

1275
NIDDITCH, P.H.
1983
P. 3-35

A 1275

The First Stage of the Idea of Mathematics: Pythagoreans, Plato, Aristotle

PETER H. NIDDITCH

C14

1. INTRODUCTION

1.1. The sense to be carried by the frame 'the idea of' in the title of this paper resembles the sense that frame has in Collingwood's use, in *The Idea of History* (and elsewhere). Collingwood's practice, though with a different terminology, had a long line of precursors, including Whewell's in his *Philosophy of Discovery: Chapters Historical and Critical*. Collingwood painted incisive portraits of a select succession of historical conceptions, i.e., of historiographies of historic moment; but his depictions were not passive replicas, for though he aimed at exhibiting the distinctive traits of his subjects, he vigorously engaged in revealing their deficiencies as perceived from his own perspectival center.

The term 'idea', then, in the frame-phrase 'the idea of', in the context of philosophical history, may be employed for a notion considered extensionally and intensionally in conjunction, or even in combination, in critico-historical chapters (note the order within the epithet): extensionally, in reference to reflective articulations of the notion in question in their historical sequence and connection, and intensionally, in reference to the nature of that notion as conceived and critically utilized by the commentator from his or her own methodical and regulative viewpoint.

1.2. The treatment here of the initial, ancient phase of the idea of mathematics is selective and abbreviative in several ways. (i) It is restricted to so-called pure mathematics (arithmetic, geometry, etc.) and so far as possible disregards mathematical physics and other fields of so-called applied mathematics. Work on the idea of applied mathematics in antiquity would be a different inquiry. (ii) I do not claim to be covering all the features of the mathematical philosophies¹ of the Pythagoreans, Plato, and Aristotle, nor, consequently, do I claim to be providing a fully balanced account of these philosophies. On the other hand, broad notice of their major preoccupations and characteristics is an objective, within the limits of a concise,

IN: MIDWEST STUDIES IN
PHILOSOPHY VOL. VIII

philosophically motivated consideration. (iii) The themes I air are relatively few, chosen out of a much more varied range of possibilities. I concentrate on issues of certain basic epistemological, logical, and ontological kinds concerning knowledge, truth, and existence, and their organizations.

Greek mathematical philosophy from its beginnings to Aristotle has exerted a great influence and attracted a lively philosophical interest down through the ages; yet, strangely, it has been attended to in an overwhelmingly piecemeal way. Overall scrutiny of it are a rarity, and, so far as I know them, they are largely intruded by rehearsals of the story of early Greek mathematics as such, with the representation of mathematical details hardly, if at all, relevant to philosophical history.² At any rate, I confine my synopsis more or less to a straight and narrow philosophical path. In the course of looking at the Hellenic philosophies of mathematics I want to bring out, not merely incidentally, what I see as significant continuity.

1.3. A silhouette of one face of the view regulating my pursuit of the present topic shadows forth the following features.³

(a) Mathematics is taken to be in general (arithmetic may be a peculiar exception) a symbolic, ratiocinative, and manifold field of discourse and thought to which there are no fixed conceptual bounds; it is a potential infinity of discursive conceptual structures, many of which themselves involve the nonfinite.

(b) That discourse is regarded as being free throughout from realist and constativist presuppositions and implications concerning knowledge, truth, and existence, and their organizations. It engages with conceivables, not reals: with conceivable relationships concerning thought-objects (*objectiva in animis*).⁴

(c) All of it, however, is in principle applicable, through interpretations either internal to mathematics itself (e.g., algebra or calculus to geometry) or external to it (e.g., algebra or calculus to physics).

To illustrate (b) plus (c): the calculus is applicable to geometry and to physics, but these facts neither presuppose nor imply any actual existence of numbers, whether the so-called reals or the so-called hyperreals,⁵ nor of operations on or relations between them. Of course, I am here not pretending to defend my antirealist and anticonstativist view, but only enunciating it so that the reader can better understand what philosophically motivates my criticisms of the ancients in what follows.

(d) The regions of mathematical discourse are characteristically founded, in logical procedure, on postulates, i.e., axioms or axiomatic definitions; these are devised to be economically capable of coping both with precise resolution of intriguing particular problems and with the need for a synthesis of associated results in a region.

(e) The mathematician's avowal and enunciation of an array of postulates constitutes a postulatory speech act, which is a performative utterance and act. This affords a quite different reason for the absence of truth-value from postulatory assertions than that which is given by a formalist. I concur with the latter in addition with respect to *what* is postulated for some regions of mathematics, e.g., the theory of abstract groups.

(f) A specification of the illocutionary force⁶ of a postulatory speech act makes emphatic use of the notions of *demand* and *commitment*: to postulate *P* is to

demand *P* and to commit oneself to the logical deployment of *P* in the relevant region. This demand and commitment by a mathematician are motivated by purposes of research and proof that interest him or her as a communicating member of the mathematical community.

2. PYTHAGOREANS

2.1. Aristotle's assertion⁷ that the Pythagoreans were the first to advance mathematics is controversial and is much controverted nowadays. Although he did not say it, what could be said more securely about them, as an adjustment of his remark, is that they were the first to advance a philosophy of mathematics. Hence, I begin with them. Some of the other early Greek philosophers—e.g., Anaximander, Parmenides, Zeno, Anaxagoras, and the Atomists—propounded doctrines or arguments that might be interpreted as contributions to a philosophical way of thought about mathematics, such as infinity, plurality, and magnitude; but it is not apparent that any of the early thinkers besides the Pythagoreans took a systematic philosophical interest in mathematical concepts, proffering an ontological and epistemological theory about mathematics as a whole.

2.2. The scrappy and, predominantly, secondary (or still further removed) evidence concerning the doctrines of Pythagoras and of Pythagoreans has generated numerous and conflicting assessments and explications swirling in a dark sky filled largely with clouds of unknowing. The scholarly storms and stresses have been especially heavy since the mid-nineteenth century, concurrently with the expanding spirit of philological criticism and of philological self-assurance. I will take it for granted (i) that Aristotle's testimony⁸ regarding the Pythagoreans yields reliable information when used cautiously and with discretion; (ii) that the contents of certain doxographical fragments ascribed to Philolaus or to Archytas⁹ are an authentic reflection of Pythagorean tenets from ca. 430 B.C., until Plato's time (when I make use of Philolaus fragment 11, commonly deemed to be dubious or spurious,¹⁰ I mention it by name; the arguments against its authenticity have not persuaded me of its unreliability); and (iii) that, on the basis of various other sources,¹¹ Pythagorean thought and teachings in that period included both some elementary algorithmic number theory, e.g., how to determine the greatest common divisor of two integers, and the recognition of some incommensurable magnitudes, such as the length of the diagonal of a square whose sides have "unit" length.

2.3. Accordingly, the following are among the chief philosophico-mathematical presuppositions or affirmations of the Pythagoreans.

(a) Numbers are the fundamental natures, causes,¹² and invariants of all bodies and modes. Unlike any one or more of the sorts of matter entertained by other Pre-Socratics as being the sources of things, the Pythagoreans reckoned that numbers, embodied in patterns, afford a precise, everywhere applicable, and adequate account of individuals of any sort; they implicitly denied that, e.g., water or fire could fulfill this requirement.

(b) Whatever is real is countable. As Locke expressed it later for his own part,

"Number applies it self to Men, Angels, Actions, Thoughts, every thing that either doth exist, or can be imagined."¹³ For the Pythagoreans, not only was whatever is real countable, it is also gradable. In regard to anything real, a definite numerical degree is involved, even if such definiteness is (somehow) conjoined with indefiniteness. As Plato expressed it later for his own part:

Our forefathers . . . passed on this tradition, that those things which are from time to time said to be are made up of one and many, with a determinant and indeterminacy inherent in them. Since this is how things are constituted we should always posit a single form in respect to every one and search for it—we shall find one there—and if we are successful, then after the one we should look for two, if there are two, or otherwise for three or whatever number is; each of these ones should be treated in the same way, until one can see of the original one not only that it is one, a plurality, and an indefinite number, but also its precise quantity.¹⁴

Whether (a) and (b) were clearly distinguished is uncertain; they may have been confounded with one another or, dimly, supposed to be mutually implicative.

(c) The physical world throughout displays orderliness and harmony. (Indeed, astronomical bodies literally produce a musical harmony: "the whole heaven [is] a musical scale and a number," and "the movement of the stars produces harmony, the sounds which they make being in accord."¹⁵) These properties are the outcome of numerical factors and regularities.

(d) Numbers have no separate existence of their own; they are all numbers of real things. As Mill expressed it later for his own part, "All numbers must be numbers of something; there are no such things as numbers in the abstract."¹⁶

(e) The integers greater than one are composed of equal and comparable units. Such a (concept of) number is called "mathematical" by Aristotle; in contradistinction to, in particular, the Platonic view according to which each member of the number sequence is a unique Idea composed of units specific to itself, different numbers being uncombinable as are likewise their respective units, "in mathematical number no one unit is in any way different from another . . . mathematical number is counted thus—after 1, 2 (which consists of another 1 besides the former 1), and 3 (which consists of another 1 besides these two), and the other numbers similarly."¹⁷ (f) Although the Pythagoreans posited and deployed a theory of "mathematical" numbers in the sense of "mathematical" elucidated in (e), numbers for them did not consist of abstract units but of *monads*—units having spatial magnitude. The Pythagoreans maintained on this basis a *geometrical* arithmetic; by this means they sought to explain the configuration of all things.¹⁸

(g) Numbers are not a mere plurality, nor are they as a plurality the ultimate principles, for these must be (i) at most very few and (ii) opposites. What underlay their belief centered on (i) is, presumably, the assumption of a certain simplicity at the root of reality. Their belief, in common with many other Greek thinkers, centered on (ii) turned, perhaps, on the assumption, first, that all sorts of things, including numbers, are penetrated by a contrasting diversity in the form of a polar duality and,

second, that, on the model of human reproduction, only such a polar duality has generative power. Aristotle reports that the Pythagoreans "consider that number is the principle both as matter for things and as forming both their modifications and their permanent states, and hold that the elements of number are the even and the odd, and that of these the latter is limited, and the former unlimited; and that the One proceeds from both of these (for it is both even and odd), and number from the One."²⁰ Two polar dualities are adduced here: the limited and the unlimited and the odd and the even. It will be suitable to approach these and their relationship via a consideration of the doctrine that number proceeds from the one, i.e., 1. For the Pythagoreans, the number sequence runs (1), 2, 3, 4, . . . ; 1 is not fully on a par with 2, 3, 4, etc., but is rather their immediate source. These divide into two types, the odd numbers and the even numbers. Members of each type were represented by geometrical patterns, possibly along the following lines for odd and even, respectively:²¹



The odd numbers 3, 5, 7, . . . and the even numbers 2, 4, 6, . . . were regarded as stemming from 1 alone: every number is a compound solely of 1s. The notion that 1 is both even and odd might, in a rational reconstruction, have been arrived at and warranted by an application of the rule: If c is the sole elementary constituent of ϕ and of ψ , then c must be ϕ and ψ . This is a sibling of a more familiar causality rule: a property of an effect must be a property of the cause.

