

THE CONSONANT ELEVENTH AND THE
EXPANSION OF THE MUSICAL TETRACTYS:
A STUDY OF ANCIENT PYTHAGOREANISM

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The interval of the eleventh, that is, the octave plus fourth, stands at the crease between rationalism and empiricism in ancient musical treatises. For convenience, I refer to the interval as the *eleventh*, and to its neighbor as the *twelfth*, although most ancient and medieval treatises refer to these intervals, respectively, as the *diapason plus diatessaron* and *diapason plus diapente*. By extracting the eleventh, I perform a biopsy on ancient Pythagoreanism and thus hope to elucidate the Pythagorean world view as well as to indicate some avenues along which ancient Pythagoreanism may have traveled to the Middle Ages.¹

By tracing the Pythagorean treatment of the eleventh, I show the role of number in early musical theory to be both causal and descriptive. The causal nature of number would seem to be more primitive and therefore older than the descriptive role played by number in Pythagorean harmonics. Modern musical theory tends to use numbers as a general, abstract system or metaphor for sound. Thus, our modern disposition invites us to see a development from a time when numbers determined the consonance or dissonance of an interval to a time when numbers simply described an interval, that is, provided a quantitative metaphor without evaluating the interval according to its consonant or dissonant character. The history of mathematics reflects this development

from numerical cause to numerical description. By studying the perception of the eleventh in antiquity, we shall see this development only in a very general and sporadic way. In fact, the causal properties of numbers remained vital for musical theory well beyond antiquity, and some would argue that this numerical vitality persists to the present.

Mathematically, the most primitive treatment of the eleventh is the exclusion of the interval from the category of consonance because it cannot be generated or accommodated by the numerical quaternary 1, 2, 3, 4. Here the numbers determine the consonance or dissonance of the interval. Nevertheless, some of the ancient theorists who treat the eleventh heard it as consonant. Thus their sensory perception confronted their rational beliefs. In some cases, the apparent paradox of rationalism and empiricism is left unreconciled. Some authors recognized the variability and uncertainty of sensory perception and therefore opted for causal numbers. Others crafted a new numerical representation that relied on the old (?) causal numbers, but one that used these numbers to justify sensory perception. In other words, some treatises synthesize two causal, numerical quaternaries (1, 2, 3, 4 and 6, 8, 9, 12) in order to characterize the consonant eleventh. By doing so, they take elements of causal number and transform them into acoustical metaphor.

A discussion of the nature of consonance falls largely outside the bounds of this investigation. I am concerned primarily with the classification of intervals as consonant or dissonant. One can study this classification from the rationalist's and empiricist's points of view without addressing the question: what is consonance? Indeed, several ancient treatises eschew the larger philosophical question in favor of classification. A few remarks about the perception of consonance, however, are in order.

In his *De sensu*, Aristotle is concerned with sensory perception in general, and in particular with his argument that a single sense cannot perceive two objects simultaneously (447b10-13).² In the course of his discussion, Aristotle queries whether a consonance is a relation, mixture, or unity. Since in most cases the notion of consonance requires two different pitches, one is tempted to rule out unity, but Aristotle observes that a consonance is like the mixture of two colors, for example, blue and yellow, to form a third (439b26ff). Just as the eyes can see only one color in the mixture, green in this case, so the ears sense one consonance and not two different pitches. In other words, because the ears cannot hear two distinct pitches at the same time, they perceive a consonance only if the two sounds have formed a single mixture. Whether two distinct but dissonant pitches would form a *single* dissonance is uncertain.

There exists some evidence in the musical treatises of antiquity,

especially the Pythagorean ones, that Aristotle's argument held true. The discussion in *Sectio canonis* of the single blend formed by consonant notes seems to stem from the Aristotelian passage (see below, p. 194). The *Sectio* stresses the unity formed from consonant notes by claiming that the notes must be related by a single name or one term (see n. 14). Furthermore, many ancient treatises—not *Sectio canonis*—use the phrase “struck at the same time” to refer to the individual pitches making up the consonance.³ This distinction is especially important in light of the essential melodic nature of ancient Greek music. Boethius goes one step further when he distinguishes between an interval (*intervallum*), the distance between two pitches struck alternately, and a consonance (*consonantia*), the mixture of two pitches struck simultaneously.⁴ Thus most ancient authors seem to associate the notion of consonance with a harmonic rather than melodic event.⁵

Finally, one can use the eleventh not only to set in relief issues of rationalism and empiricism but also to record the transmission of musical science. Treatments of the eleventh in antiquity and in the early Middle Ages may serve as markers for the transmission of theory, indicating especially what did and did not pass from Hellenistic to Latin cultures through the *De musica* manuscripts of Boethius.

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Boethius writes in the second book of his *De musica*: “What, then, if we join a fourth and an octave consonance: will they not make—according to the Pythagoreans—another consonance? Not at all.”⁶ Boethius had previously defined consonance as “a mixture of high and low sound sweetly and uniformly falling on the ears.”⁷ But here he tells us that the Pythagoreans reject the eleventh because “it falls into the superpartient variety of inequality, and it preserves neither the order of multiplicity nor the simplicity of superparticularity.”⁸ In other words, the Pythagoreans' rejection apparently rests not on the degree of sweetness or uniformity of mixture, but rather on a rationalist theory of relations, that is, Pythagorean proportional theory. As is well known, what Boethius had to say on the matter is quite important, because his treatise is the portal through which many treasures of ancient Greek musical theory flowed to the West. We should not, however, necessarily ascribe the Pythagorean theory and opinions of the *De musica* to Boethius himself. The *De musica* is certainly derived from an earlier work or works and in large part is probably a translation of Nicomachus' *On music*.⁹

In earlier chapters of the *De musica* and in its companion work, the *De arithmetica*, which is a Latin translation of Nicomachus' *Introduction to Arithmetic*, Boethius developed the proportional theory that he invokes here to dismiss the eleventh. I base my presentation of this

theory, below, on the work by Nicomachus and on Theon of Smyrna's *Expositio*.¹⁰

There exist five categories of ratios or relations between unequals that are ordered in the following manner (Diagram #1).

- (1) multiple: the greater term contains the lesser exactly some number of times greater than one.
- (2) superparticular: the greater term contains the lesser once plus one part of the lesser.¹¹
- (3) superpartient: the greater term contains the lesser once plus more than one part of the lesser.
- (4) multiple superparticular: the greater term contains the lesser more than once plus one part of the lesser term.
- (5) multiple superpartient: the greater term contains the lesser more than once plus more than one part of the lesser.

Each category also includes a reciprocal relation, the submultiple for example, which amounts to a simple restatement of the original relationship in the passive voice. This qualitative order of ratios represents a departure from both unity and equality found in the ratio 1:1. or $x:x$.

By relating any two terms from the quaternary known as the *tetrad*, that is, 1, 2, 3, 4, we see that the resultant ratios all fall into the first two categories, multiple and superparticular (Diagram #2).¹² This is precisely how the Pythagoreans defined the numerical characterization of musical consonance. The Euclidean treatise *Sectio canonis*, which is purported to date from as early as 300 B.C.,¹³ states:

And we know regarding notes that some are consonant and some are dissonant, and that [two] consonant [notes] produce a single blend from the two, and dissonant [notes] do not. And so this being the case, it is reasonable that consonant notes, since both produce a single blend of sound, when related numerically to one another by a single name, are either multiple or superparticular.¹⁴

Here we find the connection between the notion of a blend or mixture and categories of ratios. The famous astronomer Ptolemy, in his *Harmonics*, confirms this to be the Pythagorean position on consonance,¹⁵ and all this appears in the *De musica* (1.4-7) of Boethius and thus was transmitted to the Middle Ages.

To understand exactly why the eleventh was rejected, let us review the Pythagorean association of consonance with numerical ratios. Our earliest authority for this is Philolaus, a Pythagorean of the fifth century B.C., who states: "The measure of *harmonia* is the *syllaba* and the *dioxeian*. . . . The *syllaba* has a sesquitertian ratio, the *dioxeian* a sesquialter ratio, and the diapason a double ratio."¹⁶ Certainly the terminology used by Philolaus, *syllaba* for fourth and *dioxeian* for fifth, is

All ratios are reduced to their lowest terms.

NAME	ALGEBRAIC FORM	EXAMPLES
(1) multiple	$nx : x, n > 1$	2:1, 3:1, 4:1, 5:1
(2) superparticular	$(x + 1) : x, x > 1$	3:2, 4:3, 5:4, 6:5
(3) superpartient	$(x + m) : x, x > m > 1$	5:3, 7:4, 7:5, 8:5
(4) multiple superparticular	$(nx + 1) : x, x > 1 \text{ and } n > 1$	5:2, 7:2, 7:3, 10:3
(5) multiple superpartient	$(nx + m) : x, x > m > 1 \text{ and } n > 1$	8:3, 11:3, 11:4, 15:4

Diagram 1. Pythagorean Proportional Theory

$$\text{tetrad} = \{1, 2, 3, 4\}$$

$$\text{multiple ratios} \quad \left\{ \begin{array}{l} 2:1 = \text{octave} \\ 3:1 = \text{octave} + \text{fifth} (= \text{“twelfth”}) \\ 4:1 = \text{double octave} \end{array} \right.$$

$$\text{superparticular ratios} \quad \left\{ \begin{array}{l} 3:2 = \text{fifth} \\ 4:3 = \text{fourth} \end{array} \right.$$

Diagram 2

striking. These remarks serve not only as the earliest numerical characterization of musical intervals in the West, but also as the earliest grammatical or linguistic representation of music.¹⁷ Returning to the issue at hand, we can compose an eleventh from Philolaus' remarks. The sum of 4:3 plus 2:1—one "adds" ratios by treating them as fractions and multiplying—holds the ratio 8:3. Boethius composes an eleventh using 8:6 and 6:3, arriving also at 8:3.¹⁸

$$\begin{array}{l} \text{fourth} + \text{octave} = \text{eleventh} \\ 4:3 + 2:1 :: 8:3 \end{array}$$

$$\begin{array}{l} \text{Boethius, } \textit{De musica}, 2.27 \\ \text{fourth} + \text{octave} = \text{eleventh} \\ 8:6 + 6:3 :: 8:3 \end{array}$$

Thus the eleventh is a multiple superpartient, the category of ratio recognized by the Pythagoreans as the furthest removed from unity and equality. In addition to the quality of ratio involved, the Pythagoreans rejected the eleventh for another reason, that is, because the ratio involved the number 8. Orthodox Pythagorean theory recognizes five consonances: fourth, fifth, octave, twelfth, and double octave; and these are represented by the multiple and superparticular ratios derived from the tetrad (Diagram #2). The number 8 obviously does not belong to the tetrad.

