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Poles and walls in Mesopotamia and Egypt

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Abstract

The problem of a pole or reed leaning against a wall is an example of a mathematical “riddle,” or exercise, that occurs in many cultures, among other “Pythagorean”-type problems. There are a few known instances of the problem in Mesopotamia and Egypt. We analyze the way the problem is treated in these cases, paying particular attention to the structure of the algorithms used to find the solutions.

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Résumé

Le problème d'un poteau ou d'un roseau penchés contre un mur est une énigme mathématique, de type «pythagoricien» comme on en trouve dans beaucoup de cultures. Il en existe quelques exemples en Mésopotamie et en Égypte. Nous analysons la manière dont le problème est traité dans ces cas, en prêtant une attention particulière à la structure des algorithmes employés pour arriver aux solutions.

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MSC: 01A16; 01A17

Keywords: Pythagorean problems; Old Babylonian; Seleucid; Demotic; Reed and wall

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1. Introduction

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period in particular, are mostly school texts (see, for example, Robson [1999, 2000, 2002], Veldhuis [1997]). Høyrup admirably summed up these developments in the field of mathematical Assyriology up to the early 1990s in Høyrup [1996a].

In the past 10 years or so, the themes or trends identified by Høyrup have continued and been joined by others. Høyrup himself has devoted much concern to identifying and separating various strata in ancient mathematics. In particular, he has been interested in what he has termed a “lay,” or “surveyors,” tradition of problems or riddles, in contrast to, and connection with, the scholastic scribal tradition [Høyrup, 1990a, 1996b, 1997a, 1997b, 1997c, 1999, 2001, 2002a, 2002b]. Among the problems whose origins Høyrup identifies in this lay tradition are a number based on Pythagorean triples, of which the most famous in the problem of the pole or reed leaning on a wall. In [1999], Høyrup collected and discussed all instances of the Pythagorean “rule” in Mesopotamian mathematics. Here, we consider only the pole and wall problem and caution against making any wider Pythagorean conclusions without considering how the problem compares with other Pythagorean-type problems.

When the same, or similar, problem occurs in different cultures, the question of transmission of ideas inevitably arises. Pythagorean problems, whether derived from triangles or rectangles, have a wide diffusion and, given the relative simplicity of the underlying ideas, may have appeared independently in many contexts. However, the closer the “clothing” of an abstract problem, that is, the greater the similarity in dressing the problem in practical guise, the greater the suspicion of contact between the authors. Unfortunately, scarcity of sources often militates against proving direct chains of transmission, especially over large reaches of space and time. The question of transmission of mathematical ideas among various ancient cultures is addressed at length in Dold-Samplonius et al. [2002], where, in particular, Høyrup considers the Seleucid disjunction in techniques with older Mesopotamian practices and connections with mathematics in Egyptian, Greco-Roman, Islamic, Indian, and Chinese societies. In his comments on the Mesopotamian and Egyptian pole and wall problems Høyrup notes only that they are “solved similarly” [Høyrup, 2002b, p. 20]. The purpose of this paper is to probe that similarity from a somewhat different direction.

Mesopotamian and Egyptian mathematics have been interpreted in three primary ways: algebraically, geometrically, and algorithmically. As algebra is so deeply embedded in modern culture, it is natural for modern commentators to recast problems in algebraic symbolism and view the ancient procedure as solving an equation. Algebra is a powerful and compact language for communicating mathematical ideas and such an approach provides a convenient means for bridging the gap between ancient mathematics and modern readers. However, the algebra itself is alien to the ancient cultures and obscures questions of how people in those cultures viewed the mathematics they were doing.

As a closer approximation to the conceptual framework of ancient scribes, Høyrup in particular has argued extensively [1990b, 1996a, 1996b, 1997a, 1997b, 1997c, 1999, 2001, 2002a] and persuasively for a “naive,” or cut-and-paste, geometric underpinning for the recorded procedures. His analysis depends on a close reading and technical interpretation of the vocabulary of particular texts and analyzing steps of a procedure as geometric actions. The reconstructed diagrams are not attested in the original sources and remain speculative in certain particulars, although the thrust of his interpretation is convincing.

