in Croton.

Evaluation of Pythagoreanism is made difficult by the absence of fragments of the earlier disciples. The Master himself followed the Eastern usage of transmitting his views by word of mouth; his spoken judgment was held as authoritative. Moreover, the rules of the Brotherhood prevented its members from divulging the teachings of the Master or even their own opinions, so Pythagoras is credited with discoveries which may have been those of his disciples. The Pythagorean doctrines are known chiefly through their later representatives, such as Philolaus of Croton (fifth century B.C.), who is considered the

first to have written on these matters, and Archytas of Tarentum (fifth century B.C.); from the Platonist Eudoxus of Cnidus (fourth century B.C.), whose lost works are known through extracts preserved by later writers; and from references in the works of Aristotle. Plato, who owes much to Pythagorean influences, gives only a trivial reference in the *Republic* (600B) to the founder of the school.

The esoteric members of the Brotherhood were called "mathematicians" because the term merely indicated the various subjects they had to learn with the Master. The modern specific meaning of "mathematics" and "mathematicians" is due mainly to Aristotle. This is shown by the wider significance of the word in the following fragment of Archytas:

As the mathematicians seem to have arrived at correct conclusions, it is not surprising that they have a true conception of the nature of each individual thing. For having reached such correct conclusions about the nature of the universe, they were bound to see in its true light the nature of particular things as well. Thus they left us a clear knowledge about the speed of the stars, their risings and settings; about geometry, arithmetic and sphaeric; and last but not least, about music; for these studies (mathematics) seem to be sisters.¹

Notwithstanding subsequent classifications of the sciences, arithmetic and geometry should not be separated from the rest of the Pythagorean teachings. The Master could not consider them independent of the other branches of knowledge, since they were a portion of wisdom, like music or the "bad arts" he had practiced. Moreover, Pythagoras could not study them simply as pleasant or useful subjects. A natural philosopher and religious reformer, he must have considered them the best introduction to higher wisdom: they helped the liberation of the soul from the dominion of the senses, as well as the under-

¹ Hermann Diehl, *Die Fragmente der Vorsokratiker: Griechisch und Deutsch*, 5th ed. (Berlin, 1933), 47 B 1, III, pp. 431-432; (hereafter referred to as Diehl, Voss.; trans. by Thomas Greenwood).

standing of nature and the world beyond. But the technical interest of Pythagoras in mathematics was probably conditioned by his intuitive realization that number is the essence of things. In the absence of any relevant records, we can only surmise the origin of this remarkable conception, which seems to have inspired Plato in shaping his own philosophy.

The partial solutions of Ionian thinkers concerning the nature of the primordial substance must have urged the Pythagoreans to find a more fundamental cause of things, discover a principle underlying the four elements, and give a deeper explanation of the Milesian systems. Furthermore, since nature is only a part of human experience, a more universal essence was needed to explain nature, reason, and religion as well. If the empirical aspects of knowledge could somehow be satisfied with the operations of the natural elements, its rational and mystical aspects required a different principle of explanation. Because of its rationality and permanence, mathematics could provide such a principle readily. The universal value of mathematics suggested by the naturalistic account of knowledge was confirmed by the religious requirements of action. If the emotive Orphic rituals were to satisfy the positive mind of the Pythagoreans, some rational basis was needed for their religious implications. A comparative analysis of Eastern traditions would have shown that mathematical, especially numerical, relations were essential in mystical speculations and in the interpretation of the world.

Numbers were also indispensable in many practical fields, such as commerce and everyday social intercourse. At the time of Pythagoras, numbers were not the object of a separate science. They were considered to be almost as material as the ultimate principles of things—earth, air, fire, water—of the Milesians. In fact, they were merely used for practical purposes without being considered as purely rational entities. This distinction is illustrated by Plato when he says that "logistic and arithmetic are wholly concerned with numbers,"² and by the

following passage of the scholium to the *Charmides*, wherein the object of the former is explained:

Logistic is the science dealing with numbered objects, not numbers; it does not consider number in its essence, but it presupposes 1 as the unit, and the numbered object as number; that is, it regards 3 as a triad, 10 as a decad, and applies the theorems of arithmetic to particular cases. Thus, logistic investigates what Archimedes called the cattle-problem and also "melite" and "phialite" numbers, the latter relating to bowls, the former to flocks. In other aspects, too, it investigates the numbers of material bodies, treating them as absolute. Its subject-matter is everything that is numbered.³