In the preceding dot diagrams, the geometrical shape representing the odd numbers as it is expanded remains a square, but the geometrical shape of the diagram representing the even numbers as it is expanded undergoes infinite variation as an oblong ($3:2 \neq 4:3 \neq 5:4 \neq \dots$). It seems that in some such way the Pythagoreans perceived the odd and the even to be kinds, respectively, of the limited and the unlimited, this pair of opposites comprising their ultimate and most pervasive principle: "Nature in the cosmos was constructed out of unlimited and limiting factors, as was the whole cosmos and all things in it."²²

(h) The whole of mathematics is, or is reducible to, arithmetic, which concerns primarily the positive integers but also secondarily their ratios. This arithmetic, however, has geometrical aspects.

(i) Certain operations on and relations between arithmetical quantities, e.g., the determination of the greatest common divisor of two integers, can be computed by rule: there are definite constructive numerical procedures (algorithms).

(j) Some magnitudes are irrational, i.e., incapable of reduction to an integer or a ratio of integers; e.g., the length of the diagonal of a square with a side of "unit" length is irrational, and the magnitude of the diagonal's length and that of the side's length are incommensurable.

(k) The Pythagoreans developed an arithmological epistemology conformable with their numerical ontology. (Wrongly, many scholars have bypassed or barely glanced at this conception.) From the outset, starting with Pythagoras himself, they laid special emphasis on mathematico-theoretical learning as a vital method for becoming "wise" and free. Thus, Pythagoras was early on described as a "polymath" and as one engaged in scientific inquiry. The Pythagoreans of the fifth century supposed that the natures and causes of things, as mathematical, are within human grasp: they can be exactly thought, understood, and expressed by human beings, though not by all human beings in the ordinary course of life. Exact wisdom, and the spiritual detachment that accompanies it, are attainable only by those who are specially gifted and purified, practicing the requisite detachment: for the Pythagoreans "life is like a festival; just as some come to the festival to compete, some to ply their trade, but the *best* people come as spectators, so in life the slavish men go hunting for fame or gain, the philosophers for the truth."²³ (There is evidently a resonance of this in Plato's *Republic*.) *Proof* in the discipline of mathematics also had a general significance for them.²⁴

The epistemological side of the Pythagorean philosophy of mathematics is clearly attested by fragments ascribed to Philolaus;²⁵ it was a major key for them. "Actually, everything that can be known has a number; for it is impossible to grasp anything with the mind or to recognise it without this [number]"; "it would be impossible for any existing thing to be even recognised by us if there did not exist the basic Being of the things from which the universe was composed, [namely] both the Limiting and the Non-Limited"; and more fully in fr. 11:

[Without] the power of the decad . . . all things are unlimited, obscure, and indiscernible. For the nature of number is the cause of recognition, able to give guidance and teaching to every man in what is puzzling and unknown. For none of existing things would be clear to anyone, unless there existed number and its essence. But in fact number, fitting all things into the soul through sense-perception, makes them recognisable and comparable with one another . . .²⁶

(l) The Pythagoreans matched truth with, and wholly dissociated falsehood from, number. "Falsehood can in no way breathe on number; for falsehood is inimical and hostile to its nature, whereas truth is related to and in close natural union with the race of number" (Philolaus, fr. 11).

2.4. Why did the Pythagoreans produce a philosophy that gives a predominant role to the mathematical? Aristotle, in a sketch of his metaphysical predecessors, asserts that the Pythagoreans' study of mathematics itself was the genetic ground of their philosophical principles. Citing no *reasons* on their behalf, he says that "the so-called Pythagoreans, who were the first to take up mathematics, not only advanced this study, but also *having been brought up in it* they thought its principles were the principles of all things."²⁷ This version of events is singular in suggesting that the Pythagoreans began their school as mathematicians only and that it was afterward, and solely because mathematics was their frame of intellectual interest, that they turned to philosophizing, in which (he proceeds to elucidate) they fundamentally

and universally mathematicized—more precisely, arithmetized—the nature of things. A modern historian, keeping to Aristotle's track but with a modern readiness to use a more flamboyant vocabulary, might put it this way: the Pythagoreans, who at first engaged in the study of mathematics, were carried away by their successes in it and enthusiasm for it into supposing it to be *the* universal science.

Aristotle's *causal* answer to the "why" question is too simple and one-sided. Like smoke, it obscures from view and even chokes off an inquiry into the intended considerations likely to have been compelling in the minds of the Pythagoreans. Evidence of the grounds conceived—maybe no better than partially or blurrily, and far from articulatedly—by the early philosophical thinkers as warranting their respective doctrines is, notoriously, mutilated and scarce. This general shortfall holds also for the Pythagoreans. Nevertheless, some definite and coherent indications survive of what served as rational support for the Pythagoreans' mathematicism.

Three branches of this rational support may be singled out of the historically more ramified complex. First, discoveries, some of them then recent, had been made of the remarkable and exciting applicability of mathematics to natural or manufactured systems/collections/items/modes, e.g., in astronomical predictions, in the correlation of numerical ratios with the principal intervals of the musical scale (a discovery attributed to Pythagoras himself),²⁸ in the determination of distances by triangulation, and in the exact construction of the Egyptian pyramids. Second, at a more mundane level, there was the widespread recognition, established in the thought, language, and practice of humankind, of quantitative aspects of common things and the knowledge of how to count them or how to calculate some dimension of them. Aristotle, in a part of the *Metaphysics* other than the one cited at the beginning of this section, comes close to acknowledging this broader background to the formation of Pythagoreanism: "Again, the Pythagoreans, *because* they saw many attributes of numbers belonging to sensible bodies, supposed real things to be numbers—not separable numbers, however, but numbers of which real things consist. But why? Because the attributes of numbers are present in a musical scale and in the heavens and in many other things."²⁹ Third, their cherishing mathematics was rationally self-supporting, for it was the supreme rational science through its proofs.³⁰

2.5. The Pythagoreans scored a crucially original success in bringing mathematics into distinct and fundamental intellectual prominence. Other early Greek philosophers concentrated on qualitative conceptions, focusing mostly on certain natural sorts (e.g., water and air) or on abstracts from any natural types (e.g., Anaximander's Infinite and Parmenides' Being) and on quasi-passionate or quasi-intellectual cosmic agencies (e.g., Love or Reason). They did not pursue a positive and systematic reckoning with the numerical or the figured.

Further, mathematical methods had been largely, in Greek states no less than in Mesopotamia and Egypt, motivated by practical needs—agricultural, astronomico-religious, commercial, constructional, geographical, and medical. These methods had lacked any overt degree of theoretical organization or allure; still less had they received any philosophical conceptualization. The Pythagoreans initiated the latter and made considerable contributions to the new development of mathematics as an

abstract and liberal, as well as applicable, science. Perhaps, indeed, they were the founders of mathematics as an abstract, liberal science, which is what Proclus attributed to them in the person of Pythagoras: "Pythagoras transformed this study into the form of a liberal education, examining its principles from the beginning and tracking down the theorems immaterially (*aulōs*) and intellectually."³¹ Perhaps it was as such founders of "pure mathematics" that Aristotle had them in mind when he remarked, in the historical introduction to the *Metaphysics*, that the Pythagoreans "were the first to take up mathematics"; this possibility is strengthened by the fact that Aristotle's own conception of mathematics was as an abstract, theoretical science.

The intellectual prominence of mathematics that the Pythagoreans originated and promoted survived throughout subsequent antiquity; and even when not under their banner, yet it was under their radiating influence—via Plato and Aristotle³²—that it increasingly regained that status from the thirteenth century (when it was heralded anew by Roger Bacon)³³ until its full restoration in the seventeenth century and after.

2.6. A solid basis for the Pythagoreans' historic success lay in their idea of the pervasively mathematical character of reality; through this idea they rough-hewed the foundation stone of the later towering achievements of mathematical physics/astronomy. Correspondingly, they pioneered the powerful methodological policy according to which one understands a species or a rhythm of fact if, and only if, one grasps it in appropriate quantitative terms. The seed of their mathematicism developed in its maturity into a threefold faith "justified" by many works. The first strand of the faith is that mathematics's vitality depends on its renewing itself by seeking an ever more comprehensive description of nature in mathematical terms, thereby drawing on the inherently mathematical sources of nature and that in the absence of such physical contacts mathematics degenerates into idle symbolism. The second is that the adequate and clear expression of physical conceptions requires the use of appropriate mathematical language, with its precision and abstractness and with its ratiocinative—and so precisely predictive—power. The third strand is that physical conceptions, when formulated in an appropriate mathematical way, exhibit a beautiful economy matching nature's own.

2.7. Unlike some of the Pythagoreans' other teaching, e.g., that on the transmigration of souls, their philosophemes regarding numbers were no frenzied folly and no crude relic of animistic or contagious-magical modes of thought. On the contrary: they made a decided intellectual advance and became the ancestors of many gifted intellectual offspring. Those philosophemes were not wholly sweetness and light, however. Besides having various other deficiencies and defects, the mathematical philosophy of the Pythagoreans has a disturbing vagary at its metaphysical core, just where it might seem, to an antirealist concerning abstract objects, to be solid and stable. This vagary occurs at the place—the central fire³⁴ of the system—where the Pythagoreans posited the inherence of numbers in things; the vapors of this realism concerning mathematical objects were inhaled and assimilated by the mathematical philosophies of Plato and Aristotle.

2.8. Most of the Pythagoreans' doctrines are lacking in clear and distinct ideas and in clear and distinct logical concatenation. Consider, for example, their proposition that numbers are the fundamental natures and causes of the bodies and modes in the cosmos. The Pythagorean adoption of mathematical-type notions and standards by reason of their exactitude and plenitude is markedly at odds with their proposition's vagueness and inadequacy. *How* numbers are responsible for the *properties* of bodies (e.g., their mass, texture, color) and for such modes as justice, reason, and opportunity (examples cited by Aristotle) is left obscure (see p. 14 below, however). This obscurity also surrounds the way in which numbers were thought to be responsible for *change*, including motion and animate development. The Pythagorean analysis of bodies into geometrico-arithmetical figures does not by itself have the capacity to explain change and nonextensional properties, and the invocation of geometrico-arithmetical *transformations* of one figure into another, as a kind of quantitative stereochemistry, brings with it the objection that it requires *principles* of transformation and that these principles must lie beyond the static—because purely mathematical—concepts of their ontology.

Aristotle goes further in criticizing the Pythagorean assertion that bodies are composed of numbers. In his characteristically concise and adversative style he says:

[T]hat bodies should be composed of numbers, and that this should be mathematical number, is impossible. For it is not true to speak of indivisible spatial magnitudes; and however much there might be magnitudes of this sort, units at least have not magnitude; and how can a magnitude be composed of indivisibles? But mathematical number, at least, consists of abstract units, while these thinkers identify number with real things; at any rate they apply their propositions to bodies as if they consisted of those numbers.³⁵

Aristotle here is pinpointing two objections: first, that bodies essentially possess spatial magnitude, which cannot be accounted for on the basis of numerical units, for these have no magnitude; second, that there is a categorial confusion in the Pythagoreans' identifying mathematical numbers, which are essentially abstract, with physical realities as such.