On a much more general level, quaternaries such as the tetrad played a role in Pythagoreanism that equals and perhaps surpasses that of trinities in Christianity. Several authors from around the second and third centuries A.D. inform us that the quaternary 1, 2, 3, 4 assumed the position of an oath in Pythagorean circles. The Pythagoreans named the number 10 *tetractys* because it was the sum of the first four numbers. The most common oath associated with the tetractys was:

Nay, [I swear] by him who bequeathed to our soul the tetractys,
Source containing the root of everlasting nature.¹⁹

Another oath, transmitted by Iamblichus, states:

What is the oracle at Delphi? The tetractys;
that is, the harmony in which the Sirens sing.²⁰

The Sirens must be those of Plato's myth of Er (*Republic* 616Bff).

Authors of late antiquity specify that the tetractys referred to in these oaths is the quaternary 1, 2, 3, 4. Several centuries earlier, Aristotle cited Plato's *Discourses on Philosophy* (lost) as the source for connecting reason (*νοῦς*) with unity, science (*ἐπιστήμη*) with two, opinion (*δόξα*) with the number of a plane surface, and sensory perception

(αἰσθησις) with the number of a solid.²¹ The author of the *Theologoumena arithmeticae*, Nicomachus, Iamblichus, or both, who may be quoting from Speusippus (ca. 407–339 B.C.), makes overt the connection between the tetrad and spatial dimensions (from none to three): one with the point, two with the line, three with the triangle, and four with the tetrahedron.²² Thus the veneration of the tetrad, so common in late antiquity, stretches back at least to Plato's Academy if not to an earlier time.

Besides the tetrad, ancient writers concerned themselves with the important quaternaries of the four elements (fire, air, water, earth) and the four geometric figures associated with these elements (tetrahedron, octahedron, icosahedron, cube). Theon lists eleven such quaternaries, using the word *tetractys* to refer to them,²³ and to these should be added the four mathematical sciences. Archytas, a Pythagorean scientist and politician of the fourth century B.C., was the first to specify the four mathematical sciences.²⁴ Several centuries later, Iamblichus sanctified this organization of science by quoting from Pythagoras: "And indeed there are four steps in set order to ascend to wisdom: (1) arithmetic, (2) music, (3) geometry, (4) spherics."²⁵ Finally, Boethius christened this quaternary *quadrivium* at the beginning of his *De arithmetica*.²⁶

Within the realm of music, one can easily find additional quaternaries such as the tetrachord—the basic system or arrangement of notes in ancient Greek music—and the helicon—a four-string device used to test intervals. According to the Pythagoreans, however, the most important musical quaternary besides the tetrad was the musical tetractys 6, 8, 9, 12. To appreciate the importance of this quaternary, let us delve a bit more into Pythagorean arithmetic.

In addition to ratios, which involve two terms, the Pythagoreans, in their discussions of sound, employed numerical means, which require three terms. Archytas tells us that there are three means peculiar to music (Diagram #3).²⁷

- (1) The arithmetic mean between two terms is such that the first term exceeds or falls short of the mean by the same amount that the mean exceeds or falls short of the second term.
- (2) The geometric mean between two terms is such that the first term is related to the mean in the same way as the mean is related to the second term.
- (3) The harmonic mean between two terms is such that by whatever part of itself the first term exceeds or falls short of the mean, the mean exceeds or falls short of the second term by the same part of the second.

The connection between the consonant ratios of the tetrad and the musical means of Archytas occurs in the musical tetractys 6, 8, 9, 12,

Suppose $x > z$.

(1) y is an arithmetic mean between x and z if: $x - y = y - z$.

Example: $x = 3, y = 2, z = 1$ $3 - 2 = 2 - 1$

(2) y is a geometric mean between x and z if: $x \div y = y \div z$.

Example: $x = 4, y = 2, z = 1$ $4 \div 2 = 2 \div 1$

(3) y is a harmonic mean between x and z if: $(x - y) \div x = (y - z) \div z$.

Example: $x = 6, y = 4, z = 3$ $(6 - 4) \div 6 = (4 - 3) \div 3$

Diagram 3. Musical Means

for this quaternary embodies not only three of the five consonant ratios, but also two of the three musical means.

12:6 holds the duple ratio, 2:1.

12:8 and 9:6 hold the sesquialter ratio, 3:2.

12:9 and 8:6 hold the sesquitercian ratio, 4:3.

9 is the arithmetic mean between 6 and 12, and 8 is the harmonic mean.

Finally, the four numbers embrace a geometric proportion or analogy, 12:9::8:6.

There exists evidence that this quaternary may also antedate the Academy. Iamblichus informs us that Pythagoras himself brought the ultimate proportion (*τελειωτάτη ἀναλογία*), 12:9::8:6, from Babylonia to Greece, and this proportion, called (*mousikē*),²⁸ contains the same consonant ratios as those mentioned by Philolaus.²⁹ The Pythagoreans added together the numbers of 6, 8, 9, 12, as they had 1, 2, 3, 4, deriving special numerological significance from the sum. We read in the “Excerpta ex Nicomacho”:

The first tetractys, even the root of these tetrachords, is in a way the source of all the divisions by genera, perfecting, as I said, the number 36 in conformity with the addition of the stable monad.³⁰

Here the sum of the four numbers 6, 8, 9, 12 is 35, but the author adds the monad, making the sum 36, which is the square of the first perfect number, 6. Rather than adding the monad, Aristides Quintilianus multiplies the sum 35 by the perfect number 6, obtaining 210, which he observes to be the gestation period in days for seven-month fetuses. Similarly, 35 plus the tetractys 10 equals 45, and Aristides multiplies this sum by 6, obtaining 270, the gestation period of nine-month fetuses.³¹ Ancient lore held that seven- and nine-month fetuses survived birth whereas eight-month fetuses did not.

The tetractys called *mousikē* appears in very many musical treatises of late antiquity and the Middle Ages, often in the form of a delightful myth that has Pythagoras discover the numerical ratios by investigating the weights of hammers used in a forge. Recorded for the first time during the second century A.D., the myth exists in many versions, all of which contain an acoustical fallacy.³² Without actually relating the myth, Ptolemy observed that the substances and methods involved—hammers and suspended weights—were unreliable.³³ That the acoustical experiments of the myth did not work out must have only enhanced the perception of Pythagoras as magical. After all, the experiments did work when he performed them.

Boethius recounts the myth and also provides a discussion of why the Pythagoreans rejected the eleventh from the category of consonance.

Boethius underscores his role as reporter, not theoretician, here by noting that Nicomachus had much to say about this interval (no such remarks exist in the extant writings of Nicomachus), and Boethius distinguishes the Pythagorean position of Nicomachus from that of Ptolemy (see below).³⁴ Thus the Pythagorean position on the matter was passed on to the Middle Ages. All this would not amount to very much for medieval theorists or for us if there were not a sense and a theory that judged the eleventh to be consonant. Such a theory of consonance was developed by Ptolemy in his *Harmonics*, where he established four categories of intervals: homophonic, symphonic, emmelic, and ekmelic.

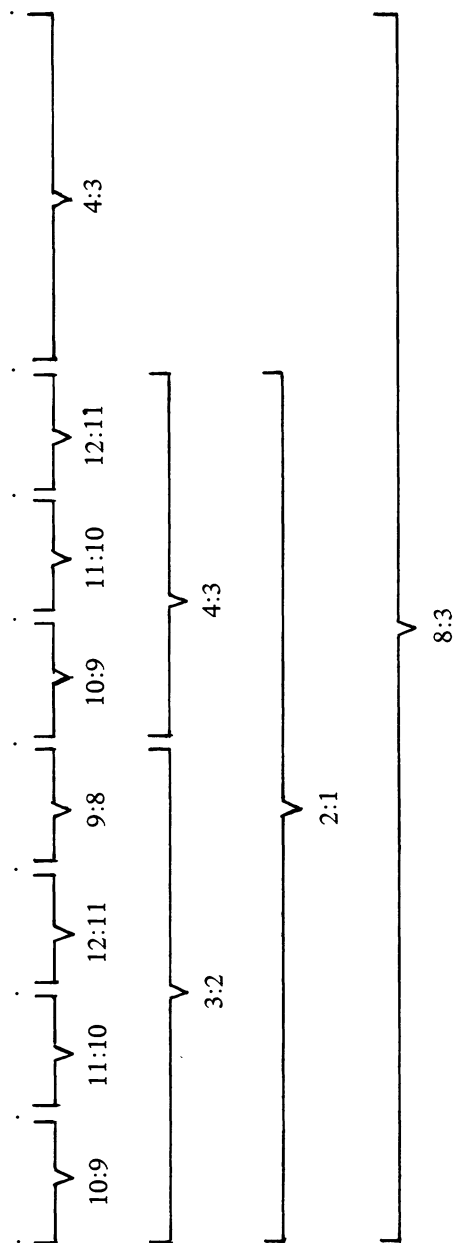
He states: "We define homophones as the [intervals] which, taken at the same time, produce a single impression on the hearing."³⁵ In this category he includes the octave and the double octave. Ptolemy's position here is similar to the one taken in the Aristotelian *Problems* (918a7-8), where the lower note of the octave interval is said to contain the higher. Ptolemy continues:

Next after homophones come symphones such as the fifth and the fourth, and they are put together from themselves and from homophones. Next after symphones come emmelic [intervals] such as whole tones and all the rest of such kind. Just as homophones are put together from symphones, so are symphones from emmelic [intervals].³⁶

Later Ptolemy tells us: "The emmelic are those of the superparticulars after the 4:3."³⁷ All other intervals are ekmelic.

