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CHAPTER ONE

Fermat's Enigma BD

The Quest to Solve the World's Greatest Mathematical Problem

By SIMON SINGH

Walker and Company

[Read the Review](#)**"I Think I'll Stop Here"**

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G. H. Hardy

June 23, 1993, Cambridge

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The rumors had started the previous day. Electronic mail over the Internet had hinted that the lecture would culminate in a solution to Fermat's Last Theorem, the world's most famous mathematical problem. Such gossip was not uncommon. The subject of the Last Theorem would often crop up over tea, and mathematicians would speculate as to who might be doing what. Sometimes mathematical mutterings in the senior common room would turn the speculation into rumors of a breakthrough, but nothing had ever materialized.

This time the rumor was different. When the three blackboards became full, the lecturer paused. The first board was erased and the algebra continued. Each line of mathematics appeared to be one tiny step closer to the solution, but after thirty minutes the lecturer had still not announced the proof. The professors crammed into the front rows waited eagerly for the conclusion. The students standing at the back looked to their seniors for hints of what the conclusion might be. Were they watching a complete proof to Fermat's Last Theorem, or was the lecturer merely outlining an incomplete and anticlimactic argument?

The lecturer was Andrew Wiles, a reserved Englishman who had emigrated to America in the 1980s and taken up a professorship at Princeton University, where he had earned a reputation as one of the most talented mathematicians of his generation. However, in recent years he had almost vanished from the annual round of conferences and seminars, and colleagues had begun to assume that Wiles was finished.

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The lecturer was Andrew Wiles, a reserved Englishman who had emigrated to America in the 1980s and taken up a professorship at Princeton University, where he had earned a reputation as one of the most talented mathematicians of his generation. However, in recent years he had almost vanished from the annual round of conferences and seminars, and colleagues had begun to assume that Wiles was finished. It is not unusual for brilliant young minds to burn out, a point noted by the mathematician Alfred Adler: "The mathematical life of a mathematician is short. Work rarely improves after the age of twenty-five or thirty. If little has been accomplished by then, little will ever be accomplished."

"Young men should prove theorems, old men should write books," observed G. H. Hardy in his book *A Mathematician's Apology*. "No mathematician should ever forget that mathematics, more than any other art or science, is a young man's game. To take a simple illustration, the average age of election to the Royal Society is lowest in mathematics." His own most brilliant student, Srinivasa Ramanujan, was elected a Fellow of the Royal Society at the age of just thirty-one, having made a series of outstanding breakthroughs during his youth. Despite having received very little formal education in his home village of Kumbakonam in South India, Ramanujan was able to create theorems and solutions that had evaded mathematicians in the West. In mathematics the experience that comes with age seems less important than the intuition and daring of youth.

Many mathematicians have had brilliant but short careers. The nineteenth-century Norwegian

Niels Henrik Abel made his greatest contribution to mathematics at the age of nineteen and died in poverty, just eight years later, of tuberculosis. Charles Hermite said of him, "He has left mathematicians something to keep them busy for five hundred years," and it is certainly true that Abel's discoveries still have a profound influence on today's number theorists. Abel's equally gifted contemporary Evariste Galois also made his breakthroughs while still a teenager.

Hardy once said, "I do not know an instance of a major mathematical advance initiated by a man past fifty." Middle-aged mathematicians often fade into the background and occupy their remaining years teaching or administrating rather than researching. In the case of Andrew Wiles nothing could be further from the truth. Although he had reached the grand old age of forty he had spent the last seven years working in complete secrecy, attempting to solve the single greatest problem in mathematics. While others suspected he had dried up, Wiles was making fantastic progress, inventing new techniques and tools that he was now ready to reveal. His decision to work in absolute isolation was a high-risk strategy and one that was unheard of in the world of mathematics.