A third approach, to view the written descriptions of mathematical procedures as algorithms and akin to computer programs has a long history (see, for example, Knuth [1972]) and appeal in an age of computerization. Such an approach focuses on the formal aspects of the procedures, rather than the underlying conceptions. In the context of locating scribal mathematics in Egypt and Mesopotamia within a domain of professionalization encompassing medicine, divination and jurisprudence, Ritter has

developed techniques for exposing the rhetorical construction of the texts [1995, 1998]. This approach has also been adopted by Imhausen in her explorations of the algorithms used in Egyptian mathematics [2002]. Here, we adapt Ritter's ideas to provide a formalism suited to describing the structure of algorithms in a way that allows comparison between procedures from widely differing sources. To provide focus to the technique, we concentrate on a single problem, and consider only its appearance in Mesopotamia and Egypt.

The problem of a ladder, or pole, leaning against a wall has an extensive history and is still a staple exercise in mathematics. Nowadays, it can be found in almost every calculus text, where it is typically used as a related rates problem. Here is an example from a recent version of a calculus book by the ubiquitous Stewart:

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall? [Stewart, 1998, p. 269]

The exercise uses related rate techniques and the Pythagorean theorem. Note that since the ladder is 10 ft long, and the base is 6 ft from the wall at the moment of interest, the resulting Pythagorean triple is 6–8–10, or twice a 3–4–5 triangle.

In the years B.C. (Before Calculus), the rate of motion was not of interest, just the position; ladders were moved, but did not slide. In this form, the problem has a long and venerable tradition and appears in various guises in many cultures. Here, we restrict ourselves to a discussion of the ways in which the problem was used and solved in the mathematics of Mesopotamia and Egypt.¹ The sources come from the Old Babylonian period (ca. 2000–1600), Seleucid Mesopotamia (third or second century), and Ptolemaic Egypt (early third century).

2. Old Babylonian poles

Only one example of this problem survives in the corpus of Old Babylonian mathematics, where it occurs as Problem 9 on the tablet BM 85196, a large (19×12.3 cm) tablet containing 18 problems on a variety of subjects. It is an example of what Friberg calls “recombination texts,” those compiled from a collection of thematic problem texts [Friberg, 1996]. Bought by the British Museum, it does not have a secure archaeological context. However, it shares a number of features with a group of tablets that Robson suggests all come from Sippar and are the work of one Iškur-mansum.² The tablet was originally published by Thureau-Dangin [1935] in copy, photograph, transliteration, translation, and commentary. It was then included by Neugebauer in Volume 2 of *Mathematische Keilschrifttexte* [Neugebauer, 1935–

¹ Fibonacci (ca. 1200) included a version with a pole leaning against a tower as the first problem in the geometric portion of the last chapter of *Liber Abaci* [Sigler, 2002, p. 543]. He used a 12–16–20 triple and presumably took the problem from an Islamic source. In China, the *Nine Chapters* (ca. 100 B.C.) includes a number of similar problems, only one of which actually features a pole and a wall. Another has a reed growing out of a pond that just touches the water when pulled to the bank, a variant which also appears in India [Shen Kangshen et al., 1999, pp. 469–473]. Sesiano traces some of the European history of the ladder and the wall in Sesiano [1987].

² Høyrup also identifies Sippar as the source [2002a, p. 332], including BM 85196 in his Group 6A of groups of texts organized by geographical origin; Friberg also suggests Sippar as the origin of the Group 6A texts, though with less force [2000, p. 169]. The tablet has not yet been dated more precisely than Old Babylonian (ca. 2000–1600).

1937] and commented on again by Thureau-Dangin in *Textes Mathématiques Babyloniens* [1938]. The various problems on the tablet have been the subject of extensive commentary since. A recent translation and transliteration appear in Høyrup [2002a, p. 275].

Small portions of Problem 9 are missing or damaged, including, unfortunately, a part of the first line, but the structure of the problem can be safely restored. The problem reads as follows:

A pole. 30 (a reed). From its . . .
 Above, it has descended 6, below, how far has it moved away?
 You. Square 30: 15 you see. Subtract 6 from 30: 24 you see.
 Square 24: 9,36 you see. Subtract 9,36 from 15:
 5,24 you see. What is the square root of 5,24? 18 is the square root. 18
 on the ground it has moved away. If 18 on the ground
 above, what did it descend? Square 18: 5,24 you see.
 Subtract 5,24 from 15: 9,36 you see. Of 9,36,
 what is the square root? 24 is the square root. Subtract 24 from 30:
 6 you see descended. Such is the procedure.

The problem begins by stating its subject—a pole. Next, the length is given, with a gloss to provide scale. The pole is 30 (nindan) long, that is a reed-length, and so the 30 should be read as 0;30, that is, half a nindan, or approximately 3 m.³ As the rest of the first line is badly damaged, it is not possible to restore completely the statement of the problem. Hence, we cannot be sure which elements were explicitly stated, and which remained implicit. Certainly, the pole begins upright, but there is no explicit mention of a wall.⁴ Much like Stewart, the problem's author is interested in the point at which the top of the pole has moved down 6 (units). The problem is to find how far the base of the pole has moved.