Ptolemy's remarks here are confusing, especially when he tells us that symphones are put together from themselves and from homophones, and then a sentence later, that they are composed of emmelic intervals. Ptolemy's intentions, however, are clear. Of the four categories of intervals, three concern melody: homophonic, symphonic, and emmelic. Of these three categories, two are consonant: homophonic and symphonic. Ptolemy wants to construct a system in which the smallest melodic intervals combine to form larger intervals. In particular, emmelic intervals such as 10:9, 11:10, and 12:11 combine to form symphonic intervals such as 4:3. Symphones such as 4:3 and 3:2 in turn combine to form homophones such as 2:1 (Diagram #4). At the same time, Ptolemy wants to classify as consonant those intervals that are formed by joining the octave to a consonance, a classification that had been established several centuries earlier by Aristoxenus.³⁸ Thus we read that symphones are composed from emmelic intervals as well as from themselves and homophones.

Ptolemy stated in his definition that some symphonic intervals are composed of homophones and symphones, and this is the category,

Diagram 4. Ptolemy, *Harmonics*, 1.7

symphonic, into which the eleventh and the twelfth—octave plus fifth—fall. Ptolemy observes:

In general the consonance of the octave—since the notes do not distinguish is in function from a single note—, when added to some other [interval], preserves the form of that interval unaltered, just as the decad does, that is, when added to other numbers.³⁹

In other words, if we add the number 10 to some other number, let us say 2, the 2 is preserved in the result, 12.⁴⁰

From this theoretical standpoint, Ptolemy criticizes the Pythagoreans for their rejection of the eleventh from the category of consonance. He maintains that they allow their rationalism, based on the tetrad and qualitative proportional theory, to interfere with sensory perception. According to Ptolemy, the eleventh sounds consonant because the fourth is consonant—all ancient theorists agree on this interval—and because the octave, when combined with the fourth, preserves the sound of the fourth. We have here, then, the elements of what appears to be a confrontation between rationalism and empiricism. In addition to the orthodox Pythagorean view regarding the eleventh, Boethius transmits Ptolemy's argument that the eleventh is consonant. Thus the Middle Ages had both points of view from which to choose, and the composite nature of Boethius' *De musica* reflects the equivocal status of the eleventh.⁴¹

Although one can clearly separate the two positions on the eleventh, one cannot divide up the ancient authors neatly into Pythagorean and Ptolemaic camps. For instance, some Pythagorean authors such as Nicomachus and Iamblichus do not mention the eleventh at all. Neither do some Latin authors such as Censorinus and Macrobius, who are working from a mixture, if not a synthesis, of Pythagorean and Aristoxenian musical theory.

We have seen that Ptolemy and Boethius are quite specific in ascribing the rejection of the eleventh to the Pythagoreans. Plutarch in his *On the E(psi)lon at Delphi*, written around the turn of the second century and thus before the treatises of either Ptolemy or Boethius, also rejects the eleventh explicitly, although he neither cites the Pythagoreans nor specifies the mathematical principles that underlie his rejection. Plutarch states (389D-E):

For the primary business of harmonics, so far as it can be said in words, is concerned with consonances. Common sense makes clear to anyone who wishes to seek these things on strings and borings by means of sensory perception in lieu of reason, that there are five [consonances] and no more; for they all originate from numerical ratios. And the ratio of the fourth is sesquitercian, that of the fifth is sesquialter, that of the octave is duple, that of the octave plus fifth is

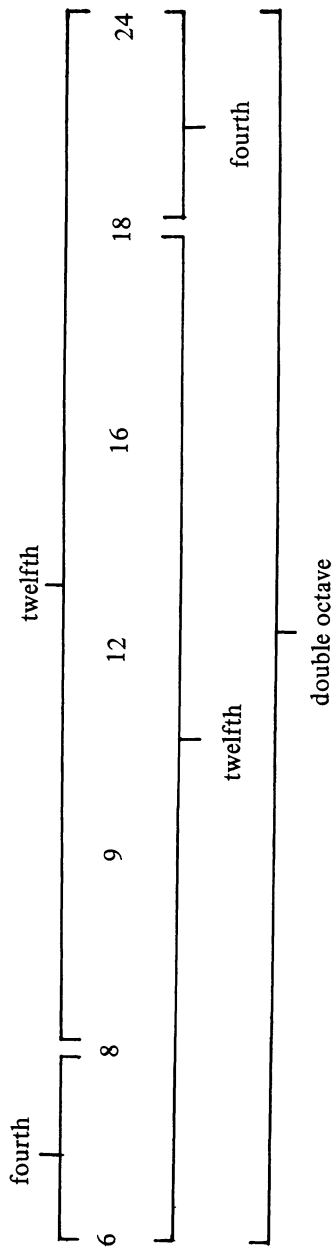
triple, and that of the double octave is quadruple. And that which the harmonists introduce, naming it the octave plus fourth, extending beyond measure, is not worthy of acceptance, because by doing so we would be favoring irrational hearing over—as it were—rational law.⁴²

Plutarch's position is Platonic, that is, a priori mathematical. From a purely mathematical standpoint, one adopted by Socrates (Plato) in the *Republic*, there is no need to discuss the eleventh, and it is noteworthy that Plato does not do so, not even in the *Timaeus*. Only if the subject matter is real sound, as was the case with the Pythagoreans, would the eleventh be a suitable topic for discussion.⁴³

There are several ancient authors besides Ptolemy who classify the eleventh as consonant.⁴⁴ Of these, Theon is the most exasperating; for he tells us that consonances are numerically related according to multiples and superparticulars,⁴⁵ and that: "The tetractys [1, 2, 3, 4] encompasses all the consonances."⁴⁶ But Theon also claims that by joining the octave to other consonances, one produces new consonances, and as one of his examples he gives the eleventh.⁴⁷

We have seen already that Theon presents several quaternaries in his *Expositio*, one of which is *mousikē*, 6, 8, 9, 12. After this, he observes that 8:3 represents the eleventh and 3:1 the twelfth,⁴⁸ and then he incorporates, partially, the two musical quaternaries 1, 2, 3, 4 and 6, 8, 9, 12 (Diagram #5).⁴⁹ He does this by doubling the 12, thereby obtaining 24. He notes that 24:6 represents the double octave, that is, a quadruple ratio, and then he proceeds to the triple ratio, 24:8 or 18:6. Theon notes further that the twelfth plus a fourth yields the double octave, and he gives two numerical characterizations of this composition. He neglects, however, to discuss the eleventh in this passage, and so does Chalcidius, who, in his commentary on Plato's *Timaeus*, follows Theon.⁵⁰

Charging Theon with inconsistency here does little to enlighten us about the classification of consonances and their numerical representations. With Theon's *Expositio*, we may have several historical layers of Pythagorean harmonics, which are inconsistent only when they are comprehended simultaneously. For instance, the tetractys 1, 2, 3, 4 may very well have delimited the numerical characterization of consonant intervals at one time. It is possible that from this quaternary came the general rule stated in *Sectio canonis* about multiple and superparticular ratios. In a different acoustical realm, and probably at a different time, Pythagoreans directed their attention to the incomposite consonances of the fourth and the fifth, their composition to form the octave, and their numerical representation in the musical tetractys 6, 8, 9, 12. From this state of musical theory, one can easily imagine the



twelfth + fourth = double octave

$$24:8 + 8:6 :: 24:6$$

$$18:6 + 24:18 :: 24:6$$

Diagram 5. Theon of Smyrna, *Expositio*, H. 63

acoustical experiments that would cause consideration of composite intervals such as the octave plus fourth, octave plus fifth, and double octave. Two of these intervals were represented already by the tetractys 1, 2, 3, 4, leaving the status and perception of the octave plus fourth open to dispute and neglect.

I turn now to Gaudentius, an ancient author about whom we know virtually nothing except that he lived sometime between the first and sixth centuries A.D. His *Introduction to Harmonics* is a remarkably sectional treatise, the first nine chapters being devoted to Aristoxenian theory, the next eight to Pythagorean theory, and the last five to species of consonances and to notation. In the Aristoxenian portion of the treatise, we read that it is through hearing alone that one recognizes consonance and dissonance, and reason will be of no help in making such judgments.⁵¹ The tone of the treatise changes dramatically at chapter ten where the author takes up Pythagorean theory. In this second section we are treated again to the myth about Pythagoras at the forge, although Gaudentius injects an equivocal note regarding the validity of these experiments. "But being dissatisfied with this attempt alone [hammers and suspended weights], he [Pythagoras] tested the method in another way."⁵² This other way was, of course, by measuring lengths of string and was, therefore, acoustically sound.

The first Pythagorean chapter is the most remarkable of all, because here the author not only shifts abruptly from Aristoxenian to Pythagorean theory, but also expands the musical tetractys 6, 8, 9, 12 in order to accommodate the five consonances of the tetractys 1, 2, 3, 4 as well as the eleventh. We read:

Ratios of consonances are found in numbers and are tested in every way, of which [consonances]: the fourth is sesquitercian, which is 24:18; the fifth is sesquialter, which is 24:16; the octave is double, which is 24:12; the octave together with the fourth is multiple superbipartient, which is 24:9. And again the octave plus fifth is triple, which is 24:8; the double octave is quadruple, which is 24:6.⁵³

24:18 = fourth	24:9 = eleventh
24:16 = fifth	24:8 = twelfth
24:12 = octave	24:6 = double octave

Thus Gaudentius incorporates, or synthesizes the tetractys 1, 2, 3, 4 with the one called *mousikē* and transmitted by the myth. In so doing, he creates a musical heptactys that represents numerically the consonant eleventh, thereby abrogating the old Pythagorean rule regarding multiple and superparticular ratios. Such a synthesis straddles the fence between rationalism and empiricism, and pricks the modern sense of logical

theory. By admitting the eleventh into the category of consonance, Gaudentius apparently acknowledges the empirical fact of the eleventh's consonance. By creating the numerical heptactys that accommodates the eleventh, he provides a numerical metaphor for apparent acoustical truth. In so doing, he relies on those same numerical quaternaries that had excluded the eleventh by their causal nature. Thus the characterization of the *consonant* eleventh as 24:9 indicates the demotion of numerical ratios from cause to metaphor in Pythagorean harmonics.

