

AB: This paper reads Zeno's paradoxes of motion against the Pythagorean doctrine that the principles of things are numbers, to argue that the application of mathematical concepts to physical reality results in paradox. Zeno is interpreted as showing that to ask whether motion is continuous is to apply a property that properly belongs to the number series to physical reality, and thus to confuse the ideal with the real. Zeno's point challenges modern assumptions that the universe is written in the language of mathematics, especially quantum theory, wherein theorists are unable to agree upon a physical interpretation of the mathematical formalism.

Zeno Against Mathematical Physics

Trish Glazebrook

Galileo wrote in *The Assayer* that the universe "is written in the language of mathematics," and therein both established and articulated a foundational belief for the modern physicist.¹ That physical reality can be interpreted mathematically is an assumption so fundamental to modern physics that chaos and superstrings are examples of physical theories developed on the basis of success in mathematics. The mathematics came first, then the physical theory. Quantum theory is an awkward and similar case in point. Despite the fact that there is much agreement among physicists about the mathematical theory, its physical interpretation remains a matter of controversy. There are several interpretations, all of which challenge our everyday assumptions about reality. This led Bohr in 1935 to call for "a radical revision of our attitude towards the problem of physical reality."² Arthur Fine, originally a staunch defender of realist interpretations of quantum theory, saw the success of Bohr's Copenhagen interpretation as the end of Einsteinian realism, and subsequently Fine declared that "realism is well and truly dead."³ Mathematical operators simply do not correspond with real entities in any conceivable way.

Ilkka Niiniluoto has argued convincingly, however, that realism is alive and well in quantum theory.⁴ The problem is that Bohr is right: quantum theory can be considered to concern the real only upon radical revision of ordinary conceptions of reality. Quantum theory does not describe the real in any meaningful sense of "the real." Rather, it is mathematical projection: ideal rather than real.

¹ *Discoveries and Opinions of Galileo*, tr. Stillman Drake (London, 1957), 238.

² Niels Bohr, "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?" *Physical Review*, 48 (1935), 696-702. Repr. *Quantum Theory and Measurement*, ed. J. A. Wheeler and W. H. Zurek (Princeton, 1983), 146.

³ Arthur Fine, "The Natural Ontological Attitude," *Scientific Realism*, ed. J. Leplin (Berkeley, 1984), 83.

⁴ Ilkka Niiniluoto, "Varieties of Realism," *Symposium on the Foundations of Modern Physics*, ed. Pekka Lahti and Peter Mittelstaedt (New Jersey, 1987), 459-83.

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Aristotle did not hold that the physicist deciphers nature mathematically. He distinguished the physicist from the mathematician at *Physics* 2.2 by analogy to the definitions of “snub nose” and “curved.”⁵ The definition of “snub nose” requires the mention of a nose, that is, of matter which is inseparable from motion, whereas “curved” and other mathematical concepts like “‘odd’ and ‘even’, ‘straight’ ... ‘number,’ ‘line,’ and ‘figure,’” which are separable from matter and from motion, are investigated by the mathematician.⁶ Aristotle claims that physics is different from mathematics because the mathematician investigates physical lines, but not *qua* physical, and the physicist investigates mathematical lines “but *qua* physical, not *qua* mathematical.”⁷

A peculiar consequence of my argument is that it establishes a common principle between two thinkers whose beliefs are otherwise so fundamentally in opposition. Zeno’s paradoxes are commonly read as a denial of motion. For Aristotle, that τὰ φυσικά move is axiomatic to the physicist. At 185a15 he considers arguments against this assumption outside the realm of objections to which the physicist must reply. If I am right that Zeno’s paradoxes of motion challenge the very notion of mathematical physics, then he and Aristotle agree, *contra* Galileo, that the universe is not written in the language of mathematics. They simply come to this point from opposite directions, Zeno from an Eleatic thesis that physical reality is illusion, Aristotle from the natural scientist’s pragmatic commitment to a truth about nature.

I intend to use Zeno’s paradoxes of motion to argue that the application of mathematical concepts to the physical world results in paradox. Zeno’s arguments are limited within mathematics to geometry. This paper is accordingly a beginning from which I hope more work can be done later. Furthermore, Zeno’s paradoxes are standardly read as directed against anyone who disagrees with Parmenides’ thesis that there is but one thing which is motionless and indivisible. The paradoxes achieve this end by means of a *reductio* against the possibility of dividing space and time. That there are four paradoxes is accordingly explicable by the fact that there are four possibilities for the divisibility of space and time. The Achilles argues against the possibility that space and time are both infinitely divisible; the Dichotomy, against the infinite divisibility of space and finite divisibility of time; the Arrow, against the infinite divisibility of time and the finite divisibility of space; and finally, the Stadium shows that time and space cannot both be finitely divisible.

This interpretation has the advantage not only of answering why there are four paradoxes, but also of being holistic. A holistic approach resolves the contradiction apparent in denying contradictory theses. Yet this interpretation is

⁵ Aristotle, *Physics*, tr. P.H. Wicksteed and F.M. Cornford, Loeb Classical edition (Cambridge, Mass., 1934), 194a6.

⁶ Aristotle, *Physics*, 194a4-5.

⁷ Aristotle, *Physics*, 194a10.

easiest to follow when the order in which the paradoxes are presented is ignored and a new order is imposed. At *Physics* 6.9 Aristotle gives the Dichotomy, followed by the Achilles, the Arrow, and the Stadium, in that order. This arrangement could of course simply be Aristotle's. Yet Brochard argues that the logic of Aristotle's ordering suggests that the order belongs to Zeno. The holistic interpretation that follows this ordering picks out from Zeno's general defense against Parmenides' critics an attack that makes much sense when read specifically against the Pythagorean doctrine that the principles, elements, and causes of things are numbers.

