

C3
Philosophy

Fresh Breezes in the Philosophy of Mathematics[†]

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Since Pythagoras, philosophy of mathematics tried to account for mathematical existence and the nature of mathematical objects.

Numbers, circles, n -dimensional manifolds, all are different from everything else we think about. They're neither physical nor mental. Not mental, because the Pythagorean theorem or any other well-established mathematical fact is independent of what you or I think. Whether we know it and believe it or don't know it and don't believe it, the Pythagorean theorem is still true. Yet it's not physical either! Plato and Aristotle explained that the triangles and circles of the geometer are not physical triangles or circles, but something "ideal."

Spiritual, empirical, psychological, formalist, and logicist explanations have been offered. None give a credible account of what we do when we do mathematics. Presently some authors are constructing a humanist answer.

An Israeli mathematics education researcher, Anna Sfard, recently found an interesting insight. In learning a mathematical concept, children first learn it as algorithm—procedure, or method. Later, the algorithm is transformed into an object. She calls this "reification." It's difficult to achieve, often needing help from teacher. This story is close to theories of the Russian psychologist, Lev Vygotsky.

For example, subtraction is an algorithm. It isn't hard. It reifies into negative numbers—very hard!

Which mathematical entities are frozen algorithms? What's the interaction between doing and being, algorithm and entity? *This is a question in philosophy of mathematics based on mathematical practise, on seeing mathematics as a human activity. It's not a foundationist question.*

FOUNDATIONS LOST. In books on philosophy of mathematics (Korner, or Benacerraf & Putnam) you read of the leading problem, "foundations." How can we establish mathematical knowledge as certain, indubitable, free of any possible doubt? Three historically important solutions to this problem were logicism (Platonism), formalism, intuitionism. All were unsuccessful. For logicism and formalism, no major new idea has come up in over half a century. Intuitionism and its daughter constructivism did strive to carry out the program of Brouwer streamlined by Bishop. But their goal of remaking mathematics constructively is more remote today than 60 or 70 years ago.

The surviving scrap of foundationalism was named "neo-Fregeanism" by Philip Kitcher. This notion still dominates the philosophy of mathematics. It says:

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1366
HERSH, R.
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"Philosophical thinking about mathematics need not concern itself with anything but sets, and set theory's twin sister, logic." But most researchers, users, teachers, historians of mathematics aren't primarily interested in sets. So philosophers of mathematics ignore mathematics and mathematicians, and mathematicians find nothing of interest in philosophy of mathematics.

Deplorable! The principal problem in philosophy of mathematics left in paralysis for over half a century! Mathematicians and philosophers of mathematics ignorant of each other's existence! A Harvard philosopher, Hilary Putnam, published a foundationalist paper titled "Mathematics Without Foundations." Is philosophy of mathematics pointless and unnecessary? Or is it time for a fresh start?

PHIL / M AND PHIL / SCI. One weird phenomenon of modern philosophy is that philosophy of science and philosophy of mathematics are almost disjoint. Authors in philosophy of science rarely refer to philosophy of mathematics, and vice versa. An author who writes on both subjects, in any one article sticks to one or the other. It's like baseball and football—play one or the other, but not both at the same time.

I like to compare philosophy of mathematics today to philosophy of science in the 30's and 40's. That subject was dominated by logical positivists: Rudolf Carnap and his friends of the "Wiener Kreis" (Vienna Circle). As a result of taking Bertrand Russell and Ludwig Wittgenstein too seriously, they believed they knew the correct methodology for scientific work: (1) state the axioms; (2) give correspondence rules between words and physical observables; (3) derive the theory, as Euclid derived geometry, or Mach derived mechanics.

It was noticed after a while that what logical positivists said had little in common with what scientists did or wanted to do. New ideas in philosophy of science came from Karl Popper, Tom Kuhn, Imre Lakatos, Paul Feyerabend. These subversives disagreed with each other. But they all thought philosophers of science could think about what scientists actually do, not bring presuppositions and instructions for scientists to ignore.

Philosophy of mathematics is overdue for its Popper, Kuhn, Lakatos, and Feyerabend. It's overdue for analysis of what mathematicians actually do, and the philosophical issues therein.