One might wonder now if the synthesis of Gaudentius was passed on to the Middle Ages. If it was, Boethius did not do it, nor did Martinus Capella. Boethius relies on Pythagorean proportional theory, not quaternaries, when he classifies the eleventh as dissonant, and at no place in the extensive *De musica* does he effect a synthesis of the two quaternaries 1, 2, 3, 4 and 6, 8, 9, 12. The *De nuptiis philologiae et mercurii* of Martianus, which draws its discussion of music in part from the *De musica* of Aristides Quintilianus, serves as another source for the transmission of Greek musical theory to the Middle Ages. In the *De nuptiis*, Martianus takes up the matter of the eleventh and recognizes it as consonant, but like Boethius he does not treat the interval in terms of musical quaternaries.⁵⁴

Besides Gaudentius, I am aware of only one author in late antiquity, Cassiodorus, who uses the number 24 in an attempt to characterize the eleventh. Cassiodorus is also the only author in all of antiquity to mention Gaudentius by name, citing him as an authority on music.⁵⁵ And the eleventh? Cassiodorus judges it to be consonant and lists all six consonances with their characteristic ratios.⁵⁶

The fourth is 4:3.

The fifth is 3:2.

The octave is 2:1.

The eleventh is 24:8.

The twelfth is 3:1.

The double octave is 4:1.

Of course, the eleventh is 24:9, not 24:8. If we can trust the manuscript tradition of the *Institutiones*, Cassiodorus has apparently made this interval *too* consonant by rendering it as a triple ratio.⁵⁷ It is noteworthy that Cassiodorus represents the other five consonances with ratios drawn from the tetrad, that is, with the smallest integers capable of expressing such ratios. In the case of the eleventh, however, he has chosen 24:9, or 24:8, rather than 8:3, or even 16:6. Rather than choosing the smallest integers capable of representing the eleventh, Cassiodorus has chosen a ratio that involves the number 24. As we have seen already, Cassiodorus did not get this representation of the eleventh from Boethius' *De musica*, at least not in the form that we now know the treatise. There are, however, numerous connections between Cassiodorus and Boethius, although all of these may have been superficial.

Both Cassiodorus and Boethius were members of the Roman government under Theodoric, the Ostrogoth king, and from Cassiodorus' letters we learn that Boethius' services were highly valued for a time by Theodoric.⁵⁸ Cassiodorus tells us also that Boethius translated works on: arithmetic by Nicomachus, the *De arithmetica*; geometry by Euclid, presumably lost; astronomy by Ptolemy, presumably lost; and music by Pythagoras, presumably the *De musica*.⁵⁹ Thus there exists the possibility that Cassiodorus derived some of his information about musical science from writings and translations of Boethius that no longer exist.⁶⁰ Of all the extant ancient sources, however, only Gaudentius, in his synthesis of the two musical quaternaries, uses the number 24 to represent the eleventh.

* * *

The several centuries that we call medieval comprise significant musical creativity as well as musical adoption and adaptation. From *Musica disciplina* and *Musica enchiriadis* to the treatises of Jacques de Liège and Marchetto da Padova, the musicologist tries to piece together the musical history of these centuries, although frequently lacking all the pieces to do so. Commensurate with the wide span of time and profound musical developments of the Middle Ages, the musical treatises of this period are varied both in scope and in purpose.⁶¹ My intent here is not to tackle the entire range of medieval musical theory, nor even to trace the treatment of the eleventh throughout this period, although such a trophy hunt might be the kind of study that ultimately would facilitate the writing of a history of medieval music theory—not an imminent achievement. Rather than these ambitious projects, I confine my attention here to the appearance and treatment of the eleventh in a few musical treatises or manuscript traditions of the Middle Ages. The treatment of the eleventh in these works proves to be of little philosophical interest. A study of the eleventh in these treatises, however, may shed some light on the transmission of musical science from late antiquity to the Middle Ages. My investigation, primarily theoretical or conceptual, cannot replace the more tangible work of manuscript studies, but rather may serve as a guide to subsequent work with primary sources copied during those centuries.

The matter of the eleventh appears in many medieval manuscripts. A perusal of Martin Gerbert's *Scriptores* produces no fewer than fifteen "treatises" that discuss the matter.⁶² These discussions vary widely from mere mention of the interval by Aurelian and Regino to extensive treatment in *Musica enchiriadis*. Some authors never mention the eleventh, and the extent to which the eleventh is discussed in a treatise may be a function of the author's intention for writing the treatise. Thus Guido,

who is concerned primarily with teaching the singing of chant, makes no mention of the matter, although he does transmit the Pythagorean myth of the hammers at the end of the *Micrologus*.⁶³ The mixture in Guido's writings of Pythagorean harmonic science with the practical, *cantus* tradition connected to the church marks a middle period in medieval musical theory. The synthesis of music as skill (*peritia*) with music as knowledge (*scientia*) had its roots *at least* in the reappearance on the continent of Boethius' *De arithmetica* and *De musica* during the ninth century.⁶⁴ I show below that the two works by Boethius plus the *De nuptiis* of Martianus as we know them do not account entirely for the transmission of *scientia* to the Middle Ages.

The *Epistola de harmonica* assigned to Regino treats the eleventh in cursory fashion, rejects it from the category of consonance, and repeats the standard Pythagorean argument received through Boethius.⁶⁵ Hucbald, on the contrary, identifies six consonances in his *De harmonica institutione*. In other words, he classifies the eleventh as consonant, although he provides no special justification for his classification. Hucbald adopts the notion of simple and compound consonances from Boethius' *De musica* (5.12), which in turn derived the classification of consonance from Ptolemy's *Harmonics* (1.7). But whereas Ptolemy and Boethius divide the six consonances into three groups (octave and double octave, fourth and fifth, eleventh and twelfth), Hucbald divides the six into two groups. The simple consonances are the fourth, fifth, and octave. The compound consonances are the eleventh, twelfth, and double octave.⁶⁶ By the time of Hucbald, that is, the early tenth century, the *De musica* of Boethius not only had reappeared on the continent but also had permeated the musical theory of the time.⁶⁷ The treatment accorded the eleventh in three treatises preceding the *De harmonica institutione*, that is, *Musica disciplina*, *Musica* and *Scolica enchiriadis*, provides us with a more perplexing picture of the eleventh's status, a picture that appears to depend less than wholly on the *De musica* of Boethius.

With the *Musica disciplina* of Aurelian, mid-ninth century, we have an early confluence of musical theories and traditions. As is well known, the *Musica disciplina* is devoted primarily to the understanding and classification of the *cantus* tradition. In these endeavors, Aurelian relies on modal classification from Byzantium, and indeed his work is the first theoretical treatment of such kind in the West.⁶⁸ Aurelian also presents the scientific theory of music that he had learned from Boethius and apparently from Cassiodorus. *Musica disciplina* is the first medieval treatise to quote substantial portions of Boethius' *De musica*. We have earlier references to the *De musica*, however, by Remigius of Auxerre, and the *De arithmetica* of Boethius was discussed by medieval writers at a time no later than the beginning of the ninth century.⁶⁹

In the sixth chapter of *Musica disciplina*, Aurelian appears to vacillate between five and six consonances. He states: "Music, moreover, consists of all six consonances, fifteen sounds, and eight tones."⁷⁰ He follows this with a list that includes the eleventh. Who is Aurelian's source here? A few sentences later, Aurelian refers to Boethius, lists the consonances with their characteristic ratios, but omits the eleventh. Unless Aurelian had been relying on the fifth book of Boethius' *De musica*—and he certainly was not—, there is no reason to expect a numerical treatment of the eleventh as a consonance at this point. Upon concluding his foray into Pythagorean arithmetic, however, Aurelian again refers to the six consonances. The six consonances, I think, come ultimately from Cassiodorus, the five from Boethius. I have shown already that Cassiodorus did not derive all of his musical science from the *De musica* of Boethius, but rather relied in part on the lost Latin translation of Gaudentius' *Introduction to Harmonics*. Therefore, when Aurelian classifies the eleventh as consonant, he is relying ultimately on Gaudentius and not on the fifth book of Boethius's *De musica*, which in turn was a loose translation of the first book of Ptolemy's *Harmonics*.

I turn now to the perplexing *Musica enchiriadis* and its equally perplexing companion work, the *Scolica enchiriadis*. With the *Musica enchiriadis* we find a self-conscious mixture of the *cantus* tradition with the harmonic science of antiquity.⁷¹ We find also the famous early discussion of organum, and one interval at which the treatise postulates organum is the octave plus fourth. The treatise even refers to the interval as the *eleventh (undecimum)*.⁷² As for the status of the eleventh, the author of our treatise determines it to be consonant and cites Ptolemy as the ultimate authority on the matter. In fact, *Musica enchiriadis* refers specifically to Ptolemy's fifth book on music, quotes from the fifth book of Boethius' *De musica* (ch. 9), and concludes the quotation with a citation of Boethius: "Haec quidem Boethius."⁷³ This portion of Boethius' treatise is explicitly based on Ptolemy's *Harmonics* (1.7.15–16). Although there exist a few soft spots in the quotation from Boethius,⁷⁴ we can safely conclude that Boethius is indeed the source for the argument presented in *Musica enchiriadis*. Thus the *Musica enchiriadis* claims that the eleventh sounds the same as the fourth because the octave preserves any consonance with which it is joined, just as does the decad when joined to another number.⁷⁵ Furthermore, one of the diagrams that accompanies the eleventh chapter of *Musica enchiriadis* presents the two-octave system and a poem that refers to intervals of the system. Regarding the eleventh we read: "Take care so that Ptolemy's dictum prevails here."⁷⁶

The *Scolica enchiriadis* is divided into three parts, the first of which is devoted to the *cantus* tradition as represented in the companion *Musica enchiriadis*. The second section introduces some ancient harmonic

science and the third, even more. The second part begins with a division of the *six* consonances into two groups, simple and compound.⁷⁷ This classification is the same as that found later in Hucbald's *De harmonica institutione*, which in turn was an adaptation or distortion of the classification found in Ptolemy's *Harmonics* and transmitted by Boethius. As this section of the *Scolica* proceeds, we find reliance on the *De arithmetica* of Boethius⁷⁸ and on the *Institutiones* of Cassiodorus.⁷⁹ The *Scolica* then expands the musical tetractys in the same way that Gaudentius did, and discusses all the possible numerical characterizations of the six consonances (Diagram #6),⁸⁰ referring to the interval in question as the *eleventh (undecima)*. One must wonder whether these numerical characterizations are original with the *Scolica*, whether they represent an elaboration and correction of those characterizations found in Cassiodorus' *Institutiones* (see above, p. 207), or whether they are based on yet other sources.