The exercise follows the conventions of Old Babylonian mathematics, where first the problem is stated and then the instructor, addressing the student, “you,” explains the procedures to be followed. As the structure is very formulaic, the terminology often becomes quite abbreviated. Here, all that is left of the phrase, “You, in your workings, . . .” is the imperative, “You.”

Since Mesopotamian problems always use specific numbers for the various quantities, there are two topics of interest: the general algorithm or procedure used, and the specific numbers chosen. We will take these topics in turn.

The organization of the procedure is typical of such Old Babylonian texts. The task is divided into steps. Each step is one simple calculation, usually using the result of the previous step, and the result of each calculation is given. As the student is working with particular numbers, the procedure invokes these data. However, to get a general understanding of the underlying algorithm, it is helpful for the modern reader to have a more symbolic representation. Here, we follow an approach adapted from that introduced by Ritter [1995]. The convention is the following. We denote by D_i the i th piece of data given in the statement of the problem, and by R_j the result of the j th step of the solution procedure. This technique has the advantage of focusing on the steps of the procedure, rather than determining an equation. This formalism represents how the algorithm proceeds, rather than presenting a proof of why it works. We take for granted that the algorithms are “correct”; a concerned reader may easily construct an algebraic proof from the algorithm as a check.

³ We follow Neugebauer's convention for writing multiplace sexagesimal numbers: sexagesimal places are separated by a comma, and, when absolute sizes are given, a semicolon represents the “sexagesimal point,” the position of units.

⁴ However, Muroi has suggested a restoration including the wall [1991].

In this particular case, the problem gives two pieces of numerical data: the length of the pole, and thus the original height, 30, which we label D_1 , and the distance the tip of the pole has moved down, namely 6, which we denote D_2 . The first instruction is to compute with the first piece of data given, although the result of that computation is not immediately used. The second instruction utilizes the second piece of data given. This is a common approach in such algorithms. The layout of the algorithm is given in Table 1. At the conclusion, it is noted that the 18 of R_5 is the quantity sought. The remainder of the problem reverses the procedure in a somewhat abbreviated form, beginning with the 18 and determining that, indeed, the pole has been moved down by 6 from its original position.

It has been common to render the results of these algorithms as algebraic formulas, a practice which obscures the procedural nature of the mathematics. However, tracing through the problem gives the induced equation $R_5 = \sqrt{D_1^2 - (D_1 - D_2)^2}$. In the notation of Fig. 1, recalling that $D_1 = d$, the length of the pole and $D_2 = b$, the amount moved down, then

$$R_5 = \sqrt{d^2 - (d - b)^2} = \sqrt{d^2 - l^2} = a.$$

Two points are important. The first is that the algorithm produces the correct result. The second is that the steps of the algorithm lead towards that solution in a natural order. Applications of the Pythagorean rule in Old Babylonian mathematics occur in a variety of problems. The entire corpus of such problems has recently been analyzed by Høyrup [1999], whose conclusion is that the rule was discovered sometime “between 2025 ... and 1825 B.C.” [Høyrup, 1999, p. 407].

As to the numbers chosen, the length of the pole is the very common 30 (one-half in the sexagesimal system) and the resulting Pythagorean triple is 18–24–30, or a multiple of the basic 3–4–5. It is not, however, the favorite 45–1–1,15 version of the 3–4–5 triple that often appears, perhaps because of a desire to use a more realistic length for the pole.

Table 1
BM 85196, No. 9

	Computation	Symbolic instruction
Step 1	Square 30, result 15	$R_1 = D_1^2$
Step 2	Subtract 6 from 30, result 24	$R_2 = D_1 - D_2$
Step 3	Square 24, result 9,36	$R_3 = R_2^2$
Step 4	Subtract 9,36 from 15, result 5,24	$R_4 = R_1 - R_3$
Step 5	Find the square root of 5,24, result 18	$R_5 = \sqrt{R_4}$

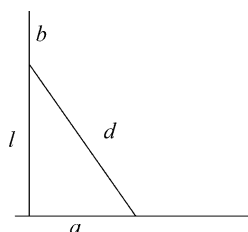


Fig. 1. The pole and the wall.