Pythagoras would have remembered the Babylonian view that each constellation had two chief characteristics—the number of stars composing it and the geometrical figure they form. Patient observation shows how to distinguish the various constellations in that way. Thus Plato wrote in *Timaeus* that "the vision of the day and night and of months and circling years has created the art of number. It has given us not only the notion of time, but also the means of studying the nature of the universe, from which has emerged philosophy in all its ranges."⁴

Further, Pythagoras would have observed that the art of music, so steadily practiced in the *Brotherhood*, was ruled by rhythm and number. This may have led him to discover the fundamental harmonic relations of a vibrating string stretched over a resounding board. By means of a movable bridge, he would divide the string into different lengths and produce various high and low notes. Though unable to determine the vibrations on which the separate sounds depended, he could measure the length of the vibrating string which was the material cause of the sound and determine the ratios corresponding to the various tones. Through number, music was thus connected with astronomy.

² Scholium to Plato *Charmides* 165E.

⁴ *Timaeus* 47B.

² *Republic* 525A.

With every constellation and every musical note characterized by a number, the study of the heavens and of sound would suggest by analogy the establishment of a number theory extending to ethics and religion. From trade to liturgy through astronomy and acoustics, man's interests would be linked together by the power of number. Merged into the things perceived by the senses as well as into the higher values of life and destiny, numbers must have appeared to Pythagoras as more universal than any other human conception. Stripped of the accidents identifying them with specific objects, or even with the astrological chart of any person, numbers could be taken as the real constituents—the very nature of the world. Thus Pythagoras was led to declare that number is the essence of things.

According to Aristotle, the Pythagoreans do not place the objects of mathematics between the ideas and material things as Plato does, they say "that things themselves are numbers"⁵ and that "number is the matter of things as well as the form of their modifications and permanent states."⁶ As the principles of mathematics, numbers are "the principles of all existing things."⁷ Therefore, numbers cannot be attributes of something else; they are the substance of all existing things⁸ and also "the causes of the reality of other things."⁹ This follows from their participation in the One, which is a substance and not a predicate of something else.

But are numbers transcendent or immanent? Whereas Plato maintained their transcendence, the Pythagoreans held that all things possess number and are numbers. Hence numbers are not separable from things: as all existing things are made up of numbers, the whole heaven is number, and even abstractions and immaterial things are numbers. The immanence of number could not be expressed in stronger terms.

⁵ *Met.* 987^b 27.

⁶ *Met.* 987^a 15.

⁷ *Met.* 985^b 25.

⁸ *Met.* 987^a 18.

⁹ *Met.* 987^b 24.

Yet, in another passage discussing assimilation and resemblance, Aristotle attributes to the Pythagoreans the opinion that numbers are affections or relations rather than substances.

They seemed to see in numbers many resemblances to the things that exist and come into being, more than in fire, in earth and in water. Such and such a modification of numbers was justice, another was soul and reason, another was opportunity, and similarly almost all other things were expressible numerically. Again, they saw that the attributes and the ratios of the musical scale were expressible in numbers. Since all things seemed in their whole nature to be modelled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heavens to be a musical scale and a number. They collected and fitted into their scheme all the properties of numbers and scales they could show to agree with the attributes, the parts, and the whole arrangement of the heavens.¹⁰

Aristotle also asserts that the Pythagoreans say things exist by imitation of numbers, while Plato says they exist by participation. But he does not elaborate on this point, because "what the participation or the imitation of the forms could be they left an open question."¹¹ This is perhaps a reference to the views of Philolaus, for whom numbers are the *paradigms* or the substantial patterns of things—a conception apparently current among the younger Pythagoreans, which could have inspired Plato. Though the two views might have coexisted among the early Pythagoreans, Aristotle must have considered the complete identification of numbers and things to be their fundamental doctrine, as he devoted to it most of his criticism.

On the other hand, when Aristotle says the Pythagoreans "supposed real things to be numbers"¹² and "did not regard number as separable from the objects of sense,"¹³ he surely

¹⁰ *Met.* 985^b 27.

¹¹ *Met.* 987^b 13.

¹² *Met.* 1090^a 20.