In the third section of the *Scolica*, the author tells us: "The principles are 1, 2, 3, 4; for the numerical quaternary relates completely all the consonances."⁸¹ Again the tetractys is expanded to 24, and even to 48, but no mention is made of the eleventh. We have seen this apparent contradiction or vacillation between five and six consonances before in *Musica disciplina*. But the Middle Ages inherited the problem from antiquity, although they did not do so from the works of Boethius that we know. Although the *De musica* of Boethius presents both the Pythagorean position (five consonances) and the Ptolemaic (six consonances), the treatise clearly distinguishes between the two positions. The same cannot be said for Theon's *Expositio* (see above, p. 204) nor for *Musica disciplina* or *Scolica enchiriadis*.

That the number of and numerical representation of consonances were confusing to the medieval theorists, as they are to us who try to read the writings of these theorists, is exemplified by Gerbert's Anonymous I. The author of the manuscript expands the tetractys to 24 but replaces 16 with 14. Thus the characterization of the eleventh—if we can trust Gerbert's edition—becomes 14:6, which the author labels a "double sesquitercian" ratio (*duplex sesquitercia* [sic]) rather than a double plus sesquitercian.⁸²

The consonant nature of the eleventh and the interval's numerical characterization attracted some attention throughout the later Middle Ages. For most of the later authors we can discern their sources—most often, Boethius—and thus the eleventh no longer serves as a guide to the transmission of musical science.⁸³ Byzantine theorists also devoted some attention to the eleventh and its numerical characterization in their revival of ancient mathematical thought, although they relied, of course, on the ancient Greek sources rather than on Boethius.⁸⁴

In contrast to early medieval theorists in the West, some later writers

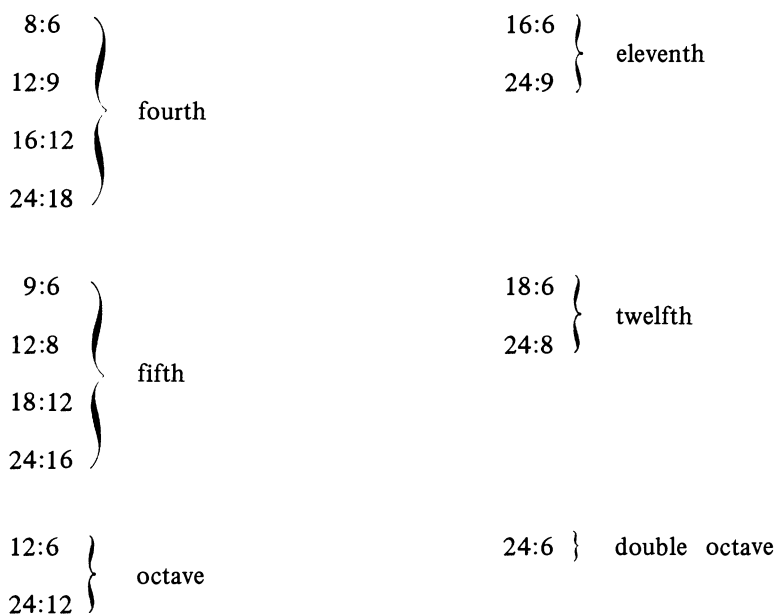


Diagram 6. *Scolica enchiriadis*, S.2.202ff

mounted *original* philosophical arguments against the Ptolemaic position. Walter Odington attacks, in particular, Ptolemy's analogy between the diapason and the decad.⁸⁵ Relying on Boethius' contribution to the analogy (see n. 40), Odington observes that the 2 is indeed preserved when added with 10 to form 12, but he counters that intervals are properly represented by ratios, not by numbers. Odington gives as an example the ratio 10:5. He notes that this is duple and thus consonant. By adding 2 to the 10 in this ratio, one produces 12:5, a superpartient ratio, which is, of course, dissonant.

Jacques de Liège takes a slightly different tack than Walter.⁸⁶ He argues that the value of an interval must be judged according to its extremes, and not according to its constituent parts. The justification for this eminently Pythagorean position stems from Jacques's refusal to allot special status to the octave. Jacques notes that two strings sounding the octave *do not* sound like one string, thus denying a basic part of Ptolemy's argument. If there is not a special category of consonance, such as Ptolemy's homophones, then there is no point in dealing with special combinations of consonant intervals, such as the octave plus fourth. Noting that the octave, represented by 2:1, is indeed consonant, Jacques observes that such an interval can be composed from the intervals 8:7 and 7:4, neither of which is consonant. Just as one does not judge the octave according to the parts 8:7 and 7:4, one should not judge the eleventh according to the octave plus fourth. Rather than this, the eleventh should be judged by its extremes, that is, 8:3, or 16:6, or 24:9. Each of these ratios, of course, is a multiple superbipartient and therefore represents a dissonant interval.

* * *

The evidence that I have presented here covers a wide span of time, from the Presocratics to the late Middle Ages, and is largely conceptual or theoretical rather than corporeal. The embodiment of these theories in manuscripts covers a much smaller span of time, beginning with the ninth century and extending to the late Middle Ages. Although the classification of intervals is a rather airy topic, I believe that the scope of my study is sufficiently narrow so as to shed light on the perception of consonance and also, perhaps, to provide some clues regarding the transmission of musical thought. We have seen that some ancient authors make a synthesis of the two musical quaternaries that lie at the heart of Pythagorean theory. Theon expands the musical tetractys and thus represents the quadruple ratio 24:6 and the triple 24:8. Gaudentius goes one step further and represents the eleventh, 24:9. It seems more than likely, furthermore, that Cassiodorus knew of this expansion.

The major sources of Greek musical theory for the medieval writers,

Boethius' *De musica* and Martianus' *De nuptiis*, treat the eleventh, but not in the context of the musical tetractys 6, 8, 9, 12.⁸⁷ And yet we find medieval treatises, especially the *Scolica enchiriadis*, that treat the eleventh in the same fashion as did Gaudentius. Now the representation of the eleventh as 24:9 is not a difficult arithmetic formulation, and any author could have happened upon it. But why use 24:9 to represent the eleventh rather than 8:3 or 16:6? The numerical heptactys that we find in ancient and medieval treatises, 6, 8, 9, 12, 16, 18, 24, could be used to represent a two-octave system, but it is ill suited to the Greater Perfect or Immutable system. If one were to attempt such a characterization according to either frequency or string length, the movable notes Paranete diezeugmenon and Lichanos hypaton would be represented in the diatonic genus at the expense of the standing notes Paramese and Hypate hypaton.

Frequency		String length
24	Nete hyperbolaeon	6
18	Nete diezeugmenon	8
16	Diatonic Paranete diezeugmenon	9
12	Mese	12
9	Hypate meson	16
8	Diatonic Lichanos hypaton	18
6	Proslambanomenos	24

The numerical sequence 4, 6, 8, 9, 12, 16 would characterize the frequencies of the Greater Perfect System better than the heptactys, omitting only one standing note, Hypate hypaton, while including no movable notes, and such a numerical sequence can be found in medieval and Renaissance treatises.

Frequency	
16	Nete hyperbolaeon
12	Nete diezeugmenon
9	Paramese
8	Mese
6	Hypate meson
4	Proslambanomenos

The original sequence from 6 to 24 would be equally ill suited to represent the medieval revision of the Greater Perfect System, which centered attention on the tetrachord of the modal finals (d, e, f, g). One must conclude, therefore, that the choice of 24:9 to represent the eleventh rather than 8:3 or 16:6 was made under the strong influence of the musical tetractys.

All this is pretty scanty stuff from which to argue that the author of a medieval treatise, for example, *Musica disciplina* or *Scolica enchiriadis*, was drawing from Gaudentius or even from Cassiodorus, although the evidence that I have presented indicates that this connection is worth pursuing further. Have portions of Boethius' *De musica* been lost in the transmission? The incomplete version of book 5 would seem to indicate so. Moreover, the letters of Cassiodorus ascribe to Boethius translations of works on the ancient mathematical sciences, the quadrivium. We have only two of these four works. These missing or incomplete works as well as the Latin translation by Mutianus of Gaudentius' *Introduction to Harmonics* may have served directly, or indirectly through Cassiodorus, as important sources for the transmission of ancient musical learning. Furthermore, Byzantine singers and sources may have provided the West with a link to the musical knowledge of antiquity, although the extant Byzantine sources are late (see n. 84). In any event, the evidence that I have presented argues for the long-lived effect of the musical tetractys, discovered by Pythagoras in Babylonia.

Regarding the perception of consonance, is the eleventh consonant? The question draws one to the center of the philosophical debate over rationalism versus empiricism. It is no surprise that some of the practical medieval treatises on music, Guido's *Micrologus* for instance, do not take up the matter at all. Unless the practitioner postulates singing in parallel elevenths, as does *Musica enchiriadis*, there is no need to discuss the interval. But if the disposition of a musical treatise is *why* rather than *how*, then the eleventh *is* an important matter.

