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MAZIARZ, E.A.
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CHAPTER 5

The Crisis of the Irrationals

There are strong reasons to believe that Pythagoras discovered the irrationals when considering the relation between the diagonal of a square and its side. His method of obtaining square-numbers may have prompted him to question why every number is not a square, when he had found some exception to his practical rule. But geometry offered the best examples of the existence of irrationals, as in the cases of the isosceles right-angled triangle and the numerical relation between the diagonal of a square and its side. The very names of *side-numbers* and *diagonal-numbers* seem to justify this view.

If the primitive treatment of the theorem of the right-angled triangle was arithmetical, the impossibility of finding a root for the square of the hypotenuse of an isosceles right-angled triangle would naturally yield the notion of $\sqrt{2}$, the first irrational. According to a scholium to Euclid, the Pythagoreans discovered the irrationals by observing numbers; for "though the unit is a common measure of all numbers, they were unable to find a common measure of all magnitudes."¹ This

¹ Thomas L. Heath, trans., *The Thirteen Books of Euclid's Elements*, ed. Heiberg (Cambridge, 1908), I, p. 415 (repub. in New York, 1956).

was because magnitudes are endlessly divisible, without leaving any part too small for further division.

After this discovery, the Pythagoreans probably investigated the properties of such magnitudes, but nothing definite seems to have been done before Theodorus of Cyrene and Theaetetus of Athens. The title of the lost work of Democritus, *On Irrational Lines and Solids*, suggests that surds were known before his time. The irrationality of $\sqrt{2}$ is alluded to in Plato's *Republic* as a well-known fact; in *Theaetetus* we are told that Theodorus was the first to prove the irrationality of $\sqrt{3}$, $\sqrt{5}$... $\sqrt{17}$, which implies that $\sqrt{2}$ has been dealt with earlier.

The traditional proof of the irrational character of certain numbers is indicated by Aristotle as an example of *reductio ad absurdum*: "All who effect an argument *per impossibile* infer syllogistically what is false, and they prove the original conclusion hypothetically when something impossible results from the assumption of its contradictory; for example, the diagonal of the square is incommensurate with its side because odd numbers are equal to evens if it is assumed to be commensurate."² This is evidently the proof interpolated in the tenth book of Euclid's *Elements*, where the fraction m/n in its lowest terms is supposed to equal $\sqrt{2}$; then both m and n must be even because $m^2/n^2 = 2$ and $m^2 = 2n^2$; yet this cannot be, because of the condition of the fraction. With this contradiction, no fraction m/n can have 2 as its square.³

Their fruitless effort to find the exact root of $\sqrt{2}$ led the Pythagoreans to prove its incommensurability, and to determine any number of successive approximations to its value by finding the integral solutions of an indeterminate equation (of the form $2x^2 - y^2 = -1$ in modern notation). The pairs of values of x and y were called *side-numbers* and *diameter-numbers* or *diagonal-numbers*, respectively; as these values increase, the ratio of y to x approximates $\sqrt{2}$ more closely. Theon of Smyrna⁴ gives an interesting explanation of the formation

² *An. Priora* 41a 23.

³ Cf. Bertrand Russell, *Introduction to Mathematical Philosophy* (New York, 1938), p. 67.

⁴ *Expositio Rerum Mathematicarum*, pp. 43-44.

of these numbers, whose names are justified by their particular function and purpose.

The Greeks treated the irrationals in general as a part of geometry rather than arithmetic, because of the difficulty of handling irrationals arithmetically and of the successful Pythagorean combination of geometry with number theory. For want of a notation, any irrational was represented by a rectilinear segment or a combination of lines. This is illustrated in the tenth book of Euclid's *Elements*, where simple and compound irrationals are dealt with geometrically. Yet the Pythagoreans could not develop the theory of irrationals on this basis, precisely because their geometry depended on their restricted number theory. Furthermore, as their arithmetic involved a theory of proportion applicable to commensurable magnitudes only, discovery of the irrationals must have dealt a severe blow to its whole structure. In fact, it involved the restriction of their method of proportion, pending the discovery of the generalized theory of proportion established by Eudoxus during Plato's time. The discovery of surds also shattered the geometrical methods of the Pythagoreans, as the proof of several geometrical theorems rested on their primitive theory of proportion.

When the invention of surds became known to members of the Brotherhood, a rift was opened between arithmetic and geometry; this led to the investigation of various problems involving irrational magnitudes. Pythagorean and other schools of mathematicians tried laboriously to find whether the rift could be bridged by purely mathematical methods. This might explain why such problems as the squaring of the circle, trisection of the angle, and duplication of the cube were popular after the death of Pythagoras. In spite of the interesting results obtained by some mathematicians, these circumstances may account for the setback suffered by mathematics at the end of the fifth century B.C.

It may be questioned whether the Pythagoreans knew the incommensurability of a circumference in relation to its diameter, although the circle played an important part in their cosmogonies. Tradition mentions Anaxagoras of Clazomenae

(ca. 500-428 B.C.) as the first to deal openly with the problem of squaring the circle. But he may have heard of it from the early Pythagoreans, of whom he was a younger contemporary, and who could not have dealt openly with this question because of their oath of secrecy. To be sure, discovery of the irrationals must have been kept secret a long time, as it had far-reaching consequences for the practical and philosophical doctrines of the Brotherhood. The esoteric disciples must have heard with awe from the Master that number, which was and explained everything, could not account for some simple geometrical magnitudes related to their number theory.