Without inventions to patent, the mathematics department of any university is the least secretive of all. The community prides itself in an open and free exchange of ideas and afternoon breaks have evolved into daily rituals during which concepts are shared and explored over tea or coffee. As a result it is increasingly common to find papers being published by coauthors or teams of mathematicians, and consequently the glory is shared out equally. However, if Professor Wiles had genuinely discovered a complete and accurate proof of Fermat's Last Theorem, then the most wanted prize in mathematics was his and his alone. The price he had to pay for his secrecy was that since he had not previously discussed or tested any of his ideas with the mathematics community, there was a significant chance that he had made some fundamental error.

Ideally Wiles had wanted to spend more time going over his work and checking fully his final manuscript. But when the unique opportunity arose to announce his discovery at the Isaac Newton Institute in Cambridge he abandoned caution. The sole aim of the institute's existence is to bring together the world's greatest intellects for a few weeks in order to hold seminars on a cutting-edge research topic of their choice. Situated on the outskirts of the university, away from students and other distractions, the building is especially designed to encourage the academics to concentrate on collaboration and brainstorming. There are no dead-end corridors in which to hide and every office faces a central forum. The mathematicians are supposed to spend time in this open area, and are discouraged from keeping their office doors closed. Collaboration while moving around the institute is also encouraged—even the elevator, which travels only three floors, contains a blackboard. In fact every room in the building has at least one blackboard, including the bathrooms. On this occasion the seminars at the Newton Institute came under the heading of "L-functions and Arithmetic." All the world's top number theorists had been gathered together in order to discuss problems relating to this highly specialized area of pure mathematics, but only Wiles realized that L-functions might hold the key to solving Fermat's Last Theorem,

Although he had been attracted by having the opportunity to reveal his work to such an eminent audience, the main reason for making the announcement at the Newton Institute was that it was in his hometown, Cambridge. This was where Wiles had been born, it was here he grew up and developed his passion for numbers, and it was in Cambridge that he had alighted on the problem that was to dominate the rest of his life.

The Last Problem

In 1963, when he was ten years old, Andrew Wiles was already fascinated by mathematics. "I loved doing the problems in school. I'd take them home and make up new ones of my own. But the best problem I ever found I discovered in my local library."

One day, while wandering home from school, young Wiles decided to visit the library in Milton Road. It was rather small, but it had a generous collection of puzzle books, and this is what often caught Andrew's attention. These books were packed with all sorts of scientific conundrums and mathematical riddles, and for each question the solution would be conveniently laid out somewhere in the final few pages. But this time Andrew was drawn to a book with only one problem, and no solution.

The book was *The Last Problem* by Eric Temple Bell. It gave the history of a mathematical problem that has its roots in ancient Greece, but that reached full maturity only in the seventeenth century when the French mathematician Pierre de Fermat inadvertently set it as a challenge for the rest of the world. One great mathematician after another had been humbled by Fermat's legacy, and for three hundred years nobody had been able to solve it.

Thirty years after first reading Bell's account, Wiles could remember how he felt the moment he was introduced to Fermat's Last Theorem: "It looked so simple, and yet all the great mathematicians in history couldn't solve it. Here was a problem that I, a ten-year-old, could understand and I knew from that moment that I would never let it go. I had to solve it."

Usually half the difficulty in a mathematics problem is understanding the question, but in this case it was straightforward--prove that there are no whole number solutions for this equation:

$$[x.\text{sup}.n] + [y.\text{sup}.n] = [z.\text{sup}.n] \text{ for } n \text{ greater than } 2.$$

The problem has a simple and familiar look to it because it is based on the one piece of mathematics that everyone can remember--Pythagoras's theorem:

In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

$$\text{Or: } [x.\text{sup}.2] + [y.\text{sup}.2] = [z.\text{sup}.2].$$

Pythagoras's theorem has been scorched into millions if not billions of human brains. It is the fundamental theorem that every innocent schoolchild is forced to learn. But despite the fact that it can be understood by a ten-year-old, Pythagoras's creation was the inspiration for a problem that had thwarted the greatest mathematical minds of history.