As already indicated, the Pythagoreans did not suppose number to be the absolutely fundamental ontological source of the quantitative; it stemmed from two "principles" or "elements,"³⁶ each consisting of a pair of contraries, namely, the limit and the unlimited, and the odd and the even. It is the principle constituted by the limit and the unlimit that is the absolutely fundamental ontological source of the quantitative, and of number specifically. Now, the sequence (1), 2, 3, 4, . . . ,³⁷ which was recognized by the Pythagoreans, is an infinite sequence; it is a case, perhaps the paradigm case, of the effect of the unlimited. Concurrently, each item in that infinite sequence is finite; this is a case, perhaps the paradigm case, of the effect of the limit. The principle of the limit and the unlimited is, however, an unsatisfactory way of accounting for the infinite sequence of positive integers, because it cannot by itself determine the actual progression, which is in terms of a unit increase at each step after the initial one. In other words, the law governing that sequence is

not covered, or at any rate is not precisely covered, by the principle of the limit and the unlimited.³⁸ An analogous defect occurs in the Pythagoreans' notion of the odd and the even as the "elements" of number, i.e., of 2, 3, 4, etc.; because although, obviously, the items in this sequence are odd or even, exclusively and exhaustively, the mentioned notion cannot by itself account for the *alternation* of odd and even numbers or for the successive unit increases in the actual progression. What is missing is the logical equivalent of: $2 = 1 + 1$, $3 = 2 + 1$; $4 = 3 + 1$; etc.

2.9. I will say a little about three further coordinates of the Pythagorean position before I recur to what I called the "disturbing vagary at its metaphysical core."

First, with respect to the thesis that "the movement of the stars produces harmony, the sounds which they make being in accord" cited in 2.3(c): this, taken prosaically, has long since been found empirically wanting. Critics from antiquity onward have noted the absence of relevant evidence supporting the claim, e.g., the fact (conceded by the Pythagoreans) that one does not hear any musical sounds attributable to celestial causes (this unawareness they ingeniously explained as being due to the continuous presence of the sound, with no contrasting silence); and they have noted observational facts incompatible with the Pythagorean astronomy or musical theory.³⁹ Therefore, in the prosaic sense the claim is a fantasy (not to say, *Fantasia*). Doubtless it is more precious when interpreted figuratively, but this lies beyond my present province.

Second, with respect to the tenet that arithmetic, in geometrical form, is the reduction base of mathematics. This tenet is implied by the overt Pythagorean ideas about number. If their tenet were correct, all affirmations in mathematics should be reducible to affirmations in their arithmetic. Stretching the sense of "their arithmetic" as generously and sympathetically as possible, it still turns out that almost all mathematics is irreducible to theirs; this applies not only to later developments, e.g., in algebra and analysis, but also to already existing mathematical results. The concepts of elementary plane geometry, as established in the fourth and fifth centuries B.C., are irreducible to ones in Pythagorean arithmetic; 'angle' and 'area' are examples. According to tradition, the Pythagoreans themselves were discomforted by the realization that their kind of arithmetic could not cope with irrational magnitudes, whose "existence" had recently been discovered—ironically, by one of their own members:⁴⁰ thus, the square root of 2 cannot be expressed in terms of a regular and "rational" pattern of positive whole numbers in an exact and terminating way, so that here the limit and harmony are overruled by the unlimited and disorder.

Third, with respect to the Pythagorean (or, at least, Philolausian) theory of knowledge: the theory's insistence on number being the cause of all grasp, perception, recognition, distinction, and comparison is understandable, given their numerical ontology whereby the characters, structure, and stability of things are a function of embodied number; but the dubiousness of this ontology proportionately threatens the tenability of their epistemology. Besides, the very plurality of cognitive terms used by them prompts an additional critical reflection: they failed to provide a philosophy of mind, conformable with their numerical ontology, that signals how the

cognitive states and processes between which they discriminated are possible and have their distinctive natures.

2.10. The Pythagoreans' naturalistic conception⁴¹ of mathematical entities and truths is unsatisfactory for the following reasons, among others. (i) Their conception would make mathematics a natural science, such as physics; whereas mathematics, including arithmetic, is not an empirical science, either in its subject matter or in its method of determining its assertibles. Patently, mathematical entities and mathematical "facts," in general (I leave aside the Pythagorean doctrine of monads in order to avoid begging the question), e.g., in algebra, analysis, and topology, are not given to us through sense perception; and the mathematician's demonstrations stand or fall not by reference to the outcome of empirical verification/falsification, but by reference to his postulates and the rules of a deductive logic. The Pythagoreans' treatment of mathematics as physics is the naive origin of the persistent view of mathematics as a body of "truths." (ii) Their conception counterproductively debars the imaginative construction and pursuit of abstract types of possibility in mathematics; yet, it is this imagination that seeds the harvest of mathematics. (iii) The Pythagoreans' view would require that the mathematician's use of language be descriptive/constative; on the contrary, his use, in general, is nondescriptive/nonconstative, for typically his language is symbolical, geared to an abstract type of possibility rather than to depicting a definite actuality. (iv) Their conception does not make clear, if indeed it allows at all, a distinction, which it is important to be able to make firmly, between, e.g., *the* number 10 and this or that particular cluster of ten things. By the Pythagoreans' ontology, "the number 10" can hardly be other than a(ny) decadal case. (For them, "the number 10 is thought to be perfect and to comprise the whole nature of numbers":⁴² $10 = 1 + 2 + 3 + 4$.) (v) Although the Pythagoreans recognized the significance of proof in mathematics, they contributed little, if anything, to the theory of proof; and the "a priori" status of mathematical assertibles was not within their grasp. It was the Academy and the Lyceum that first philosophically deliberated on and articulated a methodology of mathematics.

3. PLATO

3.1. "Let us not fail to notice, however, that there is a difference between arguments from and those to the first principles. For Plato, too, was right in raising this question and asking, as he used to do, 'are we on the way from or to the first principles?'"⁴³ Plato's mathematical philosophy is an *ascent* to the first principles: to the first principles of mathematics, and, cognately, of everything. Plato's clambering climb to the first principles in his mathematical philosophy tallies, in its direction, with Russell's characterization of mathematical philosophy.

The more familiar direction [in the study of mathematics] is constructive, towards gradually increasing complexity: from integers to fractions, real numbers, complex numbers; from addition and multiplication to differentiation and integration, and on to higher mathematics. The other direction, which is less familiar, proceeds, by analysing, to greater and greater abstractness and

logical simplicity; instead of asking what can be defined and deduced from what is assumed to begin with, we ask instead what more general ideas and principles can be found, in terms of which what was our starting-point can be defined or deduced. It is the fact of pursuing this opposite direction that characterises mathematical philosophy as opposed to ordinary mathematics.⁴⁴

There is, nonetheless, a difference between Russell and Plato in this connection, which the common term 'principles' obfuscates. Stated simply it is this: Plato's mathematical philosophy is wider than Russell's constitutive characterization, inasmuch as it, unlike Russell's, includes a broad epistemological concern.

3.2. Before I turn to an outline of some of the leading topics of Plato's mathematical philosophy, I should draw attention to two of its traits. Each is a tension, or a self-conflict. And each is already apparent in Pythagoreanism. On the one hand, the Pythagoreans wanted to pinpoint the arithmetical constitution of bodies and modes, to show how, in a definite, physical way, a body or mode is actually "nothing but" a figured arrangement. This makes their mathematicism as concrete as possible: a body or mode is a complex of monads. On the other hand, the Pythagoreans were concerned about emphasizing the mathematical structures and regularities of things, rather than about claiming the identity (without any remainder) of the things with those mathematical aspects. Thus, they did not (I think) hold that musical concords are nothing but such and such numerical ratios, still less that the concords are series of monads; their conviction was that underlying the concords, acknowledged as sounds, lay definite mathematical structures and regularities. Take another example: justice. It is probable that the Pythagoreans numbered justice as four; they did so, perhaps, because justice was regarded as involving two parties and the proportionate distribution of two parts between them.⁴⁵ But this is not an identification of justice with four, either monadically or by way of maintaining that four and justice are identical; their idea was that justice has a four-term structure. Similarly, there is a tension, or a self-conflict, in Plato's mathematical philosophy, especially (but not only) in his Unwritten Doctrine that the Ideas are numbers.⁴⁶ This doctrine runs in two directions, one of these being toward the sheer identification of the Ideas with—as nothing but—numbers, while the other direction is toward a number-type basis of the structures and regularities of the Ideas.

The other tension, or self-conflict, common to Pythagorean and Platonic mathematical philosophy concerns the scope of the One. Trends toward dualism and infinitistic pluralism are quite marked in Pythagoreanism, which tended also to maximize the role of the One: "The One is the beginning of everything," "Harmony is a *Unity* of many mixed (elements), and an agreement between disagreeing (elements)."⁴⁷ Similarly, Plato was continually driven from pillar to post as between one and many. Each property and kind of the changing, sensible things amidst which we live and move, have, actually or potentially, many instances among these things (ourselves included); he found it necessary to posit an Idea, as the appropriate unity, accounting for each such instantial multiplicity of property or kind.⁴⁸ Nevertheless, Plato's quest for unity of this Ideal sort is, in his Dialogues, constrained and

qualified by his equal acceptance of the plurality of the Ideal sorts: there are really irreducible differences; and, consequently, the Ideal, intelligible realm cannot be assimilated without comprehending the relationships of its different members. This issue is tackled in, e.g., *Republic* (in connection with the Idea of the Good), *Phaedrus*, and *Sophist*. His reality becomes layered. The study of what the scholastics called "transcendental" terms, viz., 'being', 'thing', 'something', 'one', 'true', and 'good',⁴⁹ was thus initiated. In seeking to determine the colligations and separations, the classifications and divisions, of Ideas, Plato found himself in difficulties with the status of One. He was disposed, and he was opposed, to a Parmenidean position, in the sense of the adoption of the absolute ontological primacy of the One. He may well have sensed the need for an ultimate, all-embracing unity of reality. Also, oneness is the most basic transcendental, if only because the transcendentals and the Ideas are each of them one,⁵⁰ as he supposed; whereas nothing else had for him that basic universal applicability. This disposition led him, in his Unwritten Doctrines, to allot a unique status to the One as the principle/origin of being. At the same time, he was steadily opposed to a Parmenidean position. Despite the attractiveness of a single principle/origin of reality, his reality is not uniform but inherently differentiated and layered: he is a thoroughgoing Ideal pluralist, who argued in the *Parmenides* that the One is not completely self-subsistent/coherent,⁵¹ and he is a pluralist concerning numbers. Plato's ambivalence concerning the One ran deep (and it ran long, through Neoplatonism and beyond), and the crosscurrent was not (I think) caused at all by his failure to discriminate the different connotations of the term 'one'. (The term 'the one' (*to hen*) "can mean (1) Unity or Oneness in general; (2) the unity of anything that has unity or is one thing; (3) that which has unity, anything that is one; (4) the one thing we are speaking of, as opposed to 'other ones,' and so on."⁵²)

3.3. At first, and even at second, sight, Plato's Dialogues give expression to a somewhat changing mathematical philosophy, which, however, in none of its phases, appears to be the same as the mathematical philosophy of his Unwritten Doctrines. It therefore has to be considered in a doubly divided way, one division attending to differences within the Dialogues, the other to differences between the Dialogues and the Unwritten Doctrines evidence. Despite the various differences, what is usually taken for granted as most characteristic of Platonism is indeed prevalent throughout all that is known of Plato's thought in the philosophy of mathematics: his realist and constativist assumptions about mathematics and mathematical.

Some commentators have held that Plato adhered to the Unwritten Doctrines, with its arithmetization of the Ideas, as a programmatic project and/or an esoteric theory concurrently with his production of at least the middle, as well as the later, Dialogues.⁵³ This is contrary to the usual view (with which I am inclined to agree) that the Unwritten Doctrines belong to Plato's older age only, say from his sixties. Interesting as the question of choice between these two hypotheses is, I will here proceed in neutral detachment from both of them, in the absence of decisive enough reasons for choosing either. That I begin next with the Dialogues on their own, before discussing the Unwritten Doctrines, is to be taken as a procedural order, with procedural precedence to the philosopher's own writings.