3. Seleucid walls

Although Old Babylonian mathematics is represented by a fair number of texts, the same cannot be said of subsequent periods. In particular, the dearth of texts means that it is impossible to delineate transmission of ideas through the next 1500 years that separate the Old Babylonian and Seleucid sources. All we can do when faced with a recurrence of similar problems in the Seleucid period is tentatively to decide whether to join the dots of these isolated pieces of evidence. While we have large numbers of Seleucid (and Late Babylonian) astronomical tablets utilizing intricate calculations, there are very few mathematical tablets of the Old Babylonian school type. We are fortunate in having a tablet with an instance of our favorite problem. The date of the tablet is not known; it is perhaps second century.

The example is Problem 12 on BM 34568 (see Tables 2 and 3), a tablet featuring a collection of about twenty problems mostly to do with rectangles; Problem 12 is something of an anomaly (the tablet is broken and the later problems are very damaged). The tablet was originally published by Neugebauer [1935–1937] in Volume III; an up-to-date transliteration and technical translation can be found in Høyrup [2002a, p. 394]. The translation given below is somewhat freer and thus disguises some important features of the text. One is that the vocabulary used is very different from that of Old Babylonian mathematics. While this is perhaps not surprising over a gap of 1500 or so years, the lack of continuity of vocabulary is one strand in Høyrup's argument for a lack of continuity in mathematical traditions. Besides questions of grammar and word choice, there are several other fascinating features of this text. Normalizing the vocabulary, the problem reads as follows:

A reed and a wall I erected. 3 kuš I have gone down,
 9 kuš it has moved away. What is the reed? What is the wall?
 Since you do not know,
 3 times 3: 9. 9 times 9: 1,21. Add 9 to 1,21.
 1,30 times 30 you go: 45. The reciprocal of 3 is 20. 20 times 45 you go:
 15, the reed. What is the wall? 15 times 15: 3,45. 9 times 9:
 1,21. Subtract 1,21 from 3,45, leaving 2,24.
 What times what may I go to get 2,24?
 12 times 12: 2,24. 12 the wall.

Although not entirely clear from the statement of the problem, the set-up is that the reed was moved from a vertical position so that the top descended 3 cubits, the base moved out by 9, and the tip of the reed was then resting on the top of the wall, and here there really is a wall. After the set-up, the questions are given: What is the length of the reed? and What is the height of the wall? The solutions are found in that order. That is, first the length of the reed is found and from that, the height of the wall determined. Interestingly, the height of the wall is computed using the Pythagorean rule, rather than as a subtraction. This suggests that the extended problem is built up from a simpler original. The steps for this subproblem can be summarized in Table 2, where we set D_1 to be the length of the reed, that is, the hypotenuse of the triangle, and D_2 the distance the base moved.

In terms of numbers, the wall is 12 kuš, the reed 15, and the base of the triangle 9 kuš. Hence, the problem uses a 9–12–15 Pythagorean triple, or, once again, a multiple of the basic 3–4–5.

In form, the procedure is similar to that followed in the Old Babylonian text considered above and shows the same facility with “Pythagorean” problems. The more difficult part of the problem is determining the length of the reed in the first place. The given data are $D_1 = 3$, the distance moved down, and $D_2 = 9$, the distance moved from the base of the wall. The procedure is given in Table 3.

Table 2
BM 34568, No. 12, Part 2

	Computation	Symbolic instruction
Step 1	15 times 15: 3,45	$R_1 = D_1^2$
Step 2	9 times 9: 1,21	$R_2 = D_2^2$
Step 3	Subtract 1,21 from 3,45, remainder 2,24	$R_3 = R_2 - R_1$
Step 4	Find square root of 2,24 as 12	$R_4 = \sqrt{R_3}$

Table 3
BM 34568, No. 12, Part 1

	Computation	Symbolic instruction
Step 1	3 times 3: 9	$R_1 = D_1^2$
Step 2	9 times 9: 1,21	$R_2 = D_2^2$
Step 3	9 plus 1,21: 1,30	$R_3 = R_1 + R_2$
Step 4	1,30 times 30: 45	$R_4 = \frac{1}{2} R_3$
Step 5	Reciprocal of 3: 20	$R_5 = \frac{1}{D_1}$
Step 6	20 times 45: 15	$R_6 = R_5 \times R_4$