That Zeno's paradoxes of motion are taken to concern an issue in physics, the divisibility of space and time, and not as an argument against the mathematization of nature, i.e., as an issue not in but rather about physics, is, I suggest, perfectly consistent with undervaluing their importance as an argument against the Pythagoreans. Rearranging the paradoxes both demonstrates and encourages this shift in emphasis. Furthermore, such a shift in emphasis makes perfect sense in an intellectual context that is founded on the belief that the universe is written in the language of mathematics. Zeno's paradoxes can be read in different ways. That there is at present agreement amongst physicists about the mathematical formalism of the quantum theory but not about its physical interpretation renders timely another look at Zeno's paradoxes of motion. This paper draws attention to the possibility of reading the paradoxes in order to argue that the mathematization of physical reality is not an innocuous assumption.

According to Aristotle, Zeno, besides formulating the four paradoxes of motion, is thought to be the founder of dialectic, the philosophical method in which one assumes an opponent's premise in order to render it untenable. Accordingly, the paradoxes operate by *reductio ad absurdum*. They consist in two pairs that attack opposing theses. The Dichotomy and the Achilles demonstrate the absurd conclusions that follow from the claim that distances are infinitely divisible. The Arrow and the Stadium attack atomism. Together the four arguments show that paradox results whether distance is taken to be finitely or infinitely divisible. Zeno therefore concludes, according to Aristotle, that motion is impossible.⁸ I suggest rather that Zeno's point could have been that the mathematical description of motion is problematic.

The Dichotomy

Aristotle states the problem: "that which is in locomotion must arrive at the half-way stage before it arrives at the goal."⁹ To cross a distance therefore entails traversing an infinite number of half-way points, which cannot be done in a

⁸ Aristotle, *Topics* in *The Works of Aristotle*, ed. W. D. Ross, tr. W. A. Pickard-Cambridge (Oxford, 1928), 160b7.

⁹ Aristotle, *Physics*, 239b11.

finite time. Aristotle's formulation is ambiguous: there is a "dwindling interpretation": one must first cross half a distance, but first half of that distance, yet first a quarter of it, and so on. Alternately, there is an "additive interpretation": one must first cross half the distance, and then half of the remaining distance, and so on.

Aristotle's commentators, Simplicius and Philoponus, commenting on 239b10 and 187a1 respectively, opt explicitly for the dwindling interpretation, although Simplicius tends toward the additive interpretation at 233a21 and 263a5. At these two places he draws an analogy between the Dichotomy and counting. In counting, an infinite number of terms each follow after a finite interval. Likewise, the Dichotomy is a construction by addition that yields an infinite series of half-way points, each reached after a finite interval.

The additive interpretation is clearly awkward. Since to get halfway is to complete a motion, the paradox relies on exactly what it intends to refute. Yet Zeno's method is to assume the hypothesis to be reduced to absurdity. The additive interpretation is not so much self-contradictory as methodologically innovative. That is, if Zeno is in fact the father of dialectic, his paradoxes of motion are a clear demonstration of the power of his novel method. Further, however, the additive interpretation is easily resolved: aim for some point beyond the goal; you will never reach that point, but at some stage, you will have reached the goal.

The dwindling interpretation is more difficult. G. E. L. Owen points out that it admits no first move.¹⁰ To complete a first step is already to have traversed an infinite number of points. Grünbaum formulates the problem of infinite divisibility as the question: "how can any motion or time-interval be meaningfully resolved into elements which form a mathematically dense set, i.e., which are ordered such that no event has an immediate temporal or spatial successor?"¹¹ A "mathematically dense set" is one in which between any two points, there is an infinitude of others. No point can immediately succeed any other as there must be points in between. Hence no two points can be next to each other, and subsequently one cannot get from one point to the next.

At *Physics* 6.2 Aristotle solves for the dwindling interpretation. There are two senses in which anything continuous can be called infinite, argues Aristotle: in respect of divisibility or in respect of its extremities.¹² In the latter sense, an infinite distance cannot be traversed in a finite time, but in the former sense, divisibility, it can. Kathleen Freeman draws the corollary conclusion: "the finite

¹⁰ G. E. L. Owen, "Zeno and the Mathematicians," *Proceedings of the Aristotelian Society*, 58 (1957-58), 199-222: 207.

¹¹ Adolf Grünbaum, "Messrs. Black and Taylor on Temporal Paradoxes," *Analysis*, 12 (1951-52), 144-48:144.

¹² Aristotle, *Physics*, 233a25.

length of space [is traversed] in the finite length of time.”¹³ Aristotle later explains this solution in terms of the distinction between the actual and the potential. Although the distance is potentially infinitely divisible, it is not actually so divided. Were the infinite number of points to be traversed actual, it should be impossible to traverse them in a finite time. But taken potentially, it is possible, since the distance is accidentally an infinity of halves and substantially a finite distance.¹⁴ When one crosses a stadium, the example given at *Topics* 160b7, an actual point is where one stops. Stopping is what makes the point actual. Given an infinity of actual points, one never reaches the other side; but if one does not stop, the motion is continuous, the breaks in motion remain potential rather than actual, and one reaches the other side.

Accordingly, Aristotle refutes the Dichotomy by arguing that motion is continuous. Does the Dichotomy depend, however, on discontinuity? Aristotle defines continuity as infinite divisibility.¹⁵ Yet if a line is taken to consist of an infinite number of points—hence to be infinitely divisible—it cannot be continuous. Between any two points on an infinitely divisible line there must be an infinite number of other points at which the line could be further divided. Hence no two points can touch. This is no problem for Aristotle since he does not think that a line consists of points at all. He holds that points are limits of lines, i.e., they are shared divisions between lines. Zeno requires, however, none of the complications of continuity. All he needs for his paradox to hold is that the distance is infinitely divisible, regardless of whether it is continuous or discontinuous. For the paradox asks, if the infinite divisibility of distance is granted, how is motion possible? In other words, anyone crossing any distance could first cross a smaller distance. Is there a smallest such distance that must first be crossed? The Achilles paradox demonstrates the absurd conclusion of answering “yes” to this question.