3.4. Although the Dialogues I will consider expressly are not the only ones, they include most of the most important, for studying Plato's philosophy of mathematics; they date from the earlier part of the middle, to the final, period of his production as an author.⁵⁴ In accordance with this chronology, I begin with *Meno*, whose bearing on the philosophy of mathematics is exceptional among the Dialogues in being almost exclusively epistemological and methodological.

(a) *Recollection*. A slave-boy—a prototype of ignorance—comes to recognize, through questions but (supposedly) without any information being conveyed to him by (the character) Socrates, what the correct answer is to the problem: Given a square each of whose sides is two feet long, what is the length of each side of a square having twice the area of the given square? (Answer: the length of the diagonal of the given square.) The process and success of the boy's discovery is proffered as illustration and confirmation of the prior suggestion that Socrates had heard from priests and priestesses and from many divinely inspired poets that the human soul is immortal and undergoes reincarnations and so it "has seen all things both here and in the other world, [and] has learned everything that is."⁵⁵ Sliding down the rainbow of Plato's rhetoric to a plainer level of discourse, one may say that Plato was projecting the notion that the attainment of conscious mathematical—more generally, a priori—knowledge is explicable only on the basis of direct learning previously. The person must have latently possessed the relevant knowledge; his excogitation in response to suitable questioning resuscitates that latent knowledge into consciousness. His latent possession originated in an acquaintance, in a preceding state of life, with whatever is involved in that knowledge.

Thus Plato, at a stroke, brought some major epistemological problems to the fore about the nature and genesis of a priori and, specifically, of mathematical knowledge. Unfortunately, the answers hovering on the surface of *Meno* are feeble fledglings that cannot get off the ground. What Plato tells us is itself, on the one hand, very problematical about soul and immortality and reincarnation, etc., and, on the other hand, very vague about "seen" and "all things" and "the other world" when he says of the soul that it "has seen all things both here and in the other world." He has failed to indicate the nature of the things relevant to the slave-boy's regained geometrical knowledge, e.g., whether they are objects or facts, and of what sorts. Nor has Plato catered for the evident difficulty that the slave-boy, at the terminus of the inquiry, has apparently no remembrance of the occasion or acquaintance originating his knowledge.⁵⁶

Lurking inchoately at the back of Plato's mind was the presumption, powerful in the rationalist breed of philosophies, that what is knowable and what is necessarily⁵⁷ or essentially the case coincide. Something like this presumption motivated and afforded warrant for his epistemological optimism, although it came to be modified later when Plato conjoined with it a sharply hierarchical view of humankind, whereby only a select number of people have the firm⁵⁸ native ability to gain an intelligence of truth and reality.

(b) *Characteristics of mathematical knowledge*. Plato presupposed that mathematics is a matter of strict knowledge. All strict knowledge, for him, involves what

exists eternally and independently of us; in later Dialogues, he held that whatever is known is abstract (*noētos*, "intellectual," i.e., "falling within the province of" *nous*, was how he later expressed this status). In *Meno* he emphasized what is characteristic of strict knowledge in contrasting this with "true" or "right" opinions. "[I]t is not, I am sure, a mere guess to say that right opinion and knowledge are different. There are few things that I should claim to know, but that at least is among them, whatever else is."⁵⁹ What is the difference? True opinions

run away from a man's mind, so they are not worth much until you *tether* them by working out the reason [*aitias logismō*]. This process, my dear Meno, is recollection. . . . Once they are tied down, they become knowledge, and are stable. . . . What distinguishes one from the other is the *tether* [*desmos*].⁶⁰

According to a stock modern analysis or defining set of conditions, "A knows that *p*" means (i) *A* believes that *p*, (ii) *A* can "justify" his belief, and (iii) *p* is true. This modern definition, making 'knowledge' equal 'justified true belief,' is similar to Plato's conception, whereby knowledge is true belief (or opinion) topped up by a justification, "the tether." One might extrapolate from his remarks and infer that, thinking of mathematical knowledge, he took the tether to be like a mathematical demonstration and that what he called "working out the reason," i.e., the excogitation of a rationale, was at least analogous to the mathematician's requirement and practice of a rationally demonstrative procedure in support of affirmed theorems. I would add that Plato, in assuming the truth condition for mathematical as for other knowledge, did not begin to discriminate between mathematical and any other sort of truth; for him, mathematical truth and its reality are on a par with all other truth and reality. Like the rest of the Greeks, Plato had no notion of mathematics as the disciplined exploration of *imaginative* possibilities or schemata. Mathematics was for him essentially "realistic," much as his conception of art was mimetic.

(c) *Ideas*? An insistent doctrine of the middle Dialogues in general is that the things known are ontologically altogether different from and superior to the things believed or opined: the objects of knowledge are Ideal realities, whereas the objects of belief or opinion are changeable vanities. In *Meno*, on the contrary, the different veridical cognitive states are not correlated with different objects; those states are differentiated only by virtue of the presence or absence of "the tether." This fits in with the fact that the Dialogue never mentions the term 'Idea' and says nothing implying the theory of Ideas. This is not to deny that, being wise after the event, one can discern a few green shoots from which the nascent theory will bud forth. Those⁶¹ who have found the theory of Ideas in *Meno* have read the theory into it by working backward from *Phaedo* and *Phaedrus*, where the doctrine of recollection is indeed linked explicitly to an ontology of transcendent Ideas. Unlike these and other Dialogues written after *Meno*, in this Dialogue the objects of cognition are, it seems, unconfinedly heterogeneous, within "this world" and "the realm of Hades"; not until later did Plato ascribe a purely celestial⁶² location for the knowable—the Ideas. In *Meno*, then, Plato left undetermined and unrestricted the ontological status of cognitive objects, including those of mathematics.

(d) *System*. He contributed to the furtherance of the notion, which he perhaps borrowed from and surely thought applicable to mathematics, that knowledge is systematically interconnected. "[W]hen a man has recalled a single piece of knowledge . . . there is no reason why he should not find out all the rest . . ." ⁶³ This obviously has specially forceful relevance to mathematical knowledge, whose ingredients theses in this or that setup are organized maximally under logical conditions and constraints. (I use the term 'theses' to designate axioms and theorems of a deductive system. ⁶⁴) Plato's was an incipient epistemological step toward the development of a methodology of axiomatic mathematics.

(e) *Hypotheses*. Many gallons of scholarly ink have been used up in controversy over the precise meaning of the passage in *Meno* that cites an example of geometers' use of "hypothesis" and proceeds to investigate the ethical problem in hand (viz., Is virtue teachable?) by an appropriate hypothesis (viz., Virtue is knowledge), used presumably in a way parallel to the geometrical one. Plato subjected geometers' reasoning and their use of hypotheses to a very serious critique in *Republic* (see pp. 21-22 below); to understand and assess this critique requires a grasp of what he meant there by the term 'hypothesis'. If its sense in *Meno* were about the same as it is in *Republic* or if geometers' use of hypothesis were the topic of separate philosophical comment in *Meno*, then I should need to dwell on the mentioned passage. But they are not and so I do not. Two clarifications are nonetheless called for. First, the geometrical problem in the *Meno* passage is (disregarding alternative interpretations), "whether it is possible for [a given] area to be inscribed as a triangle in a given circle"; the hypothesis, stated very obscurely, is, however, plainly ad hoc; it is not a postulate fixedly adopted and available for a whole body of geometry. Second, the hypothesis has the form: if such and such conditions are satisfied, then the result is so and so; while, if they are not, "the result is different." ⁶⁵ Plato's standpoint in the *Republic*, where, in his high dialectical mood, he contended that mathematicians cannot attain consummate knowledge of ultimate reality, would have made him judge that geometrical hypothesis wanting, with its pervasive conditionality: he was there aiming at warrantably unconditional knowledge.

3.5. *Phaedo* is the middle Dialogue in which Plato gave the theory of Ideas its first trenchant and rounded expression. The theory comprised both an ontology of *what is*, as eternal universals, paradigms, and prime causes; and an epistemology of these entities, deemed to be the sole (and the soul's) "truth," accessible only to pure thought (although the latter processes may, in some matters, be initially prompted by the senses ⁶⁶).

In this setting, various arithmetical notions are explained. ⁶⁷ (i) Plato referred to the whole positive integer series. He switched back and forth, sometimes in a few lines, between number terms and Idea-of-number terms, e.g., from "threeness" / "Idea of three" to "three" and reversely; I surmise that this arose from his taking it for granted that his doctrine of Ideas as applied to numbers would be immediately read into his simpler phraseology, in a context containing the slightly more elaborate Ideas phraseology concerning numbers. He specifically mentioned the Ideas of 1, 2, 3, and 5. He probably meant to convey, then, not only that there are Ideas of all the

positive integers, but also that all these integers are Ideas. (ii) He mentioned certain positive fractions, e.g., $\frac{1}{2}$; but he did not intimate how he conceived of their ontological status, particularly in relation to Ideas of integers. (iii) He recognized overtly that, for example, in addition a plurality of the same number may be dealt with, e.g., in adding two 1s; he did not elucidate how, despite the uniqueness of each Idea, this is possible. (iv) He remarked the distinction between identity and partitive attribution; "3 is odd" is true as a partitive attribution but false as an identity. (v) The Ideas of the odd and the even are cited. He declared them to be opposites but pointed out that this relationship does not imply that cases of them are opposites, although such cases do exclude the relevantly opposite attribute; e.g., 2 and 3 are not opposites, yet 2 excludes the odd and 3 excludes the even.

Equality is a fundamental concept in and for mathematics. Plato treated it at some length in *Phaedo* as a general relation. ⁶⁸ He argued that there is an Idea of equality: for when we think, in sense-experience, of one item being equal to another, e.g., a log to a log, we have to acknowledge that the apparent equalities are variable, relative, and deficient, not holding, utterly, permanently, and under all conditions; hence, we must, implicitly, be cognizant and make use of the paradigmatic Idea of equality. Further, since this paradigmatic notion cannot have been learned in this life, our present knowledge of equality must have come by recollection from a prebirth state of the soul. Similarly, we knew before birth into this life all the other Ideas, of the larger, the smaller, the just, the beautiful, etc.—and of oneness, twoness, and so on. ⁶⁹

Of course, a modern critic is liable to be repelled by what will strike him as Plato's naivety in invoking reincarnation and recollection to explain our possession and use of certain concepts and to presuppose, contrary to Plato, that humankind has a lively capacity for the original formation and the adaptive use of such concepts in the ordinary course of mortal life. And he is liable to be disappointed by what will strike him as the verbal and vacuous elucidation of mathematical notions by the theory of Ideas: according to the theory, (i) one must not say "that ten is greater than eight by two, and that this is the reason for its exceeding" but rather "that it's by numerosness, and because of numerosness"—which cannot explain the specific differences between, say, ten and eight, and ten and seven, and (ii) one must not say "that when one is added to one, the addition is the reason for their coming to be two . . . ; you [must] own no other reason for their coming to be two, save participation in twoness." ⁷⁰ It looks as though Plato in this part of his garden has planted the flower beds with weeds.

3.6. *Republic* is the climactic Dialogue of Plato's middle period; it brought together and developed in a fresh perspective more or less all the themes of its predecessors. Its curving paths intersect many topics of mathematical philosophy. I will comment on three of the most important intersections.