In fact, this turn is taking place. Wittgenstein and Lakatos helped start it. In recent years Michael Polanyi, George Polya, Alfred Renyi, Leslie White, Ray Wilder, Greg Chaitin, Phil Davis, Paul Ernest, Nick Goodman, Phil Kitcher, Penelope Maddy, Michael Resnik, Gian-Carlo Rota, Brian Rotman, Gabriel Stolzenberg, Robert Thomas, Tom Tymoczko, Jean Paul van Bendegem, and Hao Wang have participated.

Here are ideas some of these people hold.

1) Mathematics is human. It's part of and fits into human culture. (Not Frege's abstract, timeless, tenseless, objective reality.)

2) Mathematical knowledge is fallible. Like science, mathematics can advance by making mistakes and then correcting and recorrecting them. (This "fallibilism" is brilliantly argued in Lakatos' *Proofs and Refutations*.)

3) There are different versions of proof or rigor, depending on time, place, and other things. The use of computers in proofs is a nontraditional version of rigor.

4) Empirical evidence, numerical experimentation, probabilistic proof all help us decide what to believe in mathematics. Aristotelian logic isn't necessarily always the best way of deciding.

5) Mathematical objects are a special variety of social-cultural-historical object. We can tell mathematics from literature or religion. Nevertheless, mathematical objects are shared ideas, like Moby Dick in literature, or the Immaculate Conception in religion.

How do humanists answer the big question, "What's the nature of mathematical objects?"

The question seems difficult because of a centuries-old assumption in Western philosophy: "In the world there are two kinds of things. What's not physical is mental; what's not mental is physical." When Frege proved that mathematics is neither physical nor mental, he accounted for it by means of a third kind of entity—"abstract objects"—about which he could say nothing except that they're neither physical nor mental.

Mental is thought, individual consciousness, subjectivity; wishes, fears, perceptions, hopes, desires, private thoughts.

Matter is what takes up space, has weight, can be studied by scientific instruments. Mountains, bugs, the stars, gamma rays.

Is there anything that's neither mental nor physical? Yes! Sonatas. Poems. Churches. Religions. Diplomas. Armies. Wars. Universities. Academies of science!

Does the New York Academy of Science exist? Undoubtedly. Is it mental? If the Secretary and the President of the Academy died of amnesia, the life of the Academy would continue. The Academy isn't just somebody's thoughts! Even if the building were blown up and the trustees moved the Academy to Yonkers, it would go on. Its physical and mental embodiments are necessary, but they're not *it*. The Academy isn't just the minds and bodies of anyone. Neither is it just the stones of its building.

What is it? It's a social institution. The mental and physical aren't sufficient to describe the New York Academy of Science. Nor are they sufficient to describe most of the things that most concern us. Marriage and divorce, employment, shopping, prices and salaries, war and peace, professional sports and television shows. All have mental and physical aspects, but they aren't mental or physical entities. They're social entities.

There are not two but three basic kinds of things in the world.

Now, what about mathematical objects—let's just say numbers. If everything's either mental, physical, or social, then what are numbers? We've already seen that numbers aren't mental or physical. By the law of the excluded middle, they must be social. But let's not be peremptory. Let's consider it a hypothesis. Is mathematics social-cultural-historical?

Certainly it's historical. The history of mathematics is a developed subject. Historians have studied mathematics back to the Babylonians. We don't know the remote origin of mathematics, or the remote origin of writing, speech, religion, or the family. That origin was part of the self-creation of the human race. Archeology, linguistics, genetics, ethnology tell us a little more. Counting and talking both had their human beginnings.

Mathematics is a social entity. Mathematicians never were isolated hermits. Today they're in academic, government or industrial jobs, paid directly or indirectly by the government.

Srinivasa Ramanujan, the self-taught Indian mathematical genius, worked hard to be recognized by the English mathematics establishment. Once he was invited, he went to England, at a cost to his family, his religious commitment, and his ability to find daily food he could eat. His did so in order to work with mathematicians who understood what he was doing.

In the 16th and 17th centuries, Fermat, Huygens, Leibnitz were assiduous letter writers, constantly trading ideas with colleagues in other cities and other countries.

Today a new result is certified as part of mathematics after experts read it and pronounce it good. We monitor our product. Acceptance by the profession is essential to be recognized or accepted as a mathematician.

The overall content of mathematics and its direction of movement respond to the pressures of society. The militarization of U.S. mathematics in World War II is an example.