In the sixth century B.C., Pythagoras of Samos was one of the most influential and yet mysterious figures in mathematics. Because there are no firsthand accounts of his life and work, he is shrouded in myth and legend, making it difficult for historians to separate fact from fiction. What seems certain is that Pythagoras developed the idea of numerical logic and was responsible for the first golden age of mathematics. Thanks to his genius numbers were no longer merely used to count and calculate, but were appreciated in their own right. He studied the properties of particular numbers, the relationships between them, and the patterns they formed. He realized that numbers exist independently of the tangible world and therefore their study was untainted by the inaccuracies of perception. This meant he could discover truths that were independent of opinion or prejudice and that were more absolute than any previous knowledge.

He gained his mathematical skills on his travels throughout the ancient world. Some tales would have us believe that Pythagoras traveled as far as India and Britain, but what is more certain is that he gathered many mathematical techniques from the Egyptians and Babylonians. Both these ancient peoples had gone beyond the limits of simple counting and were capable of performing complex calculations that enabled them to create sophisticated accounting systems and construct elaborate buildings. Indeed they saw mathematics as merely a tool for solving practical problems; the motivation behind discovering some of the basic rules of geometry was to allow reconstruction of field boundaries that were lost in the annual flooding of the Nile. The word itself, geometry, means "to measure the earth."

Pythagoras observed that the Egyptians and Babylonians conducted each calculation in the form of a recipe that could be followed blindly. The recipes, which would have been passed down through the generations, always gave the correct answer and so nobody bothered to question them or explore the logic underlying the equations. What was important for these civilizations was that a calculation worked--why it worked was irrelevant.

After twenty years of travel Pythagoras had assimilated all the mathematical rules in the known world. He set sail for his home island of Samos in the Aegean Sea with the intention of founding a school devoted to the study of philosophy and, in particular, concerned with research into his newly acquired mathematical rules. He wanted to understand numbers, not merely exploit them. He hoped to find a plentiful supply of freethinking students who could help him develop radical new philosophies, but during his absence the tyrant Polycrates had turned the once liberal Samos into an intolerant and conservative society. Polycrates invited Pythagoras to join his court, but the philosopher realized that this was only a maneuver aimed at silencing him and therefore declined the honor. Instead he left the city in favor of a cave in a remote part of the island, where he could contemplate without fear of persecution.

Pythagoras did not relish his isolation and eventually resorted to bribing a young boy to be his first pupil. The identity of the boy is uncertain, but some historians have suggested that his name was also Pythagoras, and that the student would later gain fame as the first person to suggest that athletes should eat meat to improve their physiques. Pythagoras, the teacher, paid his student three oboli for each lesson he attended and noticed that as the weeks passed the boy's initial reluctance to learn was transformed into an enthusiasm for knowledge. To test his pupil

Pythagoras pretended that he could no longer afford to pay the student and that the lessons would have to stop, at which point the boy offered to pay for his education rather than have it ended. The pupil had become a disciple. Unfortunately this was Pythagoras's only conversion on Samos. He did temporarily establish a school, known as the Semicircle of Pythagoras, but his views on social reform were unacceptable and the philosopher was forced to flee the colony with his mother and his one and only disciple.

Pythagoras departed for southern Italy, which was then a part of Magna Graecia, and settled in Croton, where he was fortunate in finding the ideal patron in Milo, the wealthiest man in Croton and one of the strongest men in history. Although Pythagoras's reputation as the sage of Samos was already spreading across Greece, Milo's fame was even greater. Milo was a man of Herculean proportions who had been champion of the Olympic and Pythian Games a record twelve times. In addition to his athleticism Milo also appreciated and studied philosophy and mathematics. He set aside part of his house and provided Pythagoras with enough room to establish a school. So it was that the most creative mind and the most powerful body formed a partnership.