(a) *Mathematics and Truth*. Plato used a concept of propositional truth and attached value to such truth. ⁷¹ But his explication of it, so far as he gave one, was in terms of the theory of—nonpropositional—Ideas. He conceived of truth principally as the totality of the "intelligible" entities that eternally are and that are the objects of our knowledge through pure thought.

We must accept as agreed this trait of the philosophical nature, that it is ever enamoured of the kind of knowledge which reveals to them something of that essence which is eternal, and is not wandering between the two poles of generation and decay. And, further, that their desire is for the whole of it. . . . [Arithmetic] really seems to be indispensable for us, since it plainly compels the soul to employ pure thought with a view to *truth* itself.

What we have to consider is whether the greater and more advanced part of [mathematics] tends to facilitate the apprehension of the idea of good. That tendency, we affirm, is to be found in all studies that force the soul to turn its vision round to the region where dwells the most blessed part of reality . . . to contemplate essence.

[T]he real object of the entire study [sc. of geometry] is pure knowledge . . . [i.e.] the knowledge of that which always is. . . . Then, my good friend, it would tend to draw the soul to *truth*, and would be productive of a philosophical attitude of mind.⁷²

Plato, then, viewed truth and knowledge as principally objectual. Mathematical truth and knowledge, correspondingly, were in the main geared by him to the Ideas. So far as he regarded those as being propositional, it was in a subsidiary way, and he did not investigate or state how the propositional is related to and, especially, founded on the objectual. This relative lack of interest in the propositional aspects stemmed from his reliance on an intellectual-visual model of reality and knowledge/understanding; thus, significantly, the highest philosophical activity—that of the dialectician—is *synoptical*.⁷³ In the event, he mistook the importance, scope, and role of the evident in mathematics; and his lofty metaphysical orientation resulted in his misinterpreting the character of specifically mathematical “truth.”

(b) *Mathematicals: Ideas or Intermediates?* Aristotle tells us⁷⁴ that besides sensible things and Ideas, Plato says that there are the mathematical (*ta mathēmatika*, “the objects of mathematics”); they are between (*metaxu*) those two, being different from Ideas in that each Idea is unique whereas each mathematical is plural and different from sensible things in that the mathematical is eternal and unchangeable while (it is implied) sensible things are neither. Some scholars⁷⁵ have espied this Platonic doctrine in *Republic*. Their interpretation is an erroneous inference from two undeniable data. In one place in this Dialogue mention is made of units (in the plural) in “the numbers themselves,” i.e., the Ideas of numbers; these units are spoken of as being absolutely equal and indivisible, in contrast with sensible unit(s) that are inherently diverse and divisible.⁷⁶ So, the interpreter’s reasoning runs, Plato must have intended these ideal but plural units to occupy an intermediate position between the ideal and unique entities that are the Ideas and the plural entities that are sensible and similarly in respect of all other types of mathematical entity. There is a flimsy basis here for attributing this doctrine to Plato’s “intention.” The argument assumes that Plato realized an important distinction that he did not state, viz., between an Idea and an intelligible, ideal non-idea, or that, if he did not realize it,

this distinction has to be drawn and applied unless he is to be convicted of incoherence/inconsistency. I discern too little evidence for the former of these alternatives. And the conclusion of the argument seems to me excessively generalized, jumping from units to all types of mathematical entity. The second undeniable datum is that Plato in *Republic* marked a difference between the mathematician’s intellection (*dianoia*) and that of the dialectician (*nous* or *noēsis*), and that he maintained that the mathematician’s is intermediate (*metaxu*) between the dialectician’s, on the one hand, and belief related to the sensible, changeable world, on the other hand.⁷⁷ In the light of Plato’s analogies between ontic strata and cognitive grades, the interpreter infers that for Plato the objects of mathematical cognition must be intermediate between the objects of dialectical cognition and the objects of belief. The fallacy committed in reaching this conclusion comes from assuming that Plato’s analogies are, first, to be taken to the foot of the letter and, second, to be given more weight than Plato’s own clear indications contrary to the conclusion, in particular that the objects of mathematical cognition are not intermediates but Ideas; e.g., numbers, the square, and the diagonal are referred to in terms of Ideas.⁷⁸

Besides and beyond his appeal to the odd bit of text, the ‘intermediates’ proponent has two connected reasons to advance. One is that the Ideas are paradigmatic properties and not ideal things possessing such properties; e.g., circularity, squareness, and twoness and not the circle, the square, and the number two. But mathematics is about these things and not about the properties. Therefore, mathematics cannot be Ideas; of course, they are not sensibles either—they are between (*metaxu*) the two strata. The other reason is that theses or proofs in mathematics often refer to a plurality of mathematical entities of a single kind; ‘ $4 \times 3 = 12$ ’ invokes a plurality of 3s (and of 1s), a theorem about congruent triangles invokes a plurality of triangles, and so on. However, although these may be good reasons for what Plato ought to have claimed in *Republic*, the text shows no evidence that he wished to claim in it the intermediate status of mathematical—on the contrary. This verdict does not debar that, especially, the second of these reasons may have influenced Plato toward embracing the intermediate doctrine in his oral teaching.

(c) *A Twofold Critique of Mathematics*. Plato’s commendation of mathematics as propaedeutic to the philosophical encounter with truth/reality has been reported in (a) above. Nevertheless, he went on, in a crucial passage of *Republic*, to fault and delimit the mathematician’s quest for knowledge on two grounds. The mathematician poses his postulates as though they were absolute, when they are as knowledge still unjustified assumptions, the justification necessitating an ascent above the confined pediment of mathematics. The second criticism is that the mathematician, inescapably, proceeds by using sensible models; thus, the geometer uses diagrams or solid figures and the arithmetician uses “numbers attached to visible and tangible bodies.”⁷⁹ For Plato, then, the mathematician’s method to science is doubly defective. What prevents its practitioner from attaining complete knowledge of truth/reality is that the methodical constitution of mathematics is not properly supported or oriented. Even though the mathematician’s mind is on the exemplary Ideas, he or she keeps standing on the soil of sensible things.

Plato's attitude toward the mathematician was that the latter's knowledge is not self-supporting and cannot be left hanging in the air; nor can it be suitably grounded on empirical insights, for these are altogether too weak and unsteady a basis. Further, Plato rejected the conception of mathematics as simply the working out consistently (*homologoumenōs*) of its supposedly evident postulates ("hypotheses,"⁸⁰ i.e., axioms or axiomatic definitions); the alleged obviousness of the postulates (including that of the very existence of their subject matter) and of the immanent logic of the mathematician was unsatisfactory, philosophically speaking, because it relied on mere presumption or on sensible prompting and guidance. Plato insisted that mathematical knowledge required apposite vindication; for him that meant a complete ascent and descent, using Dialectic, through the realm of the impersonal Ideas, with all hypotheses being surmounted at the summit of the ascent when the thinker "by thought itself" apprehends "the starting-point of all," viz., "the nature of the good itself," which is the supreme "confirmation."⁸¹ Plato's objectually realist, and his constativist-veridicalist, presuppositions led him away and astray from the consistency conception of mathematics into the Dialectical labyrinth of his own making, where he was swallowed up by the Minotaur of his ambitious obscurity.

A continual factor of this obscurity was his failure to distinguish between the objectual and the propositional, to employ appropriately and clearly each of the two sides of this distinction, and to indicate how they are interrelated. He occasionally dropped hints of the relevance of the propositional: the mathematicians are only "dreaming about being, but the clear waking vision of it is impossible for them as long as they leave the assumptions which they employ undisturbed and cannot give any account (*logos*) of them."⁸² But since he demanded that the account should be in terms of the Ideas, especially of the "starting-point" (*archē*) of all "that is the (nature of the) Idea of the good, it is hard to discern in *Republic* a coherent explanation of the foundations of mathematical knowledge.

3.7. *Philebus* illustrates the continuing importance, into his later period, that Plato attached to mathematical matters, partly the influence of Pythagoreanism, as in its deployment of the notion of the limit and the unlimited (cf. p. 6 above).⁸³ In this Dialogue Plato drew, and followed up for epistemological and other purposes, a contrast between two kinds of arithmetic (and likewise of other branches of mathematics), viz., the "popular" and the "philosophical"; in the former one calculates with unequal units, e.g., two armies or two oxen (each of which may be of different size), whereas the philosophical arithmeticians proceed "on the postulate that none of the myriad units under discussion is in any way different from any of the others"—and analogously in regard to, e.g., practical mensuration versus philosophical geometry. Philosophical mathematics is superior to the popular most notably in its "precision."⁸⁴ Plato graded the ordinary arts and sciences according to their degree of such precision. He ranked Dialectic, though, even higher than philosophical mathematics, for it alone being "concerned with the final truth [to on], the real nature of things and unchanging reality is the most genuine [*alēthestatos*] knowledge"; it has "as its province the clearest, most precise, and true [*alēthestatos*]

subject-matter," on which thought and intelligence are engaged.⁸⁵ This Dialogue tells us (i) that mathematical objects and truths are "intelligible," objective, unchangingly existent, and exact; (ii) that no truly mathematical objects or perfect instances of mathematical truths exist in the sensible world; and (iii) that there is a science, Dialectic, superior even to mathematics in its concern with truth/reality in all its comprehensiveness. None of this, any more than what was said in *Republic*, can rightly be read as implying that mathematics are intermediate between sensibles and Ideas. Plato has, again, failed to clarify the relationship between mathematical truths and mathematical entities and to justify his existential assumptions about them.

3.8. Before I leave Plato's writings, reference should be made to his (if genuine) Seventh Letter. Every real being, he wrote, is knowable, and there are four relevant things distinct from it: (1) its name, e.g., "circle," though this is arbitrary and variable; (2) its definition, e.g., in the case of circle, "the figure whose extremities are everywhere equally distant from its centre," and this too is not fixed inasmuch as it is composed of items of language that are arbitrary and variable; (3) an image, e.g., a so-called circle that we draw or erase; and (4) knowledge of the being, e.g., of (a) circle: "In the fourth place are knowledge (*epistēmē*), reason (*nous*), and right opinion (which are in our minds, not in words or bodily shapes and therefore must be taken together as something distinct both from the circle itself and from the three things previously mentioned). . . . The same thing is true of straight-lined as well as of circular figures; of colour; of the good, the beautiful, the just" And he added that "every circle that we make or draw in common life is full of characteristics that contradict the [real being itself], for it everywhere touches a straight line, while the circle itself, we say, has in it not the slightest element belonging to a contrary nature."⁸⁶ No doctrine of the intermediate status of mathematics can be detected in this letter; indeed, they are absolutely on a par with the real beings—the Ideas—that are the good, the beautiful, the just, and so on.

3.9. The most convenient focal place for studying Plato's Unwritten Doctrines is Aristotle's *Metaphysics*, Book A, Chapter 6. There, Aristotle cited no oral or written sources, did not allude to this distinction, and gave no hint that Plato's philosophy underwent a change (though *Metaphysics*, Book M, Chapter 4, recognizes such a development) or that it was at all different orally from what it was in the Dialogues. It looks as though all of Aristotle's remarks in A6 were intended to apply to Plato's oral teaching. Aristotle makes the following statements.

(a) The philosophy of Plato in most respects followed the philosophy of the "Italians," i.e., the Pythagoreans (Aristotle might just possibly have meant to include Parmenides, also). Plato said that the many things exist by "participation" in the appropriate Idea; this is only a difference in terminology, however, from the Pythagoreans' saying that things exist by "imitation" of numbers. (This "imitation" is an expression of the doctrine that things are numerically structured, as against being nothing but numbers. See p. 14 above.) Aristotle's comparison clearly implies that for Plato the Ideas are, in some sense, numbers. This is an Unwritten Doctrine.