The Achilles

Practically the only detail about Zeno’s paradoxes of motion that comes to us independently of Aristotle is that Achilles races a tortoise. Aristotle discusses the quickest and slowest runners in a race. Ross suggests that the story became mixed up with one Plutarch tells about a tortoise racing the fastest horse of Adrastus.¹⁶ The Achilles is a theatrical version of the additive interpretation of the Dichotomy. It too hinges on divisibility: in a race the fastest runner cannot

¹³ Kathleen, Freeman, *The Pre-Socratic Philosophers* (Oxford, 1949), 162.

¹⁴ Aristotle, *Physics*, 263b7-9.

¹⁵ Aristotle, *Physics*, 185b10, 200b17-20, 232b24; cf. Aristotle, *De Caelo* in *Basic Works of Aristotle*, ed. Richard McKeon, tr. J. L. Stocks (New York, 1941), 268a6-7.

¹⁶ W. D. Ross, *Aristotle’s “Physics”: A Revised Text with Introduction and Commentary* (Oxford, 1936), 71.

overtake the slowest, since the pursuer must first reach a point through which the pursued has passed. By then, however, the pursued has moved to another such point. As long as the distance between pursuer and pursued is infinitely divisible into such diminishing gaps, Achilles, having started behind, can never catch the tortoise. His motion entails a series of increasingly smaller additive moves for which there is no last move.

This paradox has received an overwhelming amount of attention, particularly in the last fifty years, and there are several different strategies for solution. Some are more amusing than others, but none are uncontroversial. Alan White sends Achilles to a shooting gallery so he can learn to lead a target.¹⁷ The paradox has nothing to do with infinity, contends White, but shows that the tortoise cannot be caught by a series of gap-closures. Mayo responds that Zeno's concern is precisely that "there are an infinite number of such 'gaps,'" ¹⁸ and Hanson argues that White missed the point by conflating intercepting with overtaking.¹⁹

A popular strategy is language reform, sometimes in strange ways given that Zeno's language was Greek. For example, Ritchie argues the paradox arises out of an inaccurate way of expressing the relation between a foot and 12 inches.²⁰ Metcalf finds a mathematical fallacy concealed behind the word "never."²¹ Schwayder sorts out the difference between "any" and "all."²² Nelson wants to correct the word "motion,"²³ Quan, "quicker."²⁴ The general suspicion is that Zeno is a wolf (that is, a mathematician) in sheep's clothing (that is, ordinary language). He is either playing on some covertly introduced incorrect mathematical assumption or is himself the victim of inadequate mathematics.

Other solutions leave Zeno's language alone and go straight to his mathematics. Broad argues that the structure of the argument is deceptive: it gives only *some* of the points in the trajectory of the racers and not the one where they meet.²⁵ TeHennepe argues that what one needs to solve the paradoxes is a precise notion of infinity. The cause of the paradox is that technical, mathematical language "is uncritically transplanted into common language."²⁶ Maxwell and

¹⁷ Alan R. White, "Achilles at the Shooting Gallery," *Mind*, 72 (1963), 141-42.

¹⁸ Bernard Mayo, "Shooting It Out With Zeno," *Mind*, 73 (1964), 282-83.

¹⁹ Norwood Russell Hanson, "The Tortoise Shoots Back," *Philosophical Studies*, 16 (1965), 14-16.

²⁰ A. D. Ritchie, "Why Achilles Does Not Fail to Catch the Tortoise," *Mind*, 50 (1941), 310-11.

²¹ Wilmont V. Metcalf, "Achilles and the Tortoise," *Mind*, 51 (1942), 89-90.

²² David S. Schwayder, "Achilles Unbound," *Journal of Philosophy*, 52 (1955), 449-59.

²³ John O. Nelson, "Zeno's Paradoxes on Motion," *Review of Metaphysics*, 16 (1963), 486-90.

²⁴ Stanislaus Quan, "The Solution of the Achilles Paradox," *Review of Metaphysics*, 16 (1963), 473-85.

²⁵ C. D. Broad, "Note on Achilles and the Tortoise," *Mind*, 22 (1913), 318-19.

²⁶ Eugene TeHennepe, "Language-Reform and Philosophical Imperialism: Another Round with Zeno," *Analysis*, supplement to 23 (1963), 43-49: 47.

Feigl argue likewise that in mathematical language, Zeno's paradoxes lose their sting.²⁷ The correct way to talk is apparently mathematical.

There are indeed two mathematical treatments of the paradox. The older purports to solve it using the limit of an infinite series. The more recent refutes it by means of transfinite numbers. Gregoire St. Vincent first solved the problem using limits in 1647.²⁸ An infinite series can have a limit. The series of rational numbers, for example, between 0 and 1 is infinite, but 1 is its limit. The paradox is solved in that the infinite series of diminishing gaps between Achilles and the tortoise has a mathematical limit. That limit represents the point at which Achilles passes the tortoise.

This solution was supported by Descartes and Whitehead.²⁹ It is so simple that Peirce dismissed the paradoxes as easily soluble "to a mind adequately trained in mathematics and logic."³⁰ Russell declared it a two-thousand-year achievement in answering Zeno.³¹ In the 1950s, however, a series of objections to the limit solution were raised by James Thomson, Max Black, and J. Wisdom.

Thomson and Black both argue that Achilles is being asked to perform an infinite number of tasks.³² To complete an infinite series of tasks is to perform a super-task, says Thomson. He and Black demonstrate how this notion is nonsensical and contradictory by means of imaginary infinity machines. Thomson's is a lamp that can be turned on and off, Black's is a machine for moving marbles between boxes. Thomson thinks the mathematical solution glosses over the discomfort one should feel at the introduction of infinity to a physical process; and similarly Black argues, as did Aristotle, that there is no such thing as infinity in actuality.

Thomson has been told that his claims have no consequences for the completion of super-tasks,³³ accused of formally invalid argument,³⁴ and advised that after an infinite number of switchings his lamp is in fact on.³⁵ Ray counters that it remains paradoxical that the outcome cannot be predicted on the basis of the

²⁷ Grover Maxwell and Herbert Feigl, "Why Ordinary Language Needs Reforming," *The Journal of Philosophy*, 58 (1961), 488-98.

²⁸ Ross, 83.