¹³ *Phys.* 203^a 6.

means they must have studied numbers as external objects, not as mere auxiliaries to ordinary computation. This view is emphasized by his reference to Eurytus, a disciple of Philolaus, who expressed the nature of objects by means of pebbles or counters. "Eurytus decided what was the number of an object (for example of a man or a horse) by imitating the figures of living things with pebbles, as some people bring numbers into the forms of the triangle and square."¹⁴ Theophrastus reports the same story, which probably goes back to Archytas.¹⁵

A more detailed account of the method of Eurytus is given by Alexander:

Let us assume for example that 250 is the number which defines man, and 360 that which defines plant. Having laid this down, he took 250 counters, some green and some black, others red and of many other colors; then smearing the wall with plaster and sketching on it a man and a plant, he proceeded to fix some of the counters in the outline of the face, some in that of the hands, and some in that of other parts. Thus he completed the outline of the man he was imagining by a number of counters equal in number to the units which he said defined the man.¹⁶

This doctrine of numbers and things prompted the Pythagoreans to collect and fit into their scheme all the properties of numbers they could discover to agree with particular experiences.

Another reason for the Pythagorean doctrine that number is immanent may be found in the general character of philosophical speculations at the time. The Ionians did not inquire about the "likeness" of things, but about the "nature" of things, their ultimate essence. The objects of sense perception were not explained through their participation with water or air, but through their ultimate identification with those elements. For example, Anaximenes considered the phenomena of the external world as the various modes of one single ulti-

mate reality, air. The Ionian monism excluded a multiplication of causes.

If things imitate numbers, the unanswered question of their ultimate cause might involve a possible multiplicity of causes. Imitation explains neither essence nor existence. Mere assertion of an analogy between numbers and the objects of experience offers no account of their nature. The Pythagoreans could scarcely fail to perceive the weakness of the argument for simple imitation. The world of experience, therefore, could not *be like* numbers; it had to *be* numbers. Instead of being mere paradigms or archetypes of things, numbers had to be the things themselves, or at least the stuff out of which things are made. By word of mouth, this intuition was transmitted from one generation to another, until Philolaus announced openly that "all things which can be known have number; for it is impossible for a thing to be conceived or known without number."¹⁷

With this fundamental principle, the Pythagoreans developed their views on the nature of particular things by a more intimate study of numbers. If number is the essence of things, why turn to experience for an explanation? In tracing the cause of things, the *a posteriori* method was of little help to men already convinced that they possessed the most important clue to the solution of the mysteries of the world. If number is the basis of all knowledge and if its various transformations cause the nature of everything, it should suffice to organize and analyze a variety of numbers in order to understand rationally how the world is built. Hence Pythagoras "attached supreme importance to the study of arithmetic, which he advanced and removed from the region of commercial utility."¹⁸ This testimony is corroborated by Aristotle in his statement that "the Pythagoreans devoted themselves to the study of mathematics and were the first to advance this knowledge."¹⁹

What could be the stages of such a study? Establishment of

¹⁴ *Met.* 1092b 10.

¹⁵ Theophrastus *In Met.* p. 6a 19.

¹⁶ Alexander *In Met.* 827-829.

¹⁷ This fragment is preserved by Stobaeus; cf. Diels, *Vars.*, 44 B 4, III, p. 408.

¹⁸ Aristoxenus as quoted by Stobaeus; cf. Diels, *Vars.*, 58 B 4, III, p. 451.

¹⁹ *Met.* 985b 23.

an adequate number theory requires first the definition of number. Following the view of Thales²⁰ that number is a collection of units, the Pythagoreans "made number out of one."²¹ But they could scarcely regard one as a number, since a measure is not the thing measured and since one is "the beginning of number."²² Euclid implies this view in defining a unit as that by which an existing thing is called one, while a number is the multitude made up of units.²³ Among later Pythagoreans, Thymaridas (fourth century B.C.) defines a unit as a "limiting quantity"; Chrysippus (third century B.C.) calls it "multitude one."²⁴ Among neo-Pythagoreans, Moderatus (ca. 60 A.D.) considers number as "a progression of multitude beginning from a unit, and a regression ending in it,"²⁵ while Nicomachus (ca. 100 A.D.) defines it as "a flow of quantity made up of units."²⁶ The assimilation of units with points will be discussed later.