(b) Plato held that, besides the sensible things and the Ideas, there are the objects of mathematics, which occupy an intermediate position. (See pp. 20–21 above.)

I have argued that this Platonic view was not propounded in the Dialogues or Epistles. Aristotle's account does not answer the question: Did Plato maintain that there are mathematical Ideas on top of the mathematical intermediates, viz., one such Idea for each, many alike among the intermediates? The answer is yes and no: yes in that there are *some* mathematical Ideas, viz., Ideal numbers; no in that there is not an Idea for *each* sort of intermediates (e.g., there is no Idea of the circle, according to the Unwritten Doctrines, unlike in the original form of the theory of Ideas).⁸⁷

(c) For Plato, "the Ideas [are] the causes of all other things," i.e., "the Numbers [= the Ideal numbers] are the causes of the reality of other things";⁸⁸ *all* the Ideas are Ideal numbers. This, I surmise, is why Aristotle did not deal with the question raised in (b): a general negative answer was implied by the doctrine he reported, that all the Ideas are Ideal numbers; so, more specifically, there were no geometrical Ideas in Plato's Unwritten Doctrines.

(d) Plato diverged from the Pythagoreans in two major respects, despite the resemblance described in (a). First, he introduced the Ideas. Second, it "is his view that the Numbers exist apart from sensible things, while they say that the things themselves are numbers, and do not place the objects of mathematics between Ideas and sensible things."⁸⁹

(e) Plato accepted that the Ideal numbers are not self-sufficient/self-explanatory. They derive from two elements/principles: one is limiting—it is "essential reality, the One"; the other is the unlimited—it is the principle of plurality involving the endless alternation of the lesser and greater, e.g., the Number 3 is less than the Number 4 but is greater than the Number 2.⁹⁰

On these points (d) and (e), Plato's thought is Pythagoreanism etherealized.

4. ARISTOTLE

4.1 Plato's ontology gave pride of place to supersensible entities—primarily those universal in character—existing in their own right, although sensible particulars have merely a cavernous or shadowy subexistence. Aristotle came to be scornful of all this. He argued against Plato's transcendentalism in favor of his own metaphysical priority ascribed to individual sensible substances. (I leave aside his theology, with its strained adoption of the reality of sheer form, perfect and partless—and living.) Aristotle's philosophy of mathematics⁹¹ was shaped by his critical preoccupation with Platonism, no less than by his constructive need to show how mathematics have only a dependent reality, i.e., dependent on individual sensible substances and on our processes of abstraction and reason.

I will skim the rooftops of certain terraces of Aristotle's ontology and methodology of mathematics: in his rejection of Plato's doctrines of intermediates and of numbers; his conception of mathematics in terms of the axiomatic method; his ontology of mathematical generally and of geometricals, numbers, and infinity severally; and, associated with this ontology, his proposal of abstraction as the fundamental procedure of mathematical concept of formation.

4.2. Plato's doctrine of mathematical as purely intelligible entities existing in

a layer of reality between Ideas and sensible things was unacceptable to Aristotle, basically because the very idea of the separate existence of abstract entities savored of nonsense to him. He sublimated his distaste in a series of arguments, including the following:⁹² (i) The doctrine presupposed the reality of Ideas (distinct from the mathematical intermediates). He dismissed them because, *inter alia*, they are superfluous duplications of features of the world of our sense-perception and action, can have no causal efficacy, and are inherently useless and unknowable principles, which cannot be coherently defined nor consistently circumscribed. (ii) The doctrine of intermediates, with its commitment to the separate existence of mathematical objects, was repugnant to Aristotle because of its absurd reduplicative consequences, as he thought them. (1) According to the doctrine, the mathematician's study is aimed not at sensible solids (e.g., pyramids) but at special—perfect and nonsensible—mathematical solids (e.g., tetrahedra) and likewise not at sensible but at mathematical planes and lines supposed over and above sensible planes and lines. Those mathematical solids also contain mathematical planes and lines. There must, therefore (by consistency in appealing to what has priority, "for incomposites are prior to compounds"), be further mathematical planes and lines distinct from those in the initial mathematical solids. And each range of the so far mentioned mathematical planes must contain distinct ranges of mathematical lines, additional to the initial set of mathematical lines. "The accumulation becomes absurd. . . . With which of these [mathematical planes and lines], then, will the mathematical sciences deal?" Analogous difficulties of reduplication arise, he alleged, in regard to numbers. (2) If mathematical intermediates are required to be the objects of mathematics, then there should also, by parity of reasoning, be appropriate sorts of intermediates to be the objects of other sciences; so, for biology, there would have to be "intermediate" animals apart from ordinary animals: a reduction to the absurd. (iii) Neither, e.g., lines nor numbers have any primary existence as substances, forms, or material substrata in Aristotle's senses of these terms; since, accordingly, mathematical have only a subsidiary mode of existence, they cannot exist in their own right as the doctrine of intermediates demands.

Aristotle's irritation with Plato's theory of Ideas in its various versions spurred him to adduce a collection of criticisms against the theory of Ideal numbers, each composed of units specific to it (see 2.3 (e) above).⁹³ (i) In mathematics a number may derive, as a sum or product, from a plurality of numbers; but how can one Idea come from a plurality of Ideas? (ii) Plato was unable to explain how the unitary character of each of his Ideas is possible if these are composed of units. (iii) The relationship between an Ideal number (e.g., Three) and the correlative mathematical number (e.g., three{s})—the latter being the object(s) of the mathematician's study—was left obscure. (iv) Plato was unable to explain how the two fives in the Ideal number Ten differ from the five units in the Ideal number Five. More generally, Aristotle charged to catch Plato on the horns of dilemma. Either all the units are alike or (as Plato had supposed) they are unlike specifically. (1) If all the units are alike, then the Ideas, i.e., Ideal numbers, would be related as parts to wholes; e.g., Three would be a part of Four and of all succeeding Ideal numbers. This result is

incompatible with Plato's conception of Ideas as each having a completely distinct and self-subsistent reality (subject to its dependence on the One). (2) If the units are unlike except for those that combine to constitute a particular Ideal number, then some differentiating attribute(s) must occur among them; but this is incompatible with Plato's conception of units as absolutely qualityless indivisibles. (v) "Further, the units in 2 must each come from a prior 2; but this is impossible."⁹⁴ There are difficulties and differences of interpretation about this. I take the thrust of Aristotle's point to be that Plato's account of the nature of 2 involves him in presupposing something binary and that his account is, therefore, vitiated either by its circularity or by its entailing an infinite regress.

4.3. It may not have been long after Aristotle's matured rejection of Plato's doctrines of transcendent or separately existing abstract entities before he drafted his *Posterior Analytics*, in which he propounded a systematic methodology of scientific knowledge and understanding. "Demonstration" was his kernel idea. He placed mathematics, even focally, within the scope of this methodology, as is revealed by his many mathematical references and illustrations (alongside many nonmathematical ones). Plato and the Academy had discussed "hypotheses" and definitions as starting-points of mathematical reasoning. And the need of proof in mathematics was already recognized, and the practice of proof in mathematics well established, by this time. With these developments and discussions behind him, Aristotle pioneered the elaborated philosophical formulation of the axiomatic method. Not that his version was clear-cut; it was misty and disorderly. Neither was it specific to mathematical science; it was meant to be applicable to all fields of scientific intelligence.

For Aristotle, each science has its special subject matter in the form of a domain or kind of elements; e.g., arithmetic has numbers and geometry has magnitudes. A mathematical science is an organization of knowledge of reasoned facts; this is the most exact knowledge. It is an enclosed body of demonstrations concerning its subject matter; demonstrations across domains, e.g., a demonstration of something geometrical by arithmetic, are debarred. Further, each mathematical science inherently requires the presence of two mutually exclusive sorts of truth, viz., principles (i.e., indemonstrable truths) and demonstrable truths; the former sort provides a finite basis of the latter. The principles of science, e.g., geometry, fall into three classes: basic existential truths about its domain, definitions concerning types in its domain, and basic truths having common applicability in the sciences (such as: "If equals are taken from equals, the remainders are equal"). Knowledge of the truth of principles is obtained by, above all, the exercise of the intellect's power of "intuitive reasons" (*noûs*). Knowledge of demonstrable truths in a science is obtained by valid syllogistic reasoning relying ultimately on indemonstrable truths as premises.

Aristotle's appeal to his syllogisms in this connection was an unfortunate—and needless—restriction; it is a notorious fact that, in general, mathematical proofs cannot be satisfactorily molded into syllogistic patterns. He also laid down conditions—at least some of which are needlessly restrictive—of a proposition's being a principle in a mathematical science: it had to be true, primitive, universal, necessary, and explanatory. Thus, for instance, principles must be really true because (he claimed) we

cannot "understand" (*epistasthai*), i.e., understandingly know, what is not the case; and he emphasized that we come to understand something not by mere deduction along valid lines but only by demonstration, i.e., deduction from true premises. Aristotle's methodology of mathematics was pervaded by his epistemological interests.

4.4. For Aristotle, mathematics cannot exist apart from sensible substances. Two prospective possibilities are that mathematics are themselves sensible substances or that they are one of the factors—form or matter—of Aristotelian substances. He denied that mathematics have any such status.

How can lines be substances? Neither as a form or shape, as the soul perhaps is, nor as matter, like the solid; for we have no experience of anything that can be put together out of planes or lines or points, while if these had been a sort of material substance, we should have observed things which could be put together out of them.⁹⁵

He maintained, then, that mathematical objects neither are unchanging and eternal realities over and above sensible ones, nor are they sensible substances, nor are they the form or the matter of such substances. On the other hand, he took it for granted that mathematical assertions and knowledge are of real truths, at bottom on the same footing as the truths of physics and of metaphysics (the other two theoretical sciences, in his classification) and that mathematics have an external reality whose basis lies in individual sensible substances.

Obviously, physical bodies contain surfaces and volumes, lines, and points, and these are the subject-matter of mathematics [sc. geometry]. . . . The mathematician . . . does not treat of [figures] as the limits of a physical body; nor does he consider the attributes indicated as the attributes of such bodies. That is why he separates them; for in thought they are separable from motion, and it makes no difference, nor does any falsity result, if they are separated. . . . Geometry investigates physical lines but not *qua* physical.⁹⁶

In this way, in terms of how the mathematician considers given attributes of the sensible, Aristotle specified what is distinctive of mathematical objects. He did, in fact, distinguish two types of mathematical objects, the universal and the individual, e.g., the universal circle and individual circles; and he paid attention to the "formula" (as he called it), i.e., the essential/defining conditions, relating primarily to the former.⁹⁷ However, he did not explicitly and carefully pursue the difference between the universal and the individual among mathematical objects when he concentrated on mathematical philosophy in the last two books of his *Metaphysics*.