²⁹ Rene Descartes, *Oeuvres* (11 vols.; Paris, 1901), IV, 445-47; Alfred North Whitehead, *Process and Reality* (New York, 1929), 85.

³⁰ C. S. Peirce, *Collected Papers of C. S. Peirce*, eds. Charles Hartshorne and Paul Weiss (8 vols.; Cambridge, Mass., 1935), VI, 177.

³¹ Bertrand Russell, "The Problem of Infinity Considered Historically," *Our Knowledge of the External World* (London, 1914), 175-78.

³² James Thomson, "Tasks and Super-Tasks," *Zeno's Paradoxes*, ed. Wesley Salmon (New York, 1970), 89-102; Max Black, "Achilles and the Tortoise," *Analysis*, 2 (1951), 91-101.

³³ R. M. Sainsbury, *Paradoxes* (Cambridge, 1988), 15.

³⁴ Paul Benacerraf, "Tasks, Super-tasks, and the Modern Eleatics," *Journal of Philosophy*, 59 (1962), 765-84: 770.

³⁵ Geoffrey C. Berresford, "A Note on Thomson's Lamp 'Paradox,'" *Analysis*, 41 (1981), 1-3: 2.

infinite tasks.³⁶ Thomson, however, concedes in response to his critics that super-tasks may not be contradictory after all and appeals that nonetheless “if enough people feel a conceptual difficulty about something, that is some reason for thinking that there *is* a difficulty.”³⁷ True, but unhelpful.

Against Black, Taylor and Watling argue that to perform an infinite number of tasks is logically consistent.³⁸ It’s just difficult to imagine, says Taylor, while Watling reverts to limits to make it make sense. A year later Taylor, reminiscent of Aristotle’s solution to the Dichotomy, solves the paradox taking an opposite strategy to Black’s denial of infinity: Achilles has infinitely many times to cross the infinitely divisible distance.³⁹ Thomas argues similarly that Achilles himself is just as indefinite as the space he must traverse.⁴⁰ But these approaches seem even more confounding to common sense than the original paradox. One almost hears Zeno chuckle.

Wisdom rejects the limit solution on the grounds that a limit is “not attained by finding the sum of a finite number of terms however much this is increased, i.e. is not attained by any amount of counting and adding.”⁴¹ One can never reach the sum of an infinite series from within that series. The solution by mathematical limit assumes the very point Zeno is bringing into question: how is it possible to reach the end of an infinite series? The solution by appeal to the sum of an infinite series fails because it simply reformulates what the Achilles demonstrates: how can the limit of an infinite series be reached from within the series?

The second attempt at mathematical refutation uses Cantor’s and Dedekind’s work from the latter part of the nineteenth century.⁴² Cantor showed that there are different orders of infinity. Many infinities are denumerable, that is, each member of the infinite set can be paired with a positive integer from the number series. But some infinities are not. For example, although there are infinite rational numbers, there are more real numbers, which include both rationals and irrationals. Reals are a higher order of infinity than rationals.

Harold Lee argues that in the Achilles, Zeno mistakes density for continuity.⁴³ Density is a property belonging to a magnitude, a distance for example,

³⁶ Christopher Ray, “Paradoxical Tasks,” *Analysis*, 50 (1990), 74.

³⁷ James Thomson, “Comments on Benacerraf’s Paper,” *Zeno’s Paradoxes*, ed. Wesley Salmon (New York, 1970), 130-38: 133.

³⁸ Richard Taylor, “Mr. Black on Temporal Paradoxes,” *Analysis*, 12 (1951-52), 38-44; J. Watling, “The Sum of an Infinite Series,” *Analysis*, 13 (1952-53), 39-46.

³⁹ Richard Taylor, “Mr. Wisdom on Temporal Paradoxes,” *Analysis*, 13 (1952-53), 15-17.

⁴⁰ L. E. Thomas, “Achilles and the Tortoise,” *Analysis*, 12 (1951-52), 92-94.

⁴¹ J. O. Wisdom, “Achilles on a Physical Racecourse,” *Analysis*, 12 (1951-52), 68.

⁴² Georg Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers*, tr. Philip Jourdain (New York, 1915); Richard Dedekind, *Essays on the Theory of Numbers*, tr. W. W. Berman (Chicago, 1901).

⁴³ Harold N. Lee, “Are Zeno’s Paradoxes Based on a Mistake?” *Mind*, 74 (1965), 563-70.

that is infinitely divisible. Between any two points there are infinitely more points such that the magnitude is “dense.” The Achilles is predicated on the fact that the distance between Achilles and the tortoise is dense. A feature of density is that no two points touch, since they are always separated by more points. Density therefore precludes continuity. Furthermore, Zeno’s argument depends on divisibility into rational fractions. There was no other way to understand divisibility until Dedekind showed in 1872 that if a magnitude is cut, “there exist infinitely many cuts not produced by rational numbers,”⁴⁴ and he gave a formula for such cuts. Zeno’s account omits irrational numbers and hence the infinitely divisible distance that Achilles must traverse to catch the tortoise has gaps. It is not continuous.

Were it the case that the Achilles is predicated upon continuity, his paradox would now be refuted. But it is not. The Achilles requires that the distance between the racers is infinitely divisible. Whether it is dense or continuous is irrelevant. Perhaps the belief that Zeno conflates density with continuity arises because we have only Aristotle’s account of the paradox, and as pointed out in reference to the Dichotomy, Aristotle defined continuity as infinite divisibility. The refutation from transfinite numbers has no more success than did the solution by mathematical limit. Achilles is still running.

The Dichotomy and the Achilles are aimed at reducing to absurdity the claim that distance is infinitely divisible. The Dichotomy argues that if this is the case, then motion cannot begin; the Achilles, that it cannot end. The third and fourth of Zeno’s paradoxes of motion, the Arrow and the Stadium, attack the opposing hypothesis: distance is finitely divisible; that is, any distance divided sufficiently many times will eventually result in some indivisible magnitude.