Breaking down Ptolemy's argument into its parts, the first is that the octave is consonant. Historically in the West, this claim is irrefutable and has been for millennia. Rationalists and empiricists agree here, as do most modern students of acoustics. The second part of the argument claims that the fourth is consonant, and again from a historical point of view, this is true throughout antiquity and well into the late Middle Ages. The third part of Ptolemy's argument is *not* that a consonance plus a consonance yields a consonance, although Walter Odington, in his rebuttal to Ptolemy, demonstrates that a fifth plus a fifth produces a dissonance.⁸⁸ Ptolemy knew this and demonstrated it also in his *Harmonics*.⁸⁹ Ptolemy claims, rather, that the octave, along with the double octave, is *more* consonant than the other consonances. Finally, the octave is *so* consonant that it almost sounds like a unison, thus when added to another consonance the octave preserves the consonant character of the other consonance.

I believe that the third part of Ptolemy's argument—that the octave is more consonant than the fourth or fifth, for example—is also borne out historically in the West, although it may be going too far to claim unanimity on the matter. That the octave is so special that it preserves

the character of a consonance to which it is added is another matter entirely. Ptolemy claims that he hears this to be the case. In effect, so do Gaudentius and the author of *Musica enchiridis*. We have the basis here for an empirical fact, that is, a fact contingent upon sensory perception. Of course, sensory perception varies from person to person. From this argument, then, the status of the consonant eleventh could never attain a position higher than or different from one contingent upon unanimity of sensory perception. As we have seen, the eleventh does not even get that far.

In what amounts to a synopsis of Boethius' remarks on the Pythagorean notion of consonance, Jehan des Murs writes:

The Pythagoreans rejected the octave plus fourth [from the category of] sounding a good harmony because it fell on the hearing neither sweetly nor pleasantly. Inquiring why, they discovered [the interval] itself to be outside of the multiple and superparticular varieties.⁹⁰

In other words, the Pythagoreans reject the eleventh from the category of consonance because it is neither sweet nor pleasing to the ears. Why? Because it is neither multiple nor superparticular. This is the connection between numerical and acoustical truth first made by *Sectio canonis* (see above, p. 194).

When Ptolemy makes his empirical claim, this entails the qualitative notion of a sweet and pleasing blend or mixture of sounds. These metaphors of mixture and sweetness were used to describe consonance throughout the early history of Western music. But we have seen that ancient and medieval writers differ on whether the eleventh is a sweet mixture. This is so, in part, because such a description is both metaphorical and qualitative. The numerical ratios used by the Pythagoreans, however, are neither metaphorical nor qualitative. Clearly the quantitative aspect of the Pythagorean argument has the advantage of being immutable. But there is more than this. The orthodox Pythagoreans claim that numbers do not represent reality, but rather they *are* reality.⁹¹ Thus the Pythagoreans avoid metaphor in their discussion of consonance. This is the point made by Jehan des Murs when he states that the eleventh is not pleasing to the ears *because* it is neither a multiple nor a superparticular ratio.

The issue of consonance and dissonance was for the Pythagoreans not a matter of devising a theory that was harmonious with their hearing, but rather one of hearing the numerical truth that they discovered to be inherent to nature.

NOTES

NOTE: I read a short version of this article at the Annual Meeting of the American Musicological Society (Boston, November, 1981).

1. Barbara Münxelhaus surveys some of the information presented here in her *Pythagoras musicus. Zur Rezeption der pythagoreischen Musiktheorie als quadrivieraler Wissenschaft im lateinischen Mittelalter* (Bonn-Bad Godesberg: Verlag für systematische Musikwissenschaft, 1976), 88–94. The scope of my study is considerably more narrow than hers and, I think, more philosophical and exegetical.
2. See Carl Dahlhaus, “Ein vergessenes Problem der antiken Konsonanztheorie,” *Festschrift für Walter Wiora zum 30. Dezember 1966*, eds. Ludwig Finscher and Christoph-Hellmut Mahling (Kassel-Basel: Bärenreiter, 1967), 164–169.
3. The phrase “*μα κροβω*,” meaning to strike or play at the same time, is usually rendered with *κροβω* in a participial form. See Aristides Quintilianus, *De musica libri tres*, ed. Reginald P. Winnington-Ingram (Leipzig: Teubner, 1963), p. 10.2 and 10.4; Gaudentius, *Introduction to Harmonics*, in *Musici scriptores graeci*, ed. Karl von Jan (1895; reprint, Hildesheim: Olms, 1961), p. 337.8–9 (hereafter *JanS*); Nicomachus, *Manual of Harmony*, *JanS* 262.2–3; Porphyry, *Porphyrios Kommentar zur Harmonielehre des Ptolemaios*, ed. Ingemar Düring (1932; reprint, New York: Garland Publishing, 1980), p. 114.11; Theon of Smyrna, *Expositio rerum mathematicarum ad legendum Platonem utilium*, ed. Eduard Hiller (Leipzig: Teubner, 1978), p. 51.3. Boethius translates this phrase into Latin as *simul pulso*, *De institutione arithmetica libri duo. De institutione musica libri quinque*, ed. Godofred Friedlein (1867; repr. Frankfurt: Minerva, 1966), pp. 302.3 and 349.7.
4. *De musica*, F. 195.6–8 and 348.23 – 349.1.
5. But see the discussion of intervals by Aristides Quintilianus where he implies that consonant and dissonant intervals are melodic, *De musica* 1.7. In all matters regarding Aristides Quintilianus, see Thomas J. Mathiesen’s translation of and commentary on the *De musica*, *On Music. In Three Books* (New Haven: Yale University Press, 1983).
6. *ibid.*, F. 259.10–13. “Quid igitur, si diatessaron ac diapason consonantias iungamus, ullamne secundum Pythagoricos efficient consonantiam? Minime.”
7. *ibid.*, F. 195.6–8. “Consonantia est acuti soni gravisque mixtura suaviter uniformiterque auribus accidens.”
8. *ibid.*, F. 259.13–15. “in superpartiens inaequalitatis genus cadit, nec servat vel multipliciter ordinem vel superparticularitatis simplicitatem.”
9. See Calvin M. Bower, “Boethius and Nicomachus: An Essay Concerning the Sources of *De Institutione Musica*,” *Vivarium* 16 (1978): 1–45, and Ubaldo Pizzani, “Studi sulle fonti del *De Institutione Musica* di Boezio,” *Sacris erudiri* 16 (1965): 5–164.
10. Nicomachus, *Introductionis arithmeticae libri ii*, ed. Richard Hoche (Leipzig: Teubner, 1886), pp. 44.8 – 72 and 119.19 – 144.19. See also my “*Republic* 530C – 531C: Another Look At Plato And The Pythagoreans,” *American Journal of Philology* 102 (1981): n. 29.
11. Nearly all modern writers on the subject of Pythagorean arithmetic represent

the superparticular ratio as $(x + 1) : x$, assuming the ratio to be in its lowest terms. This representation is incorrect because it admits 2:1, a multiple ratio, into the category of superparticular. Both Nicomachus, *Introductionis arithmeticae*, H. 49.1–4, and Theon, *Expositio*, H. 76.21 – 77.2, carefully define superparticular so as to exclude 2:1.

12. The word *τετράς* means, literally, *the number four*. The word also refers to the first four integers, 1, 2, 3, 4. See: Armand Delatte, *Études sur la littérature pythagoricienne* (1915; reprint, Geneva: Slatkine Reprints, 1974), 253ff. Thus I distinguish, as do some ancient authors, between the *tetrad*, 1, 2, 3, 4, and a *tetractys* (τετρακτύς), that is, a quaternary.
13. Many of the mathematical and acoustical principles contained in *Sectio canonis* are doubtless very old. Far less certain is the date of composition of the treatise as we know it, that is, introduction and twenty propositions. *Sectio canonis* is transmitted in three versions: one ascribed to Euclid; one contained in Porphyry's commentary on Ptolemy's *Harmonics*, D. 99 – 103.25; and one contained in the *De musica* of Boethius, F. 301.6 – 308.15. A comparison of these three versions reveals an unstable, probably living tradition of Pythagorean acoustical science that may have received an embodiment in the form of the *Sectio* at a time considerably later than that of Euclid.
14. *JanS* 149.17–24. “Γινώσκουμεν δὲ καὶ τῶν φθόγγων τοὺς μὲν συμφώνους ὄντας, τοὺς δὲ διαφώνους, καὶ τοὺς μὲν συμφώνους μίαν κρᾶσιν τὴν ἐξ ἁμφοῦν ποιῶντας, τοὺς δὲ διαφώνους οὐ· τοῦτων οὕτως ἐχόντων εἰκὸς τοὺς συμφώνους φθόγγους, ἐπειδὴ μίαν τὴν ἐξ ἁμφοῦν ποιῶνται κρᾶσιν τῆς φωνῆς, εἶναι τῶν ἐν ἐνὶ ὀνόματι πρὸς ἀλλήλους λεγομένων ἀριφμῶν, ἥτοι πολλαπλασίους ὄντας ἢ ἐπιμορίους.” Immediately preceding this passage, we read: “Of these, the multiple and the superparticular are ordered one to another by a single name.” (τοῦτων δὲ οἱ μὲν πολλαπλασίοι καὶ ἐπιμόριοι ἐνὶ ὀνόματι λέγονται πρὸς ἀλλήλους, *JanS* 149.14–16.) The phrase “a single name” or “one term” (ἐνὶ ὀνόματι) is difficult to interpret here. Thomas Mathiesen maintains that “one term” means *consonant*, citing Porphyry's commentary on Ptolemy's *Harmonics* (D. 98.3–6) as support, “An Annotated Translation of Euclid's Division of a Monochord,” *Journal of Music Theory* 19 (1975): n. 12. See my discussion of this phrase in “Placing *Sectio canonis* in Historical and Philosophical Contexts,” *The Journal of Hellenic Studies*, forthcoming.
15. Ptolemy, *Die Harmonielehre des Klaudios Ptolemaios*, ed. Ingemar Düring (1930; reprint, New York: Garland Publishing, 1980), 11.
16. Hermann Diels, *Die Fragmente der Vorsokratiker*, 10th ed., rev. Walther Kranz, 3 vols. (Berlin: Weidmann, 1960), 44 B 6. “ἁρμονίας δὲ μέγεθος ἐστὶ συλλαβὰ καὶ δι' ὅξειαν . . . ἃ δὲ συλλαβὰ ἐπίτριτον, τὸ δὲ δι' ὅξειαν ἡμιόλιον, τὸ διὰ πασῶν δὲ διπλόν.”
17. The analogy between the rational combination of letters to form syllables, syllables to form words, and the rational combination of notes to form intervals, intervals to form melodies, occurs frequently in ancient treatises. See my “The Persistence of Pythagorean Mathematics in Ancient Musical Thought” (Ph.D. diss., Univ. of North Carolina at Chapel Hill, 1980), ch. 5, nn. 20 and 37, as well as *JanS* 254.4ff and 417.12–18.
18. Boethius, *De musica* 2.27.
19. Aëtius, *Placita* 1.3.8, in Hermann Diels, *Doxographi graeci*, 4th ed. (1879; repr. Berlin: De Gruyter, 1965), 282.