All "[t]heoretical] scientific knowledge is judgement about things that are universal and necessary."⁹⁸ Also, the theoretical sciences "omit the question whether the genus with which they deal exists or does not exist, because it belongs to the same kind of thinking to show *what* it is and *that* it is"⁹⁹—a doubly questionable proclamation, which he applied to mathematics by saying that the issue was "not *whether* they exist but *how* they exist."¹⁰⁰ What is peculiar to mathematics among the theoretical sciences is that, first, its objects have no real separate existence, and,

second, these objects exist qua mathematical only by mental transformation and separation of movable, concrete substances into immovable, abstract entities through a process of considering the movable, concrete substances in abstraction from their physical makeup and features; the mathematician treats the sensible substances as though they were stripped of all their sensible components and attributes, e.g., their hardness and softness, heat and cold, etc., and he investigates the resulting abstractions.¹⁰¹

The mathematician may use a similar process of abstraction in regard to representations, such as geometrical diagrams. Aristotle's thought was this: the geometer has to draw a triangle with a determinate size, but he can and may disregard the fact that the size is determinate.¹⁰²

4.5. Quantity is the immediately relevant category of mathematics.¹⁰³ "Quantum" means that which is divisible into two or more constituent parts each of which is by nature a 'one' and a 'this'. A quantum is a 'plurality' if it is numerable and a 'magnitude' if it is measurable. Geometry is the science of magnitudes. Characteristic of a magnitude is that it is divisible into continuous parts, whereas a plurality is discrete. Mere continuous magnitude is pure extension; in one dimension it is length, in two dimensions breadth, and in three dimensions depth. If limited, length is a line, breadth a plane, and depth a solid.¹⁰⁴ A point has no extension; it is that which has position and is indivisible (a numerical unit has only the latter property) or is a division and limit of a line.¹⁰⁵

Aristotle's conception of specifically geometrical objects is somewhat hazy through its invoking the notion of 'intelligible matter' without clearly explaining it. For him, as a rule, genus and matter are both correlated with potentiality; accordingly, those are in some cases equated.¹⁰⁶ A genus is a possibility of different determinate species: their commonality is their generic matter. And an individual in its actual concreteness must have a formal and a material component; the latter alone is only a possibility of individuality. Given the abstraction process already described, the matter of, e.g., an individual circle cannot be sensible; it is "intelligible matter." This is the ground—accessible to thought—common to all plane figures, whatever they are specifically. An individual circle is a composite of this intelligible matter together with the form/essence of circle. Circle itself, as a universal, is a species of the genus, plane.¹⁰⁷

Aristotle's ontology of arithmetical objects, viz., one and the integers designating plurality, was intended to be mainly analogous to that of geometrical objects, but with the contrasts I have indicated above. One of the defects of his analyses of number was his repeated failure to keep distinct the notions of, e.g., (1) a mathematical number itself, (2) (i) the occurrence or (ii) the use of such numbers in arithmetical sums or products, (3) numbered aggregates of physical, etc., things, and (4) the process of numbering such an aggregate. Typical is his mixed remark that "a number, whatever number it is, is always a number of certain things, either of parts of fire or earth or of units";¹⁰⁸ also typical is his remark that "the one is the beginning of the knowable regarding each class. But the one is not the same in all classes. For here it is a quarter-tone, and there it is the vowel . . ."¹⁰⁹

4.6. Unlike the Aristotelian views presented in §§4.2, 4.4, and 4.5, which are in his *Metaphysics* (even if partly elsewhere in his works, too), his major discussion of infinity took place in *Physics*.¹¹⁰ He dwelt on the physical bearings of the notion: on whether and, if so, in what sense, e.g., an infinite body is possible or time is infinite. He concurrently touched on the question of mathematical infinity. Besides, his treatment of physical infinity when conjoined with his theory of abstraction would be liable to have consequences for mathematical infinity.

Aristotle argued at length, deploying his special ideas of space and place, that there is no body that is actually infinite. However, he linked this with a denial that it means that the infinite does not exist in any way, for the unqualified nonexistence of the infinite would lead to such impossible consequences, he held, as time having a beginning and an end, as a magnitude not being divisible into magnitudes, and as number not being infinite. Are time, magnitude, and number, then, actual infinities? And if they are not actual infinities, how are they infinities at all? He did not supply all the requisite explanatory and justificatory details of his answers, and some of the details he gave are by no means convincing. His directing ideas are, nonetheless, straightforward enough. He maintained that neither time, magnitude, nor number has an independent—substantial—reality; none of them is a 'this', a complete, self-subsistent entity. In the substantial sense of "is," none is an actual infinite. On the other hand: "Generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different."¹¹¹ Time is infinite in being an unending process of coming to be and passing away; its ingredients are transient; they do not all exist to compose a single infinite state. A magnitude, i.e., a continuous quantum, is infinite in being infinitely divisible; continuity cannot be reduced to discreteness. "The number of times a magnitude can be bisected is infinite. Hence this infinite is potential, never actual: the number of parts that can be taken always surpasses any assigned number."¹¹² A magnitude is fixed; it does not lose its parts (like time) or grow (like number). Whereas in dividing a magnitude there is no terminus in the direction of smallness, in counting/adding number (positive integers) there is no terminus in the direction of largeness: "in the direction of largeness it is always possible to think of a larger number."¹¹³ Finally, Aristotle offered the following characterization of infinity: its conceptive character is emphatic: "a quantity is infinite if it is such that we can always take a part outside what has been already taken."¹¹⁴

4.7. The philosophical development concerning mathematics from the Pythagoreans to Plato and thence to Aristotle might be proposed, albeit as a *jeu d'esprit*, as a pellucid example of a so-called Hegelian triad. Mathematics were at first thought to be real as physical constituents or structures wholly embedded in sensible things. Plato moved to an antithesis of this, thinking mathematics to be real as immovable, immaterial entities existing quite separately from sensible things. Aristotle's position was an ingenious synthesis in which he managed to combine a physicalism derived from the Pythagoreans with an abstractionism derived from Plato. He maintained their shared assumption of an external realism; he kept to the Pythagoreans' confinement of all mathematical to an existence as aspects of sensible

things, while he, stressfully, accommodated to this confinement an insistence, learned from Plato's orientation, on the ideality—the abstractness and perfection—of all mathematical.

Aristotle took over from these predecessors two other assumptions about mathematics. One is the constativist and veridicalist nature of mathematical thought/discourse. The second is the intelligibility of mathematics. For him, it may be suggested, this feature was allied to the fact that mathematical concepts are an outcome of a certain kind of mental activity and attention by us; he made mathematical concepts our internalized constructions.¹¹⁵ This was indeed an advance. But it still had this drawback, that it presupposed the priority and the limitations imposed by the sensible world as the matrix of mathematical concept formation.

Notes

1. In my usage the term 'mathematical philosophy' has the forefront sense of 'philosophy of mathematics'; in appropriate cases, it can also carry '(a) philosophy with a mathematical content or style' as a background sense. Cf. §3.1.
2. G. Milhaud, *Les Philosophes-géomètres de la Grèce* (Paris, 1900) (pre-Socratics and Plato only), and E. A. Maziarz and T. Greenwood, *Greek Mathematical Philosophy* (New York: Alcan Ungar, 1968), are cases in point.
3. Cf. my "Preface to the Grammar of Postulates," *Aristotelian Society Supplement*, Vol. LIII (1979), pp. 1–21.
4. For the Latin phrase, see Ockham, *Philosophical Writings*, ed. P. Boehner (London: Nelson, 1957), p. 41.
5. On the system of hyperreal numbers, cf. J. M. Henle and E. M. Kleinberg, *Infinitesimal Calculus* (Cambridge, Mass., and London: M.I.T. Press, 1979), and H. J. Keisler, *Elementary Calculus* (Boston: Prindle, Weber and Schmidt, 1976).
6. The term 'illocutionary force' originates on p. 100 (cf. pp. 98–99) of J. L. Austin, *How to Do Things with Words*, 2nd ed. rev., ed. J. O. Urmson and M. Sbisà, with index by P. H. Nidditch (Oxford: Oxford University Press, 1980).
7. *Metaphysics*, A5, 985b23–24, following trans. by W. D. Ross. (A translator's name is cited in these notes only on the first relevant occasion.)
8. In *Metaphysics*, *Physics*, *On the Heavens*, and elsewhere; see W. T. Organ, *An Index to Aristotle* (Princeton, N.J.: Princeton University Press, 1949), s.v. Pythagoreans.
9. Translated in K. Freeman, *Ancilla to the Pre-Socratic Philosophers* (Oxford: Blackwell, 1947). Cf. H. Thesleff, *An Introduction to the Pythagorean Writings of the Hellenistic Age* (Åbo: Acta Academiae Aboensis, 1961), and *The Pythagorean Texts of the Hellenistic Period* (Åbo: Acta Academiae Aboensis, 1965).
10. E.g., by E. Frank, *Plato und die sogenannten Pythagoreer* (Halle: Niemeyer, 1923), pp. 313–15, footnote, and W. Burkert, *Love and Science in Ancient Pythagoreanism*, trans. E. L. Minar (Cambridge: Harvard University Press, 1972), pp. 273–75. Cf. n.25.
11. Cf. Burkert, *op. cit.*, pp. 401ff.; B. L. van der Waerden, "Die Arithmetik der Pythagoreer," *Math. Annalen*, Vol. 120 (1947–49), pp. 127–53, 676–700; B. L. van der Waerden, *Science Awakening*, trans. A. Dresden (Groningen, Noordhoff, 1954), per index s.v. "Pythagoras," "Pythagoreans;" W. R. Knorr, *The Evolution of the Euclidean Elements* (Dordrecht and Boston: Reidel, 1975).
12. "Natures": Aristotle, *Metaphysics*, A5, 985b34; "causes"; A6, 987b24–25, and A8, 990a19–20.
13. *Essay Concerning Human Understanding*, ed. P. H. Nidditch (Oxford: Oxford University Press, 1975), II.xvi.1: 205(8–10). My cross-quotations are meant to be taken as implying a sub-