The Arrow

Schofield recreates Zeno’s syllogism on the basis of Diogenes Laertius:

- (1) Anything occupying a place just its own size is at rest.
- (2) In the present, what is moving occupies a place just its own size.
- So (3) in the present, what is moving is at rest.
- Now (4) what is moving always moves in the present.
- So (5) what is moving is always—throughout its movement—at rest.⁴⁵

Aristotle argues that the statement that the arrow is stationary results from granting that time is composed of moments or instants, that it is a series of discrete “nows.”

⁴⁴ Dedekind, 13.

⁴⁵ G. S. Kirk, J. E. Raven, and M. Schofield, *The Presocratic Philosophers* (Cambridge, 1983²), 273.

This is the intent of “the present” in premises two and three. Such instants are the smallest unit of time, indivisible, and as such, of no duration. The moving arrow is, as Russell points out, “never moving, but in some miraculous way the change of position has to occur *between* instants,” i.e., at no time.⁴⁶ Bergson calls this view the “cinematographic” representation of reality, which is fully in accord with the Zenonian conclusion that motion is an illusion.⁴⁷

Aristotle rejects the thesis that time is made up of instants in *Physics* 4.11, where he argues that the “now” is not an indivisible unit of time but rather a demarcation of past from future, analogous with his view on lines and points.⁴⁸ As a line does not consist of points but rather is limited and divided by points, so time does not consist of moments but can be divided and limited by them. For Aristotle the “now” is a number that distinguishes the before from the after. Schofield argues that Aristotle is mistaken, and that he is responsible for the equally mistaken idea that the Arrow implies that space and time are not infinitely divisible.⁴⁹ The Arrow, argues Schofield, commits Zeno only to the claim that what is true of something at every moment of a period of time, holds throughout that period. Alba Papa-Grimaldi argues likewise that the Arrow is intended to make difficult the relation between the one and the many in that it makes inexplicable the connection between the instant and duration.⁵⁰ But as Papa-Grimaldi shows, in order to do so, it is predicated on the atomicity of time.

McLaughlin and Miller answer the challenge of the Arrow using infinitesimals. These are tiny intervals of space and time that are “greater than 0 and less than every possible standard real [number].”⁵¹ They are found on either side of every real number, and it is here that motion takes place. This seems a revival of the medieval notion of infinitesimal: that which is not nothing, but when added causes no increase, and no decrease when taken away. Before calculus it was used to make sense of acceleration.

Zangari argues that the Arrow is fallacious by rejecting the third premise, that in the present what is moving is at rest. He argues that the velocity of the arrow is not 0 but rather indeterminate. In any finite time interval δt the arrow travels some distance, δx . Its velocity is $\delta x / \delta t$. In Zeno’s instant, $\delta t = 0$, as does δx . Hence its velocity in the instant is $0/0$. But, argues Zangari, to have a determinate velocity of 0, δt would have to be greater than 0. In other words, to be at rest is to travel no distance over some enduring time. But an arrow travelling at

⁴⁶ Russell, 179.

⁴⁷ Henri Bergson, “The Cinematographic View of Becoming,” *Zeno’s Paradoxes*, ed. Wesley Salmon (New York, 1970), 59–66.

⁴⁸ Aristotle, *Physics*, 219b26.

⁴⁹ Kirk, Raven, and Schofield, 273.

⁵⁰ Alba Papa-Grimaldi, “Why Mathematical Solutions of Zeno’s Paradoxes Miss the Point: Zeno’s One and Many Relation and Parmenides’ Prohibition,” *Review of Metaphysics*, 50 (1996), 299–314.

⁵¹ William McLaughlin and Sylvia Miller, “An Epistemological Use of Nonstandard Analysis to Answer Zeno’s Objections Against Motion,” *Synthese*, 92 (1992), 371–84: 376.

any speed will cross no distance when $\delta t = 0$. Its velocity is indeterminate since there is “an infinite number of possible states of motion that are consistent with the information provided.”⁵² Has Zangari refuted Zeno’s Arrow? Or has he confirmed precisely Zeno’s point: if you take time to be atomic, there can be no such thing as motion. For motion can, then, only take place between instants; this is unintelligible.

The Stadium

This paradox, also called the Moving Rows, has received the least attention from commentators and critics, perhaps because it is difficult to reconstruct as a result of Aristotle’s scanty description and quick solution of an apparently bastardized version. Indeed, it seems appropriate to treat Aristotle as a hostile witness, as it were. There is a diagram accompanying the brief explanation in Simplicius’s commentary, which he attributes to Alexander:

$$\begin{array}{c} [A_1][A_2][A_3][A_4] \\ [B_1][B_2][B_3][B_4] \longrightarrow \\ \longleftarrow [C_1][C_2][C_3][C_4] \end{array}$$

Two rows of bodies (Bs and Cs) move at the same speed past each other in opposite directions on a race-course. They pass a third row (As) that remains stationary. Aristotle states that Zeno’s conclusion is that “half a given time is equal to double that time”⁵³ and he claims Zeno reaches it by assuming “that a body occupies an equal time in passing with equal velocity a body that is in motion and a body of equal size that is at rest.”⁵⁴ Aristotle thinks that Zeno’s worry is about measuring velocity and that Zeno’s “fallacy lies in maintaining that an object with an equal speed takes an equal time to pass a moving body as a stationary body of an equal length; and to maintain this is false.”⁵⁵ Aristotle is quick to dismiss the paradox, which thus understood is the silly mistake of not noticing that motion is relative.

The Moving Rows is, however, readily intelligible and nowhere near as dismissable, if it is read as an argument against the atomicity of distance. Suppose each unit of As, Bs, and Cs is a minimal unit, that is, the smallest possible unit of distance. Such a minimal is by definition indivisible. In the time one C passes one B, it has passed half an A. But this is impossible since there can be no half of the minimal. Under this reading, the Moving Rows, like the Arrow, de-

⁵² Mark Zangari, “Zeno, Zero and Indeterminate Forms: Instants in the Logic of Motion,” *Australasian Journal of Philosophy*, 72 (1994), 193.

⁵³ Aristotle, *Physics*, 240a1. This is Hardie and Gaye’s translation in *The Basic Works of Aristotle*, ed. Richard McKeon (New York, 1941).