Secure in his new home, Pythagoras founded the Pythagorean Brotherhood--a band of six hundred followers who were capable not only of understanding his teachings, but who could add to them by creating new ideas and proofs. Upon entering the Brotherhood each follower had to donate all his worldly possessions to a common fund, and should anybody ever leave he would receive twice the amount he had originally donated and a tombstone would be erected in his memory. The Brotherhood was an egalitarian school and included several sisters. Pythagoras's favorite student was Milo's own daughter, the beautiful Theano, and, despite the difference in their ages, they eventually married.

Soon after founding the Brotherhood, Pythagoras coined the word philosopher, and in so doing defined the aims of his school. While attending the Olympic Games, Leon, Prince of Phlius, asked Pythagoras how he would describe himself. Pythagoras replied, "I am a philosopher," but Leon had not heard the word before and asked him to explain.

Life, Prince Leon, may well be compared with these public Games for in the vast crowd assembled here some are attracted by the acquisition of gain, others are led on by the hopes and ambitions of fame and glory. But among them there are a few who have come to observe and to understand all that passes here.

It is the same with life. Some are influenced by the love of wealth while others are blindly led on by the mad fever for power and domination, but the finest type of man gives himself up to discovering the meaning and purpose of life itself: He seeks to uncover the secrets of nature. This is the man I call a philosopher for although no man is completely wise in all respects, he can love wisdom as the key to nature's secrets.

Although many were aware of Pythagoras's aspirations, nobody outside of the Brotherhood knew the details or extent of his success. Each member of the school was forced to swear an oath never to reveal to the outside world any of their mathematical discoveries. Even after Pythagoras's death a member of the Brotherhood was drowned for breaking his oath--he publicly announced the discovery of a new regular solid, the dodecahedron, constructed from twelve regular pentagons. The highly secretive nature of the Pythagorean Brotherhood is part of the reason that myths have developed surrounding the strange rituals that they might have practiced, and similarly this is why there are so few reliable accounts of their mathematical achievements.

What is known for certain is that Pythagoras established an ethos that changed the course of mathematics. The Brotherhood was effectively a religious community, and one of the idols they worshiped was Number. By understanding the relationships between numbers, they believed that they could uncover the spiritual secrets of the universe and bring themselves closer to the gods. In particular the Brotherhood focused its attention on the study of counting numbers (1, 2, 3, ...) and fractions. Counting numbers are sometimes called whole numbers, and, together with fractions (ratios between whole numbers), they are technically referred to as rational numbers. Among the infinity of numbers, the Brotherhood looked for those with special significance, and some of the most special were the so-called "perfect" numbers.

According to Pythagoras, numerical perfection depended on a number's divisors (numbers that will divide perfectly into the original one). For instance, the divisors of 12 are 1, 2, 3, 4, and 6. When the sum of a number's divisors is greater than the number itself, it is called an "excessive" number. Therefore 12 is an excessive number because its divisors add up to 16. On the other hand, when the sum of a number's divisors is less than the number itself, it is called "defective." So 10 is a

defective number because its divisors (1, 2, and 5) add up to only 8.

The most significant and rarest numbers are those whose divisors add up exactly to the number itself, and these are the perfect numbers. The number 6 has the divisors 1, 2, and 3, and consequently it is a perfect number because $1 + 2 + 3 = 6$. The next perfect number is 28, because $1 + 2 + 4 + 7 + 14 = 28$.

As well as having mathematical significance for the Brotherhood, the perfection of 6 and 28 was acknowledged by other cultures who observed that the moon orbits the earth every 28 days and who declared that God created the world in 6 days. In *The City of God*, St. Augustine argues that although God could have created the world in an instant he decided to take six days in order to reflect the universe's perfection. St. Augustine observed that 6 was not perfect because God chose it, but rather that the perfection was inherent in the nature of the number: "6 is a number perfect in itself, and not because God created all things in six days; rather the inverse is true; God created all things in six days because this number is perfect. And it would remain perfect even if the work of the six days did not exist."