“οὐ μὰ τὸν ἀμετέρα ψυχᾷ παραδόντα τετρακτύν,
παγὰν ἀενάου φύσεως ῥίζωμά τ' ἔχουσιν.”

Cf. Iamblichus, *De vita pythagorica liber*, ed. Ludwig Deubner (Leipzig: Teubner, 1937), p. 85.4–5, and Sextus Empiricus, *Adversus mathematicos* 4.2, in *Sexti Empirici opera*, ed. Hermann Mutschmann (vols. 1 and 2) and Jürgen Mau (vol. 3) (Leipzig: Teubner, 1912 (vols. 1 and 2), 1954 (vol. 3)) 3: 133.16–17. Theon, *Expositio*, H. 94.6–7, is identical to Sextus Empiricus above.

20. Iamblichus, *De vita pythagorica*, D. 47.15–16.

“τί ἐστι τὸ ἐν Δελφοῖς μαγεῖον; τετρακτὺς
ὅπερ ἐστὶν ἡ ἁρμονία, ἐν ᾗ αἱ Σειρήνες.”

I have adopted Walter Burkert's translation of this oath; see: Walter Burkert, *Lore and Science in Ancient Pythagoreanism*, trans. Edwin L. Minar, Jr. (Cambridge, Mass.: Harvard University Press, 1972), 187. See, also, Paul Kucharski, *Étude sur la doctrine pythagoricienne de la tétrade* (Paris: Les Belles Lettres, 1952), 75–77.

21. Aristotle, *De anima* 404b18–27. Cf. Aëtius, *Placita* 1.3.8, Diels, *Doxographi graeci*, 280–283.

22. Iamblichus, *Theologoumena arithmeticae*, ed. Victorius de Falco (Leipzig: Teubner, 1922), 82–85.

23. Theon, *Expositio*, H. 94ff.

24. Diels, *Die Fragmente der Vorsokratiker* 47 B 1.

25. Iamblichus, *Theologoumena arithmeticae*, de F. 21.8–10. “τέτταρες μὲν καὶ ταὶ σοφίας ἐπιβάδραι, ἀριθμητικὰ μουσικὰ γεωμετρία σφαιρικά, ἀ β γ δ τεταγμένα.”

26. F. 7.21ff.

27. Diels, *Die Fragmente der Vorsokratiker* 47 B 2.

28. Iamblichus, *In Nicomachi arithmeticae introductionem*, ed. Hermanegildus Pistelli (Leipzig: Teubner, 1894), pp. 118.19–24 and 122.26–27.

29. Modern scholars have tended to dismiss many of Iamblichus' remarks as fanciful, but this skeptical assessment results in part from our inability to unearth his written sources, if in fact they ever existed.

30. *JanS* 282.10–14. “ἡ δ' οὖν πρώτη τετρακτὺς, ἡ καὶ τῶν τετραχόρδων τούτων ῥίζα, τρόπον τινὰ πασῶν ἐστὶ παρεκτικὴ τῶν κατὰ γένη διαιρέσεων, ἀποτελοῦσα ὡς ἐφάμην τὸν λς ἀριθμὸν τῆς ἀπλανοῦς οἰκείου προσθήκης μονάδος.”

31. Aristides Quintilianus, *De musica* 3.18.

32. Of the many renditions of this myth, Nicomachus presents the earliest in his *Manual of Harmony*, *JanS* 245.19 – 248.26. See Barbera, “The Persistence of Pythagorean Mathematics,” 313–315.

33. Ptolemy, *Harmonics*, D. 17.19–20. See also Flora R. Levin, “πληγή and τάσις in the Harmonika of Klaudios Ptolemaios,” *Hermes* 108 (1980): 205–229.

34. Boethius, *De musica* 2.27.

35. Ptolemy, *Harmonics*, D. 15.10–12. “ὀριζέσθωσαν δὲ ἡμῖν ὁμόφωνοι μὲν οἱ κατὰ τὴν σύμφωνον ἐνὸς ἀντίληψιν ἐμποιοῦντες ταῖς ἀκοαῖς.”

36. *ibid.*, D. 15.12–17. “σύμφωνοι δὲ οἱ ἐγγυτάτω τῶν ὁμοφώνων, ὡς οἱ διὰ πέντε καὶ οἱ διὰ τεσσάρων καὶ οἱ ἐξ αὐτῶν καὶ τῶν ὁμοφώνων συντιθέμενοι, ἐμμελεῖς δὲ οἱ ἐγγυτάτω τῶν συμφώνων, ὡς οἱ τονιαῖοι καὶ τῶν τοιοῦτων οἱ λοιποί. διὸ καὶ συντίθενται πως οἱ μὲν ὁμόφωνοι τοῖς συμφώνοις, οἱ δὲ σύμφωνοι τοῖς ἐμμελεῖσι.”

37. *ibid.*, D. 16.24–25. “ἐμμελεῖς δὲ οἱ μετὰ τὸν ἐπίτριτον τῶν ἐπιμορίων.”
38. Aristoxenus, *Aristoxeni elementa harmonica*, ed. Rosetta da Rios (Rome: Typis Publicae Officinae Polygraphicae, 1954), pp. 25.18 – 26.1 and 56.10–19.
39. Ptolemy, *Harmonics*, D. 13.3–7. “καθόλου γὰρ ἡ διὰ πασῶν συμφωνία, τῶν ποιούντων αὐτὴν φθόγγων ἀδιαφορούντων κατὰ τὴν δύναμιν ἐνός, ὅταν προσα-
φθῇ τινι τῶν ἄλλων, ἀπαράτρεπτον τὸ ἐκείνης εἶδος τηρεῖ, καθάπερ ἡ δεκάς
ἔχει, φέρε εἰπεῖν, πρὸς τοὺς ὑπ’ αὐτὴν ἀριθμούς.”
40. The specific example of 10 plus 2 is Boethius’ contribution to Ptolemy’s re-
remark, *De musica*, F. 360.11–14.
41. Cf. also the intervallic tests contained in Ptolemy’s *Harmonics* 1.8 and Boe-
thius’ *De musica* 4.18. As Bower has pointed out, the chapters are related, but
whereas Ptolemy tests *all* of the consonant intervals (4:3, 3:2, 2:1, 8:3, 3:1,
4:1), Boethius tests only four (4:3, 3:2, 2:1, 3:1), omitting both the eleventh
and the double octave, “Boethius and Nicomachus,” 37–38. Bower proposes
that we have an “edition” of Ptolemy preserved in the *De musica*, undertaken
by Nicomachus, and that the omission of 8:3 in Boethius’ tests stems from
the Pythagoreans’ rejection of this interval. This may be so, although we must
question why the double octave, 4:1, is also absent from Boethius’ tests.
42. Plutarch, *Moralia*, 7 vols. (Leipzig: Teubner, 1925–1967), vol. 3 ed. W.R.
Paton, M. Pohlenz, and W. Sieveking (1929), pp. 12.22 – 13.7. “τὸ γὰρ
πλείστον ὡς ἔπος εἰπεῖν ἔργον ἀρμονικῆς περὶ τὰς συμφωνίας ἐστίν, αὐταὶ δ’
ὅτι πέντε καὶ οὐ πλείους, ὁ λόγος ἐξελέγχει τὸν ἐν χορδαῖς καὶ τρυπήμασι
ταῦτα θηρᾶν ἀλόγως τῇ αἰσθήσει βουλόμενον. πάσαι γὰρ ἐν λόγοις τὴν γένεσιν
ἀριθμῶν λαμβάνουσιν· καὶ λόγος ἐστὶ τῆς μὲν διὰ τεσσάρων ἐπίτριτος, τῆς δὲ
διὰ πέντε ἡμιώλιος, διπλάσιος δὲ τῆς διὰ τασῶν, τῆς δὲ διὰ πασῶν καὶ διὰ πέντε
τριπλάσιος, καὶ τετραπλάσιος τῆς δις διὰ πασῶν. ἣν δὲ ταῦταις ἐπεισάγουσιν οἱ
ἀρμονικοὶ διὰ πασῶν καὶ διὰ τεσσάρων ὀνομάζοντες ἔξω μέτρου βαίνουσιν,
οὐκ ἄξιόν ἐστι δέχεσθαι τῆς ἀκοῆς τῷ ἀλόγῳ παρὰ τὸν λόγον ὥσπερ νόμῳ
χαριζομένους.”
43. Barbera, “The Persistence of Pythagorean Mathematics,” 1–5 and 61–83.
44. For example, Cleonides, *JanS* 194.9–12, and Baccheius, *JanS* 294.13–16.
45. Theon, *Expositio*, H. 50.16–19.
46. *ibid.*, H. 58.13. “πάσας δὲ τὰς συμφωνίας περιέχει ἡ τετρακτὺς.”
47. *ibid.*, H. 52.1–9.
48. *ibid.*, H. 62.1 – 63.2.
49. *ibid.*, H. 63.2–24.
50. Chalcidius, in *Platonis Timaeum*, W. 113.1 – 114.19.
51. *JanS* 330.15–20.
52. *JanS* 341.12–13. “Ἄλλ’ οὐδὲ τῇ πείρᾳ τούτων ἀρκεοῦσι μόνῃ βασανίζει
τρόπον ἄλλον τὴν μέθοδον.”
53. *JanS* 339.21 – 340.3. “Λόγοι δὲ εἰσιν ἐν ἀριθμοῖς ἡρρημένοι τῶν συμφωνιῶν
καὶ δοκιμασθέντες ἀκριβῶς πάντα τρόπον τῆς μὲν διὰ τεσσάρων ἐπίτριτος,
ὃν ἔχει τὰ κδ πρὸς τὰ ιη· τῆς δὲ διὰ πέντε ἡμιώλιος, ὃν ἔχει ὁ κδ πρὸς τὸν ις·
τῆς δὲ διὰ πασῶν διπλάσιος, ὃν ἔχει ὁ κδ πρὸς τὸν ιβ· τῆς δὲ διὰ πασῶν τε-
τραπλάσιος, ὃν ἔχει ὁ κδ πρὸς τὸν θ. καὶ πάλιν
τῆς μὲν διὰ πασῶν καὶ διὰ πέντε τριπλάσιος, ὃν ἔχει ὁ κδ πρὸς τὸν οκτώ. τῆς
δὲ δις διὰ πασῶν τετραπλάσιος, ὃν ἔχει ὁ κδ πρὸς τὸν ξ.”
54. Martianus Capella, *De nuptiis philologiae et mercurii*, ed. Adolf Dick (1925;
reprint, Stuttgart: Teubner, 1978), pp. 507.14 – 509.5.