⁵⁴ Aristotle, *Physics*, Hardie and Gaye, 240a3.

⁵⁵ Aristotle, *Physics*, 240a2-4.

nies the atomicity of distance in that it refutes the possibility that both time and space are finitely divisible.

Individually, then, each of Zeno's paradoxes is a cleverly constructed *reductio ad absurdum* concerning the divisibility of space, time, and distance. Since they refute contradictory hypotheses, however, how can they be read together consistently?

Zeno against the Pythagoreans

Ross rejects on chronological grounds both Stallbaum's argument that Zeno's targets were Anaxagoras and Leucippus, and Gomperz's suggestion that the paradoxes were directed at Gorgias's "satirical exaggeration of Parmenides' views."⁵⁶ He further finds the evidence slight that the Stadium is aimed at Empedocles. Nor is he convinced that the Stadium has the Pythagoreans as its aim. In fact, although he suggests that Zeno directed his arguments against "people who ridiculed Parmenides' denial of the existence of plurality,"⁵⁷ he claims in his commentary that "it cannot well be maintained that all four of Zeno's arguments were directed against a single thinker or school; for the first two are directed against those who believe, and the last two against those who disbelieve, in the infinite divisibility of continua."⁵⁸ The inconsistent and antithetical nature of the paradoxes is, however, evidence for the view that the paradoxes are directed at a single school, if one believes that school to be the Pythagoreans.

We have it on Aristotle's authority that the Pythagoreans held that numbers are the principles and elements of everything⁵⁹ and the causes of being.⁶⁰ The Pythagoreans thus collapsed the mathematical with the physical. I suggest that Zeno is attacking that collapse: nature does not consist in mathematical entities. Zeno's paradoxes of motion demonstrate that mathematizing nature results in absurdity. If Zeno intends to refute the Pythagorean thesis that the nature of physical reality is mathematical, how is it possible to do this? As an a priori assumption, the thesis cannot be demonstrated empirically. A reasonable strategy to achieve such an end would be to take a mathematical concept and show that whatever application it is given to physical entities results in absurdity. Hence one would assume contrary hypotheses and impale the opponent on both horns. Zeno does this with divisibility. What better number to display the difference between mathematics and physical reality than infinity, a number never encountered in experience? He produces absurd conclusions from the thesis that time and distance are infinitely divisible and from the thesis that they are finitely divisible. If Zeno held these theses to be contradictory, then he should choose

⁵⁶ Ross, 656.

⁵⁷ Ross, 72.

⁵⁸ Ross, 656.

⁵⁹ Aristotle, *Metaphysics*, tr. Hugh Tredennick (Cambridge, Mass., 1933), 985b24-986b9.

⁶⁰ Aristotle, *Metaphysics*, 987b25.

one and refute the other. If he held both, he is inconsistent. If, however, he took the theses to be contrary, he could hold neither. Rather, he would hold that physical reality is neither finitely nor infinitely divisible. For infinity is a mathematical concept belonging to the ideal and not the physical realm.

Accordingly, Zeno could be perfectly content with arguments within mathematics about the divisibility of lines. In fact what the Germans (meaning Cantor and Dedekind) have shown, claims Ross, is that “the problems about space, time, and movement ... are involved in the nature of the number series itself and merely exemplified in such problems as Zeno’s.”⁶¹ What Zeno has shown goes further, however. When discussion of the mathematical issues spills over into physical analogy, absurdity results. One cannot look to physical entities to resolve issues about ideal entities. There can be no empirical investigation of the divisibility of lines. Nor is the mathematization of nature an unproblematic explanatory strategy in physics.

Taken together, the paradoxes do seem to have an ordering structure. A group of French mathematicians showed at the end of the nineteenth century that the paradoxes are arranged according to a plan. Amongst them, Brochard perhaps best expresses the intricacy of the relations between the four arguments; indeed, his overview is worth repeating:

the first and the fourth consider ... motion between given limits; the second and the third envisage them in indeterminate and indifferent lengths. In the first and the third, one single mover is charged with realizing the motion, and it finds that even the beginning of movement is impossible. The second and the fourth, through comparison of two movers in motion, making the absurdity of the hypothesis more obvious, prove that the motion, even if started, could not continue, and demonstrate the impossibility of relative motion as well as of absolute motion.—The first two establish the impossibility of motion through the nature of space ... although time is still considered to be composed in the same manner as space; in the last two, it is the nature of time which serves to prove the impossibility of motion, while space is still considered as formed also of indivisible points.—Finally, the second is only another form of the first, and the fourth relies on the same principle as the third.—The first pair of arguments is destined to combat the idea which naturally presents itself first to the mind, that of indefinite divisibility of the continuous; the second [pair] opposes itself to the conception which only offers itself to thought when it has recognized the difficulties of the first [pair].⁶²

⁶¹ Ross, 85.

⁶² Reproduced at H. D. P. Lee, *Zeno of Elea: A Text, with Translation and Notes* (Amsterdam, 1967), 103. This is my translation.

Brochard concludes that the order in which Aristotle presents the arguments must have been Zeno's order. Indeed, it hardly seems possible that the symmetries apparent in the sequence of arguments could be coincidental. The paradoxes can quite sensibly be taken, then, to form a unified whole, a single argument. As such, they defend Parmenides against his critics. One such group of critics was the Pythagoreans.