The final result, R_6 , is the length sought. The algorithm conforms well to the rhetoric of Old Babylonian mathematics we saw above: it begins by using the given data, and then most steps use as input the result of the previous step. Steps 4 and 5 introduce a certain amount of complexity into the procedure and should alert us to a closer study. The solution to the problem is really rather elegant, and I suspect that anyone stumbling through the algorithm without additional explanation would have little feeling for why the obtained result is correct. For the benefit of the algebraically minded, unravelling the procedure and setting d to be length of the reed gives the induced formula:

$$d = R_6 = \frac{1}{D_1} \frac{1}{2} (D_1^2 + D_2^2).$$

The steps in the procedure are the same as would be taken in computing this formula from the given data. Why is this formula correct? In fact, it is exactly the formula that would be obtained by modern algebraic reasoning. Using the notation in Fig. 1, set d to be the length of the reed and l the height of the wall; then $l = d - D_1$, and the Pythagorean theorem gives

$$d^2 = (d - D_1)^2 + D_2^2 = d^2 - 2dD_1 + D_1^2 + D_2^2,$$

which naturally (algebraically) simplifies to

$$2dD_1 = D_1^2 + D_2^2, \tag{1}$$

and so

$$d = \frac{1}{2D_1} (D_1^2 + D_2^2).$$

Given the data D_1 and D_2 , d would be computed by much the same procedure as that detailed in the tablet. The challenge is to provide a geometrical argument that makes this formula natural.

Høyrup rises to this challenge by noting that $D_1 = d - l$, and so Eq. (1) indicates that the sum of the square on $(d - l)$ and the square on D_2 (the base of the triangle) is equal to twice the rectangle on d and $d - l$ (see Høyrup [2002a, p. 399, 2002b, pp. 15–17]). This connection between the rectangles and squares is easily obtainable by a “naive geometrical” argument and may well underlie the construction of the problem. It also accounts for the order of Steps 5 and 6 of the algorithm. However, Høyrup notes in concluding his analysis that, “the method that is used . . . resembles nothing we have encountered in the Old Babylonian corpus (or the pre-Seleucid Late Babylonian texts), and is therefore highly unlikely to be old” [Høyrup, 2002a, p. 399]. The problem is a curious mixture of old and new. The problem is given the dress of pole and wall as in the older source. While it uses a multiple of the standard 3–4–5

triple, it is not the same multiple as the earlier case and the fact that two sources use multiples of the simplest Pythagorean triple does not provide evidence for contact between them. The organization of the procedure conforms to expectations, but the algorithm used is not old.

4. Egyptian poles

Although there is no evidence of interest in leaning poles against walls in the meagre sources we have on Pharaonic Egyptian mathematics, the problem does appear in a demotic papyrus from the early third century B.C., roughly contemporary with the Seleucid tablet studied above, probably slightly older.

A collection of mathematical papyri was published by Parker [1972]. One of these, P. Cairo J.E. 89127-30, 89137-43, in a very fragmentary state, has a legal code on the recto and some 40 mathematical problems on the verso. It was dated on paleographic grounds from a study of the legal text.⁵ Both the beginning and end of the papyrus are lost and some of the remaining problems are missing too much content to be restored, but among them is a series of eight problems to do with a pole (literally, a tree) whose top moves down or base moves out. As in the Old Babylonian problem, and against the Seleucid version, the Egyptian pole appears unaccompanied by explicit mention of a wall.⁶ While the pole problems on this papyrus are in certain cases badly damaged, textual parallelism allows us to restore the mathematical content with confidence even if there linger some technical philological questions.

As is common in texts with series of problems, here some basic sets of parameters are chosen and then the problems ring a variety of changes on which parameters are given and which are computed. The first underlying Pythagorean triple, accounting for three of the eight problems, is 6–8–10 (units are cubits), which, as we saw, was still going strong in Stewart. Both the Old Babylonian and Seleucid pole problems also used multiples of the 3–4–5 triple. Another two problems use the same triple, but with the role of the 6 and 8 reversed. Interestingly, however, the triple used in P. Cairo for the other three problems is $10-10\frac{1}{2}-14\frac{1}{2}$. Presumably this was chosen so that in the one case the largest member of the triple was 10, and in the other case, the smallest was 10, thereby linking the two sets of parameters by the number 10.

The problems are numbered 24–31 in Parker's text. We retain that numbering here for convenience. The translation is based on Parker's, but simplified. For original transliterations and translations see Parker [1972, pp. 35–40].

Problem 24 (see Table 4)

A pole is 10 cubits when erect. If its foot is moved out
6 cubits, how much is the top lowered? The working.