As the counting numbers get bigger the perfect numbers become harder to find. The third perfect number is 496, the fourth is 8,128, the fifth is 33,550,336, and the sixth is 8,589,869,056. As well as being the sum of their divisors, Pythagoras noted that all perfect numbers exhibit several other elegant properties. For example, perfect numbers are always the sum of a series of consecutive counting numbers. So we have

$$\begin{aligned} 6 &= 1 + 2 + 3, \\ 28 &= 1 + 2 + 3 + 4 + 5 + 6 + 7, \\ 496 &= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + \dots + 30 + 31, \\ 8,128 &= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + \dots + 126 + 127, \end{aligned}$$

Pythagoras was entertained by perfect numbers, but he was not satisfied with merely collecting these special numbers: instead he desired to discover their deeper significance. One of his insights was that perfection was closely linked to "twoness." The numbers 4 (2×2), 8 ($2 \times 2 \times 2$), 16 ($2 \times 2 \times 2 \times 2$), etc., are known as powers of 2, and can be written as $[2.\text{sup.}n]$, where the n represents the number of 2's multiplied together. All these powers of 2 only just fail to be perfect, because the sum of their divisors always adds up to one less than the number itself. This makes them only slightly defective:

$[2.\text{sup.}2] = 2 \times 2$	$= 4$, Divisors 1, 2	Sum = 3,
$[2.\text{sup.}3] = 2 \times 2 \times 2$	$= 8$, Divisors 1, 2, 4	Sum = 7,
$[2.\text{sup.}4] = 2 \times 2 \times 2 \times 2$	$= 16$, Divisors 1, 2, 4, 8	Sum = 15,
$[2.\text{sup.}5] = 2 \times 2 \times 2 \times 2 \times 2$	$= 32$, Divisors 1, 2, 4, 8, 16	Sum = 31,

Two centuries later Euclid would refine Pythagoras's link between twoness and perfection. Euclid discovered that perfect numbers are always the multiple of two numbers, one of which is a power of 2 and the other being the next power of 2 minus 1. That is to say,

$$\begin{aligned} 6 &= [2.\text{sup.}1] \times ([2.\text{sup.}2] - 1), \\ 28 &= [2.\text{sup.}2] \times ([2.\text{sup.}3] - 1), \\ 496 &= [2.\text{sup.}4] \times ([2.\text{sup.}5] - 1), \\ 8,128 &= [2.\text{sup.}6] \times ([2.\text{sup.}7] - 1). \end{aligned}$$

Today's computers have continued the search for perfect numbers and find such enormously large examples as $[2.\text{sup.}216,090] \times ([2.\text{sup.}216,091] - 1)$, a number with over 130,000 digits, which obeys Euclid's rule.

Pythagoras was fascinated by the rich patterns and properties possessed by perfect numbers and respected their subtlety and cunning. At first sight perfection is a relatively simple concept to grasp, and yet the ancient Greeks were unable to fathom some of the fundamental points of the subject. For example, although there are plenty of numbers whose divisors add up to one less than the number itself, that is to say only slightly defective, there appear to be no numbers that are slightly excessive. The Greeks were unable to find any numbers whose divisors added up to one more than the number itself, but they could not explain why this was the case. Frustratingly, although they failed to discover slightly excessive numbers, they could not prove that no such numbers existed. Understanding the apparent lack of slightly excessive numbers was of no practical value whatsoever; nonetheless it was a problem that might illuminate the nature of numbers and therefore it was worthy of study. Such riddles intrigued the Pythagorean

Brotherhood, and two and a half thousand years later, mathematicians are still unable to prove that no slightly excessive numbers exist.

Everything Is Number

In addition to studying the relationships within numbers, Pythagoras was also intrigued by the link between numbers and nature. He realized that natural phenomena are governed by laws, and that these laws could be described by mathematical equations. One of the first links he discovered was the fundamental relationship between the harmony of music and the harmony of numbers.

The most important instrument in early Hellenic music was the tetrachord, or four-stringed lyre. Prior to Pythagoras, musicians appreciated that particular notes when sounded together created a pleasant effect, and tuned their lyres so that plucking two strings would generate such a harmony. However, the early musicians had no understanding of why particular notes were harmonious and had no objective system for tuning their instruments. Instead they tuned their lyres purely by ear until a state of harmony was established—a process that Plato called torturing the tuning pegs.