The simplest operations performed with units are duplication and its reverse, bipartition or mediation. This immediately supplies a broad principle of classification of numbers into those which are divisible into halves and those which are not. As Philolaus says, "number is of two special kinds, odd and even, with a third even-odd arising from a mixture of the two; and there are many forms of each kind."²⁷

Nicomachus, who represents well enough the Pythagorean tradition, gives this ancient definition: "An even number is that which can be divided both into two equal parts and into two unequal parts (except the fundamental dyad which can be divided only into two equal parts); but however it is divided, its two parts must be of the same kind without share in the other kind. An odd number is that which, however divided, must in any case fall into two unequal parts belong-

ing always to the two different kinds respectively."²⁸ This statement contains a reference to the original conception of the dyad as being not a number but the beginning of the even, just as one is not a number but the starting-point of number—a conception which must be very old, as Plato already speaks of two as even.²⁹ Nicomachus gives also this other definition without mentioning the dyad: "An even number is that which admits of being divided by one and the same operation into the greatest and the least parts, greatest in size but least in number (i.e., into two halves), while an odd number is that which cannot be so divided, but is only divisible into two unequal parts."³⁰

As regards the term *odd-even*, Aristotle says that "the elements of number are the even and the odd" and that "the one proceeds from both of these for it is both even and odd."³¹ Heath explains this strange view by submitting that the unit, being the principle of even and odd numbers, cannot itself be odd and must therefore be called even-odd.³² According to a better explanation suggested by Archytas and attributed to Aristotle by Theon of Smyrna, the unit added to an even number makes an odd number, but when added to an odd number it produces an even number, and therefore must partake of both species.³³ On the other hand, Iamblichus uses this term for even numbers like 6 and 10, which yield an odd number after a first bipartition.³⁴ This conception points to Plato's distinction among "even times even," "odd times odd," "odd times even," and "even times odd,"³⁵ taken up later by Euclid.³⁶ These are examples of the possible classification of the various odd and even numbers referred to by Philolaus.

²⁰ *Introductio Arithmetica* i. 7.4.

²¹ *Parmenides* 143D.

²² *Introductio Arithmetica* i. 7.3.

²³ *Met.* 986^a 17.

²⁴ Thomas L. Heath, *A History of Greek Mathematics* (Oxford, 1921), I, p. 71.

²⁵ *Expositio Rerum Mathematicarum*, p. 22.

²⁶ *Introductio Arithmetica*, p. 22.

²⁷ *Parmenides* 143E.

²⁸ *Elements* vii. Defs. 8-10.

²⁰ Cf. Iamblichus *Introductio Arithmetica*, p. 10.

²¹ *Met.* 985^a 20.

²² *Met.* 1088^a 7.

²³ *Elements* vii. Defs. 1, 2.

²⁴ Iamblichus *Introductio Arithmetica*, pp. 11-12.

²⁵ Stobaeus *Eclogae* i. Proem. 8.

²⁶ *Introductio Arithmetica* i. 7.1.

²⁷ Diels, *Vors.*, 44 B 5, III, p. 408.

The reciprocal operations of duplication and bipartition lead naturally to the study of numbers with regard to their divisibility in general. The Pythagoreans were not slow in discovering that certain numbers can be divided by no other number than the unit. A fragment of Speusippus based upon the writings of Philolaus³⁷ distinguishes "prime" and "incomposite" numbers and "secondary" or "composite" numbers. Thymaridas calls a prime number "rectilinear," and Theon of Smyrna, "euthymetric" or "linear." Theon says further that even numbers are not measured by the unit alone, except 2, which is therefore odd-like without being prime. Hence, the neo-Pythagoreans definitely exclude 2 from the prime numbers; for their predecessors, the dyad, or 2, was not a number at all, but the principle of the even, just as the one was the principle of number. Yet in defining numbers "prime to one another" as those "measured by a unit alone as common measure,"³⁸ Euclid accepts 2 as a prime number. So did Aristotle before him, when he speaks of the dyad as "the only prime number among the even numbers,"³⁹ which shows that this change was due to the immediate successors of Pythagoras. There are no reports of the early use of prime numbers for cosmical or ethical considerations.

Another early principle of combination of numbers was that of addition. By combining the unit with itself several times in succession, the series of the natural integers is obtained. Still to be considered are the results obtained from combining addition with duplication and multiplication. Such a combination yielded the *perfect* and *friendly* numbers, which result from the relations between a given number and its component parts with respect to multiplication and addition. By taking together the factors of a given number, including one but excluding that number, it may happen that their sum is equal to the given number, which is then called a "perfect" number. The

³⁷ Kathleen Freeman, *The Pre-Socratic Philosophers* (Cambridge, Mass., 1959), p. 224.

³⁸ *Elements* vii. Def. 12.