⁵ The recto of the papyrus fragments contains the "Legal Code of Hermopolis West." The papyrus was found "in the debris of a ruined building opposite the place of mummification . . . believed to be one of the temple archives" [Mattha, 1975, p. xi]. Mattha continues that the papyrus "can be safely assigned to the reign of Philadelphus, possibly earlier," that is, probably early third century. In light of Ritter's arguments about common domains of knowledge and professionalization among mathematical and legal material, it is remarkable that the mathematics occurs on the same papyrus as a legal code. Although much of the code deals with leasing of land and property, and inheritance, there is nothing particularly mathematical about the legal procedures described. The source clearly locates the mathematics within a temple-based scribal milieu.

⁶ The apparent absence of the wall should not be given too much emphasis. The context of the problem would have been understood by its intended users and it is only the Seleucid problem that computes the height of the wall, and thus forces its explicit mention.

Table 4
P. Cairo, No. 24

	Computation	Symbolic instruction
Step 1	10 times 10, result 100	$R_1 = D_1^2$
Step 2	6 times 6, result 36	$R_2 = D_2^2$
Step 3	Subtract 36 from 100, remainder 64	$R_3 = R_2 - R_1$
Step 4	Find square root of 64, result 8	$R_4 = \sqrt{R_3}$
Step 5	Subtract 8 from 10, remainder 2	$R_5 = R_1 - R_4$

You compute 10 times 10: result 100. You compute 6 times 6: result 36.
Subtract it from 100: remainder 64. Reduce 64 to its square root: result 8.
Subtract 8 from 10: remainder 2.
You shall say, "Its top is lowered 2 cubits."

Instead of finding the height of the (missing) wall, or the height of the top of the pole from the ground, the goal is to find by how much the top has lowered when the base of the pole is moved out a specified distance. The problem is solved by a straightforward use of the Pythagorean theorem. The given data are $D_1 = 10$, the length of the pole, and $D_2 = 6$, the distance the base is moved.

As in Mesopotamia, the procedure first utilizes the data, in the order given in the statement of the problem and then makes single arithmetical steps, with the output from one step usually forming the input of the next. The final step requires recalling the initial data. The formal structure, then, of the algorithm is very similar to that of the Mesopotamian problems.

Problem 25 (see Table 5)

The second problem is formally the same as the first, but with different values for the data.

A pole is $14\frac{1}{2}$ cubits when erect.
If its foot is moved out 10 cubits, how much is the top lowered?
You compute $14\frac{1}{2}$ times $14\frac{1}{2}$: result $210\frac{1}{4}$.
You compute 10 times 10: result 100. Subtract it from $210\frac{1}{4}$:
remainder $110\frac{1}{4}$. Reduce $110\frac{1}{4}$ to its square root: result $10\frac{1}{2}$.
Subtract it from, $14\frac{1}{2}$: remainder 4.
You shall say, "Its top is lowered 4 cubits."

Table 5
P. Cairo, No. 25

	Computation	Symbolic instruction
Step 1	$14\frac{1}{2}$ times $14\frac{1}{2}$, result $210\frac{1}{4}$	$R_1 = D_1^2$
Step 2	10 times 10, result 100	$R_2 = D_2^2$
Step 3	Subtract 100 from $210\frac{1}{4}$, remainder $110\frac{1}{4}$	$R_3 = R_2 - R_1$
Step 4	Find square root of $110\frac{1}{4}$, result $10\frac{1}{2}$	$R_4 = \sqrt{R_3}$
Step 5	Subtract $10\frac{1}{2}$ from $14\frac{1}{2}$, remainder 4	$R_5 = R_1 - R_4$

Table 6
P. Cairo, No. 26

	Computation	Symbolic instruction
Step 1	10 times 10, result 100	$R_1 = D_1^2$
Step 2	6 times 6, result 64	$R_2 = D_2^2$
Step 3	Subtract 64 from 100, remainder 36	$R_3 = R_2 - R_1$
Step 4	Find square root of 36, result 6	$R_4 = \sqrt{R_3}$
Step 5	Subtract 6 from 10, remainder 4	$R_5 = R_1 - R_4$

This time the given data are $D_1 = 14\frac{1}{2}$, the length of the pole, and $D_2 = 10$, the distance the base is moved.

As is clear from Tables 4 and 5, precisely the same algorithm is used in the two problems. The problems are nicely chosen so that the answer in the first case is 2 cubits, and in the second case, 4 cubits. It is also interesting that the problem of finding the square root of $110\frac{1}{4}$ should be considered sufficiently elementary not to distract from the main purpose of the problem.

Problem 26 (see Table 6)

The next problem gives the same two pieces of data, the length of the pole and the distance the foot is moved out. It reverts to the first 6–8–10 triple, but now with the 6 and the 8 swapped around.