Newton's calculus was a tool in his theory of gravitation. His gravitation theory was a response to the need for better understanding of the motions of planets. The motions of planets were important because England was a maritime nation. Navigational methods better than those of Spain and Portugal had cash value for England.

In saying this, I don't underestimate the insistence of pure mathematicians on autonomy.

TAKING THE TEST. To test a philosophy of mathematics, ask it questions:

- (1) What makes mathematics different?
- (2) What is mathematics about?
- (3) Why does mathematics achieve near-universal consensus?
- (4) How do we acquire knowledge of mathematics, apart from proof?
- (5) Why are mathematical results independent of time, place, race, nationality and gender, in spite of the social nature of mathematics?
- (6) Does the infinite exist? If so, how?
- (7) Why does pure mathematics so often become useful?

The humanist approach gives better answers to questions 1 through 5 than the neo-Fregean, the intuitionist-constructivists, or any other proposed philosophy I know of.

Questions 6 and 7 are harder. I don't say humanism answers these questions. But neither does anybody else.

In conclusion, I want to destroy one of the most popular arrows opponents like to shoot at mathematical humanism.

$2 + 2 = 4$, they say, everywhere and always. In fact, $2 + 2 = 4$ before there were human societies, or even human beings. When 2 brontosauruses went to the water hole and met two other brontosauruses, there were four brontosauruses at the water hole. The truths of mathematics are universal, independent not only of individual consciousness but of social consciousness.

This is Platonism, the view that Wittgenstein attacked so fiercely, and the view, let's face it, that most mathematicians accept.

How can a humanist answer?

First of all, "two" plays two roles. It's an adjective and it's a noun. When you say "two brontosauruses," "two" is an adjective. "Two brontosauruses plus two brontosauruses equals four brontosauruses" is a statement about brontosauruses, not about numbers. Even if you say "Two discrete, reasonably permanent, non-interacting objects collected together with two others of the same ilk makes four such objects," you are talking about properties of discrete, reasonably permanent non-interacting objects. That's a statement in elementary physics.

The noun "two," on the other hand, as everybody since Pythagoras knows, doesn't name a physically observable thing. It names some abstract or ideal entity. Plato, Descartes, Frege knew that two is an ideal object. They explained what they meant by an ideal object only in negative terms—not mental, not physical. I'm pointing out that these abstract ideal objects are social concepts.

"But," says the Platonist, "how can you explain the fact that always and everywhere, regardless of time and place, politics or religion, race or sex, $2 + 2$ always equals 4? The only way to account for it is to say it's an objective truth, which we all recognize because it's an objective truth. Otherwise, the universal agreement that $2 + 2 = 4$ would be an inexplicable miracle."

To this I answer, "It's bad logic to say something must be true because you can't think of any other explanation. That's how philosophers used to prove the existence of a Supreme Creator—they couldn't conceive any other way for there to be a universe."

"You say that because I haven't got an explanation that satisfies you about the objectivity of mathematics, therefore I must believe in abstract entities whose relation to the physical world is obscure, which number incredibly remote uncountable infinities, and which are apprehended by our mental or physical faculties in a quite unexplained manner."

"I don't believe in them. You believe in them only by closing your eyes to their absurdity."

"I'm aware that some social or intersubjective concepts have the rigidity, the reproducibility, of physical science. The reproducibility of a mathematical calculation is comparable only to the reproducibility of a physical measurement or experiment."

Somebody might ask, "Why does the physical world have attributes which are so consistent, so reproducible? Why is the gravitational constant the same from one day to the next? Why is the speed of light in vacuum so reliable?"

No physicist or philosopher feels obliged to answer such questions. The possibility of a science of physics is something we accept. We start from there, we don't try to go back of it. Heidegger asked, "Why is there a universe?" I don't know what progress he made. Not a promising investigation.

As there's lawfulness and stability in parts of the physical world, there's lawfulness and stability in parts of the social-conceptual world. I don't know why this is so. I'm sure it's a fruitless question, as fruitless as the same question about the physical world.

Study of the lawful, predictable parts of the physical world has a name. That name is "physics." Study of the lawful, predictable parts of the social-conceptual world has a name. That name is "mathematics."

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PICTURE PUZZLE
(from the collection of Paul Halmos)



A double threat family.
(See page 619)