55. Cassiodorus, *Cassiodori Senatoris institutiones*, ed. R. A. B. Mynors (Oxford: Clarendon Press, 1937), pp. 142.13–17 and 149.16–19.
56. *ibid.*, M. 144.21 – 145.19.
57. *ibid.*, M. 145.13–15.
58. Cassiodorus, *Varie*, ed. Theodore Mommsen (Berlin: Weidmann, 1961), 1.10 and 45, 2.40. As is well known, Theodoric's devaluation of Boethius' services around 524 culminated in Boethius' execution.
59. Cassiodorus, *Varie*, 1.45. See Calvin M. Bower, "The Role of Boethius' *De Institutione Musica* in the Speculative Tradition of Western Musical Thought," *Boethius and the Liberal Arts. A Collection of Essays*, ed. Michael Masi (Berne: Peter Lang, 1981), 160–162.
60. The role played by Cassiodorus in the transmission of ancient knowledge to the Middle Ages has generated considerable interest and speculation on the part of historians. The famed library at the monastery Vivarium apparently did not survive Cassiodorus' death around 583, but the contents of this library may have been dispersed throughout early medieval Europe. In this context, see: Pierre Courcelle, *Late Latin Writers and Their Greek Sources*, trans. Harry E. Wedeck (Cambridge, Mass.: Harvard University Press, 1969), 334; James O'Donnell, *Cassiodorus* (Berkeley: University of California Press, 1979); and Günter Ludwig, *Cassiodor. Über den Ursprung der abendländischen Schule* (Frankfurt am Main: Akademische Verlagsgesellschaft, 1967).
61. See Lawrence A. Gushee, "Questions of Genre in Medieval Treatises on Music," *Gattungen der Musik in Einzeldarstellungen. Gedenkschrift Leo Schrade*, ed. Wulf Arlt, Ernst Lichtenhahn, and Hans Oesch (Bern: Francke, 1973), 365–433.
62. Martin Gerbert, *Scriptores ecclesiastici de musica sacra potissimum*, 3 vols. (1784; reprint, Milan: Bollettino Bibliografico Musicale, 1931) (hereafter *GS*).
63. Guido, *Guidonis Aretini Micrologus*, ed. Joseph Smits van Waesberghe, *Corpus scriptorum de musica* (hereafter *CSM*), 4 ([Rome]: American Institute of Musicology, 1955), 228–233.
64. See Bower, "The Role of Boethius' *De Institutione Musica*," 157–174.
65. Region of Prüm, *Epistola de harmonica*, *GS* 1.238.
66. *De harmonica institutione*, *GS* 1.107.
67. Bower, "The Role of Boethius' *De Institutione Musica*," 172.
68. See especially "De tonis octo," ch. 8, sentences 1–21, and subsequent chapters, *Aureliani Reomensis Musica disciplina*, ed. Lawrence Gushee, *CSM* XXI (1975), 78–79ff. The "De tonis octo" most likely antedates the rest of *Musica disciplina*.
69. Bower, "The Role of Boethius' *De Institutione Musica*," 167–170.
70. Ed. Gushee, 71. "Constat autem omnis musica simphoniiis sex, sonitibus quindecim, tenoribus octo."
71. Bower, "The Role of Boethius' *De Institutione Musica*," 171–172. See: *Musica et scolica enchiridis, una cum aliquibus tractatulis adiunctis*, ed. Hans Schmid (Munich: Verlag der Bayerischen Akademie der Wissenschaften, 1981), ch. 19, especially lines 33–40, and the review of Schmid's edition by Nancy Phillips, *Journal of the American Musicological Society* 36 (1983): 128–142.
72. *Musica enchiridis*, ed. Schmid, 11.40.
73. *ibid.*, 16.39.
74. Cf. Boethius, *De musica*, F. 359.1 and 359.19–20 with *Musica enchiridis*, ed. Schmid, 16.17–18 and 32–33 respectively.

75. Ed. Schmid, 16. For the analogy between the octave and the decad, see 11. 24–45.
76. Ed. Schmid, 30. “attende ut dictum Ptolomei regnet in ore.”
77. *ibid.*, 2:1–3.
78. For example, cf. *Scolica enchiriadis*, ed. Schmid, 2.165–168 with Boethius, *De arithmetica*, F. 8.8–11.
79. For example, cf. *Scolica enchiriadis*, ed. Schmid, 2^o 160–161 with Cassiodorus, *Institutiones*, M. 130.19–20.
80. Ed. Schmid, 2:202ff.
81. *ibid.*, 3:225–226. “Principales sunt I II III IV. Quaternarius enim numerus omnes symphonias perfecte absoluit.”
82. *Musica*, *GS* 1.334.
83. Berno of Reichenau presents the quaternary 1, 2, 3, 4 and notes the numerological value of its sum, 10, drawing analogies with the ten-string psaltry used in praise of God and with the four evangelists, *GS* 2.66. When it comes to the eleventh, however, he sides with Ptolemy and cites Boethius’ instance of the analogy to the decad, *GS* 2.64–65. Wilhelm of Hirsau, like Berno, deems the eleventh to be consonant, *Willelmi Hirsaugensis Musica*, ed. Denis Harbinson, *CSM* 23 (1975), 39–40. Marchetto, in the *Lucidarium*, reviews the entire matter and discusses quaternaries, referring to the four evangelists, *GS* 3.84. (In this regard, see also Aurelian’s remarks about four and the name of God, *Musica disciplina*, ed. Gushee, 79.) Marchetto ultimately sides with Ptolemy and in what amounts to a radical transformation of theory in order to accommodate empirical evidence, states: “of the five kinds of inequality in music, only three are necessary, namely the multiple, . . . the superparticular, . . . and the *multiple superbipartient*” (italics mine), *GS* 3.88. (de quinque generibus inaequalitatis in musica, omnino tria sunt necessaria, scilicet multiplex, . . . superparticulare, . . . & multiplex superbipartiens, . . .)
84. See: the eleventh-century treatise on the quadrivium traditionally ascribed to Michael Psellus, *Anonymi logica et quadrivium, cum scholiis antiquis*, ed. J. L. Heiberg, in ser. *Det Kgl. Danske Videnskabernes Selskab historisk-filologiske Meddelelser* 15, 1 (Copenhagen: 1929); Georgius Pachymeres, *Quadrivium de Georges Pachymère*, ed. Paul Tannery, text ed. E. Stéphanou, *Studi e testi* no. 94 (Vatican: Biblioteca Apostolica Vaticana, 1940), 124–126; and Manuel Bryennius, *The Harmonics of Manuel Bryennius*, ed. and trans. G. H. Jonker (Groningen: Wolters-Noordhoff Publishing, 1970), pp. 98.27ff and 144.21ff. See also: Thomas J. Mathiesen, “Aristides Quintilianus and the *Harmonics* of Manuel Bryennius: A Study in Byzantine Music Theory,” *Journal of Music Theory* 27 (1983): 31–47.
85. *Walteri Odington Summa de speculatione musicae*, ed. Frederick F. Hammond, *CSM*, 14 (1970), 72–73.
86. *Jacobi Leodiensis Speculum musicae*, ed. Roger Bragard, 7 vols., *CSM* 3 (1955–1973), 2:248–252 and especially 252–258.
87. Isidore of Seville, who relied on Cassiodorus for much of his information about music, does not take up the matter of the eleventh at all. See: “de numeris musicis,” *Isidori Hispalensis episcopi Etymologiarum sive originum libri xx*, ed. Wallace M. Lindsay (Oxford: Clarendon Press, 1911), 3.23; and Bower, “The Role of Boethius’ *De Institutione Musica*,” 162–163.
88. *Summa de speculatione musicae*, 72–73.

89. D. 12.

90. *Musica speculativa secundum Boetium*, GS 3.271–273. “Pythagorici noluerunt diapason & diatessaron sonare bonam harmoniam, eo quod nec dulciter nec suaviter veniret ad auditum. Causam inquirentes invenerunt, ipsam esse extra genus multiplicis & superparticularis.”

91. Barbera, “Interpreting an Arithmetical Error in Boethius’s *De Institutione Musica* (iii.14–16), *Archives internationales d’histoire des sciences* 31 (1981): 40–41.