A pole is 10 cubits when erect. If
its foot is moved out 8 cubits, how much is the top lowered?
You compute 10 times 10: result 100.
You compute 8 times 8: result 64. Subtract it from 100: remainder 36.
Reduce 36 to its square root: result 6. Subtract it from 10: remainder 4.
You shall say, "Its top is lowered 4 cubits."

Hence, the given data are $D_1 = 10$, the length of the pole, and $D_2 = 8$, the distance the base is moved.

Certainly there is a clear idea of the relationships between the elements of Pythagorean triples, and the dressing of the problem, as well as the order of the elements of the triple, merely provide concrete examples.

Problem 27 (see Table 7)

Problem 27 is the inverse to the first problem. The relevant triple is again 6–8–10 and the length of the pole is given. However, this time, the distance the top moves down is given and the distance the foot moves out is sought.

A pole is 10 cubits when erect.
The distance its foot is moved out will not be stated
Lower its top 2 cubits. How much is the foot moved out?
You compute 10 times 10: result 100. Subtract it, 2, from 10:
remainder 8. You compute 8 times 8: result 64.
Subtract it from 100: remainder 36. Reduce it to its square root: result 6.
You shall say, "Its foot is moved out 6 cubits."

Table 7
P. Cairo, No. 27

	Computation	Symbolic instruction
Step 1	10 times 10, result 100	$R_1 = D_1^2$
Step 2	Subtract 2 from 10, remainder 8	$R_2 = D_1 - D_2$
Step 3	8 times 8, result 64	$R_3 = R_2^2$
Step 4	Subtract 64 from 100, remainder 36	$R_4 = R_1 - R_3$
Step 5	Find square root of 36, result 6	$R_5 = \sqrt{R_4}$

The given data are $D_1 = 10$, the length of the pole, and $D_2 = 2$, the distance the top is lowered.

Whereas the first group of problems, where the distance the pole is moved out does not appear in the Mesopotamian sources, this group treats the same situation as in BM 85196. A comparison of Tables 7 and 1 shows that the Egyptian algorithm is exactly the same as the Old Babylonian one.

Problem 28

Problem 28 is parallel to Problem 27, but using the second set of data, so that it is the inverse of Problem 25. The length of the pole is given as $14\frac{1}{2}$ cubits and the top is lowered by 4 cubits. The distance the foot is moved outwards is sought and determined to be 10 cubits. The structure of the algorithm is precisely that of Problem 27.

Problem 29

Problem 29 is the third similar problem, being parallel to Problem 27, but using the third set of data, that is, the 6–8–10 data of Problem 26, where the foot is moved out 8 cubits. The given data are the length of the pole, 10 cubits, and how far the top is lowered, 4 cubits. Despite slight differences in wording (for example, this problem does not tell you what will not be stated), the solution follows exactly the same steps as in Problems 28 and 27.

Problem 30 (see Table 8)

The last two problems in the series, Problems 30 and 31, are both very heavily damaged.⁷ The reconstruction of the problems depends on assumed parallels between the two problems and requirements of the problem. Despite the extensive restoration, one can have reasonable confidence in the algorithms used.

These two problems ask for the length of the pole, when the amount the top is lowered and the distance the foot is moved out are known.

A pole. Its foot is moved out 6 cubits and
its height will not be stated. If the top is lowered
2 cubits, what was its height?

⁷ The expected ninth problem, using the third set of data, does not appear on the papyrus. Instead, the next problem is one of determining the diameter of a circle given its area.

Table 8
P. Cairo, No. 30

	Computation	Symbolic instruction
Step 1	6 times 6, result 36	$R_1 = D_1^2$
Step 2	2 times 2, result 4	$R_2 = D_2^2$
Step 3	Add 36 to 4, result 40	$R_3 = R_1 + R_2$
Step 4	Add 2 to 2, result 4	$R_4 = D_2 + D_2$
Step 5	Divide 40 by 4, result 10	$R_5 = R_3 / R_4$

You compute 6 times 6: result 36. You compute 2 times 2: result 4.
You add 36 to 4: result 40. You add 2 to 2: result 4. Take it to 40: result 10.
The height of the pole was 10.

The given data are $D_1 = 6$, the distance the foot is moved out, and $D_2 = 2$, the distance the top was lowered. The procedure is a new one.

As Problems 30 and 31 use the same algorithm, we will discuss them together.

Problem 31 (see Table 9)

This problem sets up the same situation as the previous one, but using the second set of data, the 10–14–14½ triple.