Booth argues that critics "have not shown that it is completely untenable"⁶³ that Zeno's attack was directed at the Pythagoreans, but I suggest a stronger claim. Remembering that Zeno was a student of Parmenides (in fact his adopted son according to Diogenes Laertius and Plato⁶⁴) and that Eleaticism has its beginning in Parmenides' revolt against the Pythagoreans,⁶⁵ it is readily plausible that the Pythagoreans were Zeno's target. H. D. P. Lee argues strongly that this is the case on the basis both of the arguments on plurality and the arguments on motion.⁶⁶ Burnet argues that they are the only plausible target since Zeno wrote his book in his youth, and they were the only critics of Parmenides in Italy at that time.⁶⁷ Papa-Grimaldi argues that Zeno's likely target was extremely precise: "the Pythagorean pretense to get the many of the Universe by multiplication or addition of the unit."⁶⁸ Papa-Grimaldi argues that the Arrow and the Stadium show that one cannot get from one to many or from identity to change simply by addition.⁶⁹

A difficulty arises here in that reading Zeno against the Pythagoreans contextualizes him as an Eleatic. Booth has raised the question, how can we "be sure that his arguments were valid against the 'ones' of a plurality, not against Parmenides's One"?⁷⁰ If the Arrow and the Stadium argue the impossibility of obtaining the many from the one by showing that the one is problematic, then Zeno's attack is as anti-Eleatic as it is anti-Pythagorean. On my account this difficulty falls away. I suggest that Zeno is reinforcing Parmenides' separation of truth from seeming. Parmenides' text is περὶ φύσεω, about the natures of things, and indeed the goddess promises knowledge in Fragment 10, for example, about the nature of the aether and the moon.⁷¹ She separates the way of seeming, upon which these things are found, from the way of truth, upon which

⁶³ N. B. Booth, "Were Zeno's Arguments Directed Against the Pythagoreans?" *Phronesis*, 2 (1957), 90-103: 90.

⁶⁴ Diogenes Laertius, *Lives of Eminent Philosophers*, tr. R. D. Hicks (2 vols.; London, 1925), II, IX.25, 435; Plato, *Sophist*, tr. Nicolas P. White (Indianapolis, 1993), 241d.

⁶⁵ John Burnet, *Early Greek Philosophy* (London, 1920), 82.

⁶⁶ H. D. P. Lee, 104.

⁶⁷ Burnet, 314.

⁶⁸ Papa-Grimaldi, 305.

⁶⁹ Papa-Grimaldi, 307, 312.

⁷⁰ N. B. Booth, "Were Zeno's Arguments a Reply to Attacks on Parmenides?" *Phronesis*, 2 (1957), 1-9: 1.

⁷¹ Parmenides, *Fragments*, tr. David Gallop (Toronto, 1984).

are found the truths of the One, that it is unchanging, and like a sphere, etc. I take it the Arrow and the Stadium are defending this separation. The Parmenidean One belongs not to physical reality, the way of seeming, but to the metaphysical realm of the ideal. If Zeno is demonstrating that collapsing the realms of being and becoming leads to confusion, his paradoxes can be read quite sensibly and consistently as a defense of Parmenides and an attack on the Pythagoreans. For these both follow from the same thesis: mathematics is not the language of physical reality.

Modern thinkers who come at the paradoxes from logic and mathematics have failed to see the challenge Zeno makes to the Pythagoreans precisely because they accept the modern assumption that nature can be described in the language of mathematics. If the paradoxes operate at the intersection of the mathematical and the physical, in fact bring that very intersection into question, it makes sense that they could be treated short-sightedly in strictly mathematical terms. Such competent thinkers as Peirce, Russell, and Whitehead have treated the paradoxes in exactly these terms, yet have failed to show how mathematical treatment brings definitive closure. Russell, for example, may have claimed that the work of Cantor and Dedekind on transfinite numbers solved the paradoxes; but as Ross argues, Russell fails to show how. Critics have regularly accused Zeno either of being a deceiving mathematical trickster or of not being very good at mathematics and logic. Yet his paradoxes are intensely analytic logico-mathematical puzzles, and he is taken by others to get at questions at the very root of mathematics. At the same time, however, the paradoxes are non-technical formulations that frame the questions in the popular folklore of the quotidian. Accordingly, Bergson and Whitehead, for example, interpret Zeno metaphysically, that is, as raising problems about the nature of reality. Bergson argues that the paradoxes “all consist in applying the movement to the line traversed, and supposing that what is true of the line is true of the movement.”⁷² I argue the broader thesis: Zeno is objecting to the mathematization of nature.

One argument taking the paradoxes to be more than dialectical and sophistic arguments intended to establish no positive thesis but simply show, as H. D. P. Lee puts it, that “no belief is any truer than its contradictory,” is the fact that the four arguments depend on two contradictory hypotheses.⁷³ The Dichotomy and the Achilles show the absurd consequences of taking physical reality to be constituted continuously and hence infinitely divisible; the Arrow and the Moving Rows work against taking physical reality to be constituted atomistically by the indivisible minimal and hence finitely divisible. This contradiction in primary assumption works for the claim that the paradoxes are intended to be taken together rather than as refutation by *reductio* of particular hypotheses. If Zeno is

⁷² Bergson, 64.

⁷³ H. D. P. Lee, 122.

suggesting that the qualities belonging to the mathematical should not be transferred immediately to the physical world, then it makes sense for him to challenge both that space is infinitely and finitely divisible, in order to show that the notion of divisibility has no home in the physical world. Zeno's point, I argue, is that divisibility and continuity are simply not appropriate issues to bring to physics. Mathematics and physics may have had for Zeno, as they did for Aristotle, different objects. Blurring the distinction would then be a mistake. This, I take it, is precisely Zeno's point.

According to this interpretation of the Stadium, for example, Zeno is not committed to the indivisibility of the "now" but is problematizing that very notion. This reading is consistent with the Dichotomy, which can be taken to deny motion by asserting infinite divisibility. Furthermore, such a reading is consistent with Aristotle's view that time and distance are infinitely divisible. Aristotle argues in *Physics* 3.6 that infinity is the opposite of its usual description. It is usually taken to mean that beyond which there is nothing. Rather it means, says Aristotle, "what always has something outside it."⁷⁴ For Aristotle infinity is not actual, but exists only potentially.⁷⁵ Yet he does not want to assert a beginning and end to time. Hence he explains this potentiality by suggesting that the infinite is potential in the sense that it is successive, the way a day "is" or the games "are": "one thing after another is always coming into existence," but not all parts are actualizable simultaneously.⁷⁶ As unactualizable, the infinite is unknowable.⁷⁷ This does not, however, rob the mathematicians of their science, for all his account of infinity denies is the untraversable.⁷⁸ At *Physics* 3.7 Aristotle treats infinity as a material cause, claiming that there is a greatest magnitude but not a least amount into which a thing can be divided. Although there is potentially always a greater magnitude, there is a limit to actual extension. Likewise there is no limit to divisibility, but there is a limit to divisions. Hence for Aristotle the infinite stands in the relation of part, not whole.⁷⁹ It is about divisibility into parts, not extension between extremities. So although Aristotle holds, as Hussey argues, "that the physical world exhibits mathematical relationships of various kinds," he does not hold that every mathematical concept has a counterpart in nature.⁸⁰ There is no infinite substance.