³⁹ *Topica* 157^a 35.

law of the formation of these numbers is described by Theon of Smyrna in this way: "If we take successive double numbers starting from the unit, add them until a prime and incomposite number is found, and then multiply the sum by the last of the added terms, the resulting number will be perfect."⁴⁰ The proof of this relation is given by Euclid,⁴¹ who established a general connection between prime and perfect numbers.

If the sum of any number of terms of the series: 1, 2, 2², . . . 2ⁿ⁻¹ be prime, and said sum be multiplied by the last term, the product will be a perfect number—that is, equal to the sum of all its factors; in modern notation: $S_n \cdot 2^{n-1}$ is a perfect number.

When a number is smaller than the sum of its aliquot parts, it is called an *over-perfect* number; a *deficient* number is greater than that sum. The first four perfect numbers are 6, 28, 496, and 8128; for $6 = (1 + 2 + 3)$ and $28 = (1 + 2 + 4 + 7 + 14)$. The numbers 12 and 20 are over-perfect because the sum of the aliquot parts $(1 + 2 + 3 + 4 + 6)$ of 12 is greater than 12, and the sum of the aliquot parts $(1 + 2 + 4 + 5 + 10)$ of 20 is greater than 20. The numbers 8 and 14 are deficient; for 8 is greater than the sum of its aliquot parts $(1 + 2 + 4)$ and 14 is greater than the sum $(1 + 2 + 7)$ of its aliquot parts. No reference to perfect, over-perfect, and deficient numbers is found in the fragments of Philolaus or anywhere before Euclid. We are told by Iamblichus⁴² that Speusippus had written about perfect numbers in "a neat little book entitled *On the Pythagorean Numbers*" based on the writings of Philolaus; the book has been lost.

Yet perfect numbers were probably known to Pythagoras if we accept a statement by Iamblichus⁴³ attributing to him the

⁴⁰ *Expositio Rerum Mathematicarum*, p. 45.

⁴¹ *Elements* ix. 36. The algebraic proof is given by Thomas L. Heath in *The Thirteen Books of Euclid's Elements* (Cambridge, 1908), II, p. 424 (repub. in New York, 1956).

⁴² *Theologumena Arithmeticae*, p. 82.

⁴³ *Introductio Arithmetica*, p. 35.

discovery of *friendly* numbers. These are pairs of numbers such that each is the sum of all the aliquot parts of the other, as 284 and 220: the aliquot parts of 284 ($1 + 2 + 4 + 71 + 142$) together equal 220, and the sum of the aliquot parts of 220 ($1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110$) equals 284. If Pythagoras investigated numbers reciprocally equal to the sums of their aliquot parts, he must also have considered the simpler class of numbers which are equal to the sum of their own aliquot parts. There is a story that Pythagoras defined a friend as "One who is the other I, such as 220 and 284." Discovery of such couples of numbers, which appear to have been known to the Hindus before the sixth century B.C., was a problem of considerable difficulty for the Greeks; because verification of such numbers usually requires the handling of very high digits. General interest in the friendly numbers seems to result from the belief that a good omen was attached to them.

The Pythagoreans considered development of the purely arithmetical properties of numbers as not only mathematically important, but also as preparation for wider application of numbers to the sensible world through their connection with figures or forms. To be sure, figures are the most natural link between purely numerical relations and presentations of the external world. From the pebbles of Eurytus to the simultaneous consideration of the numbers of stars of a particular constellation and the geometrical figure they trace out in space, many observations suggest an intimate connection between numbers and figures. If everything is number, then the figure of everything must be a number. These and similar considerations may have suggested to Pythagoras an essential relation between numbers and geometry.

On the other hand, the practical mensuration of the Egyptians and the mathematical generalizations of Thales, with which Pythagoras was presumably acquainted, could have led him to investigate the significance of his theories. For the Egyptians had not given any reason for their utilitarian geometry, nor were the material principles of things proposed

by the Ionians sufficiently rational to account for the abstract properties of numbers and geometrical figures. As the technique of deduction proper was not yet established, Pythagoras probably had to work out whatever rational explanation he could in order to connect the general propositions of geometry with their practical applications. Such an explanation could be suggested by number. If number is rational and material at the same time, then extension, which is also a manifestation of matter, must be number. That is why the Pythagoreans connected the unit in arithmetic and the point in geometry, by defining the unit as a "point without position," and the point as "a unit having position." The development of a number theory should therefore embrace geometry, which should in turn account for astronomy and the general study of nature.

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