A pole. If its foot is moved out 10 cubits
and its top is lowered 4 cubits, what was its height?
You compute 10 times 10: result 100. You compute 4 times 4: result 16.
You add it to 16: result 116. You add 4 to 4: result 8.
Take it to 116: result 14½. The height of the pole was 14½.

The given data are $D_1 = 10$, the distance the foot is moved out, and $D_2 = 4$, the distance the top is lowered. The procedure is as before.

As can be seen from the “Symbolic Instruction” columns of the two problems, the steps followed are the same. One needs to be a little cautious, however, due to the extensive restoration (see Parker [1972, pp. 39–40] for lacunae and commentary on the restoration). Speaking in favor of the restoration, disregarding minor textual issues, is the fact that although the precise wording changes in equivalent problems using

Table 9
P. Cairo, No. 31

	Computation	Symbolic instruction
Step 1	10 times 10, result 100	$R_1 = D_1^2$
Step 2	4 times 4, result 16	$R_2 = D_2^2$
Step 3	Add 100 and 16, result 116	$R_3 = R_1 + R_2$
Step 4	Add 4 to 4, result 8	$R_4 = D_2 + D_2$
Step 5	Divide 116 by 8, result 14½	$R_5 = R_3 / R_4$

alternative data in the series of problems above, the procedures in the algorithms are identical in each case.

This problem, in which the length of the pole is to be determined, given the distance the base is moved out and the distance the top is moved down, is precisely the situation in the first half of the Seleucid problem discussed above. In view of Høyrup's declaration that the procedure followed in the Seleucid problem is of recent discovery, it is interesting to compare the two algorithms.

First, note that although the same pieces of data are given, their order differs in the two texts. In BM 34568, the distance the top has moved down is given first, followed by the distance the foot has moved away from the wall. In both Problems 30 and 31, the distance the foot has moved out is given first, followed by the amount the top is lowered. The numbers used, the values of the data, are also different in the two sources.

Each of the three algorithms begins by finding the squares of the two initial pieces of data, in the order given, and then summing the squares. The remainder of the algorithms appears superficially different. In the Seleucid text, the result of the initial steps, R_3 , is first multiplied by 30 (i.e., divided by 2), and then multiplied by the reciprocal of D_1 . That is, R_3 is divided by $2D_1$, but in two steps, and of course division in Mesopotamia is accomplished by multiplication by the reciprocal. Recalling that the order of the initial data is reversed in P. Cairo, the two steps of the algorithm show that R_3 is divided by $2D_2$, where the doubling of D_2 is accomplished by adding it to itself. Thus the two approaches to the problem feature two multiplications in Mesopotamia, and an addition and division in Egypt (instead of the two divisions that would have occurred if R_3 had been first halved before being divided by D_2). The two algorithms are thus quite similar, given that multiplication by a reciprocal was not a standard part of Egyptian arithmetic. Certainly, both approaches utilize the crucial identity of the sum of the squares and twice the rectangle.

There is no doubt that both algorithms are derived from the same conceptual map of the problem, whether or not it is the precise geometrical interpretation suggested by Høyrup. Despite differences in data and ordering of initial data, the underlying structural similarities imply a common source for the algorithms, whether in Mesopotamia, Egypt, or in some third location. Certainly there were extensive contacts between Mesopotamia, Egypt and other regions during the Hellenistic period.

The fact that the Egyptian source may well be older than the Seleucid one might be taken to support an argument for transmission from Egypt to Mesopotamia, but it seems unlikely that either source is original in documenting the problems it contains. If both sources are compilations or copies of earlier documents, little can be said about the relative ages of the origins of the problem in the two contexts. Similarly, the fact that Problems 27, 28, and 29 of P. Cairo use precisely the same algorithm as that in BM 85196, while using different data, suggests some form of unrecorded transmission over the intervening 1500 years, but as yet we can not speculate on the means of that transmission.

5. Summary of the problems

Table 10 indicates the given data and data sought in the problems discussed above. Note that only BM 34568 actually has a wall.

We let d denote the length of the pole or reed, l the height of the wall, a the distance the foot is moved out, and b the distance the top is moved down.

Table 10
Data summary

Source	Date	Given data	Sought data
BM 85196	ca. 2000–1600	d, b	a
BM 34568	ca. 300–100	b, a	d, l
P. Cairo Nos. 24, 25, 26	ca. 300–250	d, a	b
P. Cairo Nos. 27, 28, 29		d, b	a
P. Cairo Nos. 30, 31		a, b	d

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