⁷⁴ Aristotle, *Physics*, 207a1; cf. 206b14.

⁷⁵ Aristotle, *Physics*, 206a17. Cf. William Charlton, "Aristotle's Potential Infinites," *Aristotle's "Physics": A Collection of Essays*, ed. Lindsay Judson (Oxford, 1991), 129-49.

⁷⁶ Aristotle, *Physics*, 206a22. Cf. Jaakko Hintikka, *Time and Necessity: Studies in Aristotle's Theory of Modality* (Oxford, 1973), 116.

⁷⁷ Aristotle, *Physics*, 207a26.

⁷⁸ Aristotle, *Physics*, 207b29.

⁷⁹ Aristotle, *Physics*, 207a25.

⁸⁰ Edward Hussey, "Aristotle's Mathematical Physics: A Reconstruction," *Aristotle's "Physics": A Collection of Essays*, ed. Lindsay Judson (Oxford, 1991), 213.

I argue that Zeno likewise is uncomfortable with the notion of infinity too readily applied to physical reality. In fact Zeno is rendering problematic the application of mathematical concepts to the material world. Aristotle takes him instead to be denying motion. Aristotle, whose most significant work is not in the realm of mathematics, takes the paradoxes to have a physical thesis rather than to be directed at the coincidence of the physical and the mathematical.

Zeno's paradoxes can be taken, then, to set forth the horns of a dilemma. If one takes motion to be continuous, it is infinitely divisible, and the absurdities of the Dichotomy and the Achilles follow. If one takes motion to be discontinuous, that is, composed of discrete, indivisible minimals, the Arrow and the Moving Rows result. Hence, to ask whether motion is continuous is to apply a property that properly belongs to the number series to physical reality. It is to confuse the ideal with the real. Black claims that Zeno's argument "rests upon subtle confusions between two ways of talking about space, time and motion"⁸¹—the mundane common-sense way and the precise mathematical way. Infinite divisibility is not a notion belonging to everyday life, but has its home rather in "the more exacting tasks of technology and pure science."⁸² Alternatively, tasks that belong to the quotidian and hence to common-sense discourse have no home in mathematics. To talk of an infinite number of tasks to be performed is therefore to confuse two distinct realms of discourse.

Furthermore, Wisdom dismisses the Achilles by arguing that Zeno is simply wrong. A physical point, unlike a mathematical point, must have some size, and therefore there cannot be an infinite number of physical points in a finite distance. Likewise, refutations of the mathematical solution to the Achilles by means of the limit of an infinite series, notably the refutations laid out by Thomson and Black, only work because the paradox operates precisely at the point where mathematics and physical reality coincide.⁸³ Mathematically, the limit of an infinite series is *not* a member of the series. Physically, Achilles beats the tortoise. Neither of these points is an issue until they are taken together. As Thomson has pointed out, what Zeno forces one to consider is "*applications* of the concept of infinity."⁸⁴ Infinity is not problematic as a mathematical concept. But its application to physical reality is inappropriate. Mathematical descriptions of physical reality fail, as apparent from the paradoxical results they engender.

⁸¹ Max Black, *Problems of Analysis* (London, 1954), 125.

⁸² Black, *Problems of Analysis*, 126.

⁸³ Thomson, "Tasks and Super-Tasks"; Black, "Achilles and the Tortoise."

⁸⁴ Thomson, "Tasks and Super-Tasks," 101.

Conclusion

Max Planck argued in 1936 that theoretical physics “substitutes a new world in place of that given to us by the senses.”⁸⁵ He calls this new world the physical world image. Through the world image the physicist leaves behind the inaccuracy of the physical world and works in a precisely defined ideal realm. An object is taken from the world of the senses and symbolized in the world image. Later the symbols are translated back into the world of the senses. The hope of classical physics was that the discrepancy between the two worlds would increasingly diminish as measuring apparatus became more accurate. Quantum physics poses, however, a different challenge. The discrepancy between the two worlds remains, but further it is not clear how, for example, the wave function of quantum mechanics can be understood to correspond to a physical process, perceptible by the senses, when it “denotes no more than the probability that a certain state exists.”⁸⁶ The interface between pure mathematics and physical theory is no longer as straightforward as Newtonian physics would imply. This development may give good reason to look again at Zeno’s paradoxes.

Over two thousand years ago, Zeno attacked the Pythagorean belief that everything is a number and demonstrated the paradoxical consequences of mathematizing physical reality. Quantum theory, despite long-standing acceptance of its mathematical formalism, lacks sufficient agreement about its physical interpretation even to establish its philosophical significance. Arthur Fine argues, in what Niiniluoto calls “philosophical despair,” that there are simply not adequate resources in quantum theory to settle the philosophical issues of realism that plague the physical interpretation of the formalism.⁸⁷ Twentieth-century quantum theory shows that Zeno was right: the mathematization of nature is not as insightful as the physicists would wish. For as Nancy Cartwright once said of the formal mathematics of quantum theory, “One may know all of this and not know any quantum mechanics.”⁸⁸ Quantum theory is a superb and sophisticated mathematics, but what does it tell one of nature, of physical reality?

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⁸⁵ Max Planck, *Philosophy of Physics*, tr. W. H. Johnston (New York, 1936), 53.

⁸⁶ Planck, 67.

⁸⁷ Niiniluoto, 471.

⁸⁸ Nancy Cartwright, *How the Laws of Physics Lie* (Oxford, 1983), 135.