14. *Philebus*, 16c-d, trans. J. Gosling (Oxford: Oxford University Press, 1975).
15. Aristotle, *Metaphysics*, A5, 986a2–3; Aristotle, *On the Heavens*, ii.9, 290b12–14, trans. T. L. Heath, *Aristarchus of Samos* (Oxford: Oxford University Press, 1913), p. 105.
16. *System of Logic*, II.vi.2. J. Klein, *Greek Mathematical Thought and the Origin of Algebra*, trans. E. Brann (Cambridge, Mass., and London: M.I.T. Press, 1968), Part I, stresses the 'number of' sense of *arithmos*. Cf. p. 28 below.
17. *Metaphysics*, M6, 1080a22–33. Italics in quotations are mine unless the contrary is noted.
18. *Metaphysics*, M6, 1080b16–21; M8, 1083b8–19.
19. My term 'configuration' covers the Pythagorean doctrine that solid bodies are formed of planes, planes of lines, and lines of monadic points. Relevant details are given in, e.g., W. K. C. Guthrie, *History of Greek Philosophy* (Cambridge: Cambridge University Press, 1962), Vol. 1, pp. 259ff.
20. *Metaphysics*, A5, 986a15–21.
21. Cf. W. D. Ross, *Aristotle's Physics* (Oxford: Oxford University Press, 1936), pp. 542–45.
22. Philolaus, fr. 1, trans. Guthrie, *op. cit.*, p. 330.
23. Diogenes Laertius, viii.8, trans. G. S. Kirk and J. E. Raven, *The Presocratic Philosophers* (Cambridge: Cambridge University Press, 1957), p. 228. Cf. Aristotle, *Protrepticus*, fr. 11.
24. Cf. Iamblichus, *Comm. math. sc.*, 25: "The Pythagoreans, having devoted themselves to mathematics, and admiring the rigour of its arguments, because it alone of the studies men undertake contains proofs . . .," trans. J. Barnes, *The Presocratic Philosophers* (London: Routledge, 1979), Vol. 2, p. 78. The text is given in Burkert, *op. cit.*, p. 50, n.112, who argues that it is derived from a now lost work by Aristotle, *On the Pythagoreans*. Archytas, fr. 4, links proof primarily with arithmetic, which is therefore superior to geometry.
25. The stock criticisms of the fragments' authenticity, which stem mainly from Bywater and Frank and are conveniently adduced by Kirk and Raven (*op. cit.*, pp. 308–11), are poorly reasoned, as was recognized by Guthrie (*op. cit.*, pp. 331–32). Cf. G. de Santillana and W. Pitts, "Philolaus in Limbo," *Isis*, Vol. 42 (1951), pp. 112–20.
26. Philolaus, frs. 4, 6, 11, trans. Freeman, *op. cit.*
27. *Metaphysics*, A5, 985b23–26.
28. See Burkert, *op. cit.*, pp. 369ff.
29. *Metaphysics*, N3, 1090b20–25.
30. Cf. Philolaus (Diels-Kranz) A29, and n.24 above. A. Szabò, *The Beginnings of Greek Mathematics* (Dordrecht and Boston: Reidel, 1978) is interesting but is one-sided in stressing the role of dialectic in the rise of the deductive method.
31. I. Thomas, *Greek Mathematical Works* (London and Cambridge, Mass.: Harvard University Press, 1939), Vol. 1, pp. 148–49.
32. Aristotle treated quantity as an important category; mathematics has an important place in his scheme of the sciences; and in his *Posterior Analytics* it has, in some respects, a prototypical role among the sciences.
33. See, e.g., A. C. Crombie, *Robert Grosseteste* (Oxford: Oxford University Press, 1953), pp. 110ff., 139ff.
34. Cf. Aristotle, *On the Heavens*, 293a21–22.
35. *Metaphysics*, M8, 1083b11–19.
36. Aristotle uses both terms in his exposition in *Metaphysics*, A5, 986a.
37. Of course, the Pythagoreans had no cognizance of zero or of negative integers.
38. "Harmony" as the agency of the combination of the limit and the unlimited in the universe is too general and indefinite (cf. Philolaus, fr. 6) to provide a means of defense against the criticism made in the paper.
39. Cf. Aristotle, *On the Heavens*, ii.9.
40. Cf. the Euclid scholium quoted in Thomas, *op. cit.*, (in n.31), pp. 214–17.
41. Cf. Aristotle, *Metaphysics*, A8, 989b33–34.
42. Aristotle, *Metaphysics*, A5, 986a8–9.

44. *Introduction to Mathematical Philosophy* (London: Allen and Unwin, 1919), p. 1.
45. Cf. Aristotle, *Nicomachean Ethics*, 1131a20ff.
46. Cf., e.g., Aristotle, *On the Soul*, 404b24-25, and the Aristotelian references in W. D. Ross, *Plato's Theory of Ideas* (Oxford: Oxford University Press, 1951), p. 216 n. The term 'Unwritten Doctrines' comes from Aristotle, *Physics*, 209b15. A compendium of translations of source passages on the Unwritten Doctrines is in J. N. Findlay, *Plato: The Written and Unwritten Doctrines* (London: Routledge, 1974), Appendix I.
47. Philolaus, frs. 8, 10, trans. Freeman. The Parmenidean One should be kept in mind as a part of the background to Pythagoreans and Plato on One.
48. Cf., e.g., *Republic*, 507b, 596a; *Parmenides*, 130a-d.
49. Cf. G. Leff, *William of Ockham* (Manchester: Manchester University Press, 1975), pp. 164ff. Sameness and Difference are among the important higher-level kinds or Ideas in some later Dialogues. 'Number' is a "transcendental" at *Sophist*, 238a-c.
50. Cf., e.g., *Republic*, 476a2-4.
51. *Parmenides*, 137aff.
52. F. M. Cornford, *Plato and Parmenides* (London: Routledge, 1939), p. 111. Locke, *Essay*, II.xvi.1, 2, slides from one to another of "one," "unity," and "unit."
53. Findlay, *op. cit.*, is the fullest committed presentation in English. Guthrie, *History of Greek Philosophy* (Cambridge: Cambridge University Press, 1978), Vol. 5, Chapter 8, contains references to the relevant literature – and a sharp critique.
54. I follow the chronology in J. B. Skemp, *Plato* (Oxford: Oxford University Press, 1976), pp. 13ff., 52ff. Ross, *Plato's Theory of Ideas*, p. 10, dates *Meno* earlier, to before Plato's first Sicilian visit, 389-388 B.C.
55. *Meno*, 81a-d, trans. Guthrie (Harmondsworth: Penguin Books, 1956). On "the other world" (the realm of Hades), cf. *Phaedo*, 68a-b; and note "Hades under the earth" at *Republic*, 596c.
56. 'Recollection' is hardly re-collection: the slave-boy does not remember the original acquaintance or occasion.
57. Cf., e.g., *Republic*, 458d.
58. Cf. *Republic*, 491.
59. *Meno*, 90b.
60. *Meno*, 90a.
61. E.g., J. A. Stewart, *Plato's Doctrine of Ideas* (Oxford: Oxford University Press, 1909), p. 28. (The word *eidōs* at *Meno*, 72c, does not mean "Idea.")
62. Cf., e.g., *Phaedo*, 109e; *Republic*, 500c; *Phaedrus*, 247c.
63. *Meno*, 81d.
64. This sense of 'thesis' was introduced by Leśniewski and made well known by Łukasiewicz. Cf. Łukasiewicz in S. McCall, ed., *Polish Logic, 1920-1939* (Oxford: Oxford University Press, 1967), p. 44 n.
65. *Meno*, 86c-87a. On the whole matter, see R. S. Bluck, *Plato's Meno* (Cambridge: Cambridge University Press, 1961), pp. 75ff., 321 ff., 441ff.; and R. Robinson, *Plato's Earlier Dialectic*, 2nd ed. (Oxford: Oxford University Press, 1953), Chapter 8.
66. *Phaedo*, 65c-e, 74a-75b, etc.
67. See especially *Phaedo*, 101c, 104-5, and the commentary *ad locc.* in D. Gallop's translation (Oxford: Oxford University Press, 1975). (Gallop's criticism of Vlastos, on p. 186, is [I think] partly wrong.)
68. *Phaedo*, 74a-75b.
69. *Phaedo*, 75c, 104a, 105c.
70. *Phaedo*, 101b, c, trans. Gallop. My objection about specific differences at the end of (i) is made *pace* Cook Wilson, Ross, Gallop, et al.
71. *Republic*, e.g., 382bff., 389b-c, 485c; cf., e.g., *Timaeus*, 29c. I use 'proposition' in a broad sense, to contrast with 'objectual'.

72. *Republic*, 485b, 526a-b, 526d-e (cf. 508-9), 527a-b, trans. P. Shorey (Cambridge, Mass., and London: Harvard University Press, 1935). Cf. *Sophist*, 238a.
73. *Republic*, 531d, 537c.
74. *Metaphysics*, A6, 987b14-17.
75. E.g., J. Adam, *The Republic of Plato* (Cambridge: Cambridge University Press, 1902), Vol. 2, pp. 115 n., 159ff.; A. Wedberg, *Plato's Philosophy of Mathematics* (Stockholm: Almqvist and Wiksell, 1955), p. 124; J. A. Brentlinger, "The Divided Line and Plato's Theory of Intermediates," *Phronesis*, Vol. 8 (1963), pp. 146-66. (Brentlinger's interpretation of *Rep.*, 534a, which he regards as decisively showing that the objects of *dianoia* and *noēsis* are on different ontological levels, is [I think] unfounded.)
76. *Republic*, 526a; cf. *Philebus*, 56d-e.
77. *Republic*, 511; cf. 533-34.
78. *Republic*, 510d, 525d-e.
79. *Republic*, 510a-511b, 525d.
80. *Republic*, 510-11.
81. *Republic*, 511, 532, 533.
82. *Republic*, 533b-c.
83. In *Timaeus* (somewhat earlier among the later Dialogues than *Philebus*), numbers, proportions, and shapes play prominent roles cosmologically. The influence of Pythagoreanism on this is apparent.
84. *Philebus*, 55d-57e, trans. Gosling.
85. *Philebus*, 58a-59d.
86. Seventh Letter, 342-43, trans. G. R. Morrow, *Plato's Epistles* (Indianapolis and New York: Bobbs Merrill, 1962).
87. Aristotle, *Metaphysics*, M4, discusses what it claims was the original form of Plato's theory of Ideas.
88. Aristotle, *Metaphysics*, A6, 987b18-25, Ross's trans. adapted; see also N2, 1090a4-6, and frag. 4 (Ross). Cf. Ross, *Plato's Theory of Ideas*, Chapter 15.
89. Aristotle, *Metaphysics*, A6, 987b27-29, Ross's trans. adapted.
90. Cf. p. 12 above. For elucidations of Plato's principles of generation of numbers, see Ross, *Plato's Theory of Ideas*, pp. 182ff.; and J. Annas, *Aristotle's Metaphysics, Books M and N* (Oxford: Oxford University Press, 1976), pp. 42ff.
91. Useful sources and references include: Aristotle, *Metaphysics*, Books Δ, E, Z, K, M, and N; T. Heath, *Mathematics in Aristotle* (Oxford: Oxford University Press, 1949); J. Barnes, *Aristotle's Posterior Analytics* (Oxford: Oxford University Press, 1975), with bibliography.
92. I select from *Metaphysics*, M2, 1076b11-1077a36.
93. I select from *Metaphysics*, A9 and M6-8.
94. *Metaphysics*, A9, 991b31-992a1.
95. *Metaphysics*, M2, 1077a32-36.
96. Aristotle, *Physics*, ii.2, 193b24-194a12, trans. R. P. Hardie and R. K. Gaye (Oxford: Oxford University Press, 1930).
97. *Metaphysics*, Z10, 11.
98. Aristotle, *Nicomachean Ethics*, vi.6, 1140b31-32, trans. Ross.
99. *Metaphysics*, E1, 1025b16-18.
100. *Metaphysics*, M1, 1076a36-37.
101. See *Metaphysics*, Z11, 1036a11-1036b3, 27-28, and K3, 1061a28-35.
102. Aristotle, *On Memory*, 450a1-8.
103. Cf. the notes on Chapter 6 of Aristotle's *Categories* in J. L. Ackrill's edition (Oxford: Oxford University Press, 1963).
104. *Metaphysics*, Δ13, 1020a7ff.
105. *Metaphysics*, Δ6, 1016b24ff., and K2, 1060b10ff.
106. Cf. A. C. Lloyd, *Form and Universal in Aristotle* (Liverpool: Cairns, 1981), pp. 32ff.

107. Cf. *Metaphysics*, Δ28, Z10, 11.

108. *Metaphysics*, N5, 1092b19-20.

109. *Metaphysics*, Δ6, 1016b20ff.; Aristotle's whole chapter is a valuable discussion of uses of 'one'.

110. Book III, Chapters 4-8. Valuable discussions are in Ross's editions, pp. 48ff., and in J. Hintikka, *Time and Modality* (Oxford: Oxford University Press, 1973), Chapter 6 (though its main purport is [I think] mistaken).

111. *Physics*, iii.6, 206a27-29.

112. *Physics*, iii.7, 207b10-13.

113. *Physics*, iii.7, 207b10.

114. *Physics*, iii.6, 207a7-8.

115. Aristotle, *On the Soul*, iii.4-7, is relevant in this connection.