This situation must have weakened the cosmic and moral applications of the number theory as a whole. The square, a most fundamental and beautiful type of number, was found to bear within itself an element of irrationality. By using a square number as a physical explanation of facts, reason was appealing, so to speak, to "unreasonable" elements. It may also have been observed that many rules of harmony apparently subject to number entailed irrationals. When the irrationality of π was established later, many must have thought that the laws of the heavens themselves could not be as true as Pythagoras said they were, for if the distances between the heavenly bodies were proportional to the lengths of the vibrating strings which produced the musical scale, the perfect circular paths of these bodies had no common number with the proportional distances between the center of the world and the various bodies moving in space. Consequently, it was impossible to maintain any longer that number was the essence of all existing things, or that all things were made of number.

This awkward crisis encouraged the Pythagorean esoterics to maintain their oath to withhold from the public the existence of irrationals. This accounts for the legend that the first Pythagorean who made it public, whether it was Hippasus or another, perished at sea for his impiety. According to Proclus,⁵ the unutterable and the formless were to be concealed; those

⁵ Cf. Iamblichus *De Vita Pythagorica* xviii, 88.

who uncovered and touched this image of life were instantly destroyed and shall remain forever the play of the eternal waves.

Meanwhile, destruction of the primitive mathematical balance between the cosmos and man called for new conceptions to satisfy man's yearning for truth. If nature contained elements beyond reason, then man himself should be studied in order to find out his limitations and their eventual remedy. This task was performed by the Socratic schools, although the serious interest of the Pythagoreans in the practical rules of life was originally responsible for introducing ethics and social theory into the range of philosophical inquiry. This interest was intensified with Xenophanes and Heraclitus, and it reached its highest mark with Democritus and the Sophists.

The Pythagorean experiment was certainly discussed at the time. If the existing fragments of the pre-Socratic thinkers were not so scanty, we could probably trace many more references to Pythagorean doctrines before the time of Plato. The founder of the Academy must have borrowed a good deal from them, although he scarcely mentions them in his *Dialogues*. We have to turn to Aristotle, who disagreed with both Platonism and Pythagoreanism, for the first serious criticism of these views. In the first book of his *Metaphysics*, Aristotle says the Pythagoreans treat of principles and elements stranger than those of the Ionian philosophers, for these principles are taken from non-sensible things, since the objects of mathematics are things without movement. Yet the Pythagoreans claim to discuss and investigate nature, for they generate the heavens and explain their parts and functions by observing the natural phenomena and referring them to principles and causes. This attitude was shared by the Ionian philosophers, for whom the real is all that is perceptible and contained by the so-called heavens.

For Aristotle, the causes and principles mentioned by the Pythagoreans may lead gradually to the higher levels of reality, but they are less suited to theories about nature. Elaborating this view, he criticizes the Pythagoreans for neglecting to make

clear whether the limit and the indefinite, the odd and the even are the only principles assumed. As a result, they are unable to account for the existence of motion and the facts of generation and destruction. Consequently, they offer no explanation of the particular movements of the heavenly bodies and of the difference between light and heavy objects. Although Plato assimilated the elements with the regular solids, the Pythagoreans apparently said nothing about fire, earth, air, or water. Engrossed with the problem of motion, Aristotle fails to see how number and its attributes are causes of what exists and of what happens in nature, when the only acknowledged numbers are those out of which the world is composed. Similar difficulties may have led Zeno to formulate his famous arguments.

Referring to some aspects of Pythagorean mysticism, Aristotle cannot reconcile its implications with his logical vision of the world. He fails to see why opinion and opportunity are placed in one particular region, while injustice and mixture are above or below. He also disagrees with the alleged proof that each of these is number, and that a plurality of the extended bodies composed of numbers is already in those places because the attributes of numbers are attached to various places. In this case, one cannot determine whether or not the number identified with each of these abstractions is also the number exhibited in the material universe. Although Plato says it is different, for him both the bodies and their causes are numbers.⁶

Notwithstanding the effectiveness of Aristotle's criticism, the Pythagorean doctrines and methods have suggested a basic approach to the problems of the world which has greatly influenced the development of philosophy and science. Commenting upon the Pythagorean doctrine that mathematical entities are the ultimate stuff of existence and experience, Whitehead asserts that, with this bald and crude statement, Pythagoras had hit upon

⁶ *Met.* 900a 8-30.

a philosophical notion of considerable importance, a notion which has a long history, and which has moved the minds of men and has even entered into Christian theology. About a thousand years separate the Athanasian Creed from Pythagoras, and about two thousand four hundred years separate Pythagoras from Hegel. Yet for all these distances in time, the importance of definite numbers in the constitution of the Divine Nature, and the concept of the real world as exhibiting the evolution of an idea, can both be traced back to the train of thought set going by Pythagoras. . . . So today, when Einstein and his followers proclaim that physical facts such as gravitation are to be construed as exhibitions of local peculiarities of spatio-temporal properties, they are following the pure Pythagorean tradition.⁷

In short, the Pythagorean conception of science corresponds definitely to a fundamental attitude of the mind in its search for truth.

⁷ A. N. Whitehead, *Science and the Modern World* (New York, 1941), p. 36.

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