Iamblichus, the fourth-century scholar who wrote nine books about the Pythagorean sect, describes how Pythagoras came to discover the underlying principles of musical harmony:

Once he was engrossed in the thought of whether he could devise a mechanical aid for the sense of hearing which would prove both certain and ingenious. Such an aid would be similar to the compasses, rules and optical instruments designed for the sense of sight. Likewise the sense of touch had scales and the concepts of weights and measures. By some divine stroke of luck he happened to walk past the forge of a blacksmith and listened to the hammers pounding iron and producing a variegated harmony of reverberations between them, except for one combination of sounds.

According to Iamblichus, Pythagoras immediately ran into the forge to investigate the harmony of the hammers. He noticed that most of the hammers could be struck simultaneously to generate a harmonious sound, whereas any combination containing one particular hammer always generated an unpleasant noise. He analyzed the hammers and realized that those that were harmonious with each other had a simple mathematical relationship—their masses were simple ratios or fractions of each other. That is to say that hammers half, two-thirds, or three-quarters the weight of a particular hammer would all generate harmonious sounds. On the other hand, the hammer that was generating disharmony when struck along with any of the other hammers had a weight that bore no simple relationship to the other weights.

Pythagoras had discovered that simple numerical ratios were responsible for harmony in music. Scientists have cast some doubt on Iamblichus's account of this story, but what is more certain is how Pythagoras applied his new theory of musical ratios to the lyre by examining the properties of a single string. Simply plucking the string generates a standard note or tone that is produced by the entire length of the vibrating string. By fixing the string at particular points along its length, it is possible to generate other vibrations and tones, as illustrated in Figure 1. Crucially, harmonious tones occur only at very specific points. For example, by fixing the string at a point exactly half-way along it, plucking generates a tone that is one octave higher and in harmony with the original tone. Similarly, by fixing the string at points that are exactly a third, a quarter, or a fifth of the way along it, other harmonious notes are produced. However, by fixing the string at a point that is not a simple fraction along the length of the whole string, a tone is generated that is not in harmony with the other tones.

Pythagoras had uncovered for the first time the mathematical rule that governs a physical phenomenon and demonstrated that there was a fundamental relationship between mathematics and science. Ever since this discovery scientists have searched for the mathematical rules that appear to govern every single physical process and have found that numbers crop up in all manner of natural phenomena. For example, one particular number appears to guide the lengths of meandering rivers. Professor Hans-Henrik Stolum, an earth scientist at Cambridge University, has calculated the ratio between the actual length of rivers from source to mouth and their direct length as the crow flies. Although the ratio varies from river to river, the average value is slightly greater than 3, that is to say that the actual length is roughly three times greater than the direct distance. In fact the ratio is approximately 3.14, which is close to the value of the number π , the ratio between the circumference of a circle and its diameter.

The number π was originally derived from the geometry of circles, and yet it reappears over and over again in a variety of scientific circumstances. In the case of the river ratio, the appearance of π is the result of a battle between order and chaos. Einstein was the first to suggest that rivers

have a tendency toward an ever more loopy path because the slightest curve will lead to faster currents on the outer side, which will in turn result in more erosion and a sharper bend. The sharper the bend, the faster the currents on the outer edge, the more the erosion, the more the river will twist, and so on. However, there is a natural process that will curtail the chaos: increasing loopiness will result in rivers doubling back on themselves and effectively short-circuiting. The river will become straighter and the loop will be left to one side, forming an oxbow lake. The balance between these two opposing factors leads to an average ratio of π between the actual length and the direct distance between source and mouth. The ratio of π is most commonly found for rivers flowing across very gently sloping plains, such as those found in Brazil or the Siberian tundra.

Pythagoras realized that numbers were hidden in everything, from the harmonies of music to the orbits of planets, and this led him to proclaim that "Everything Is Number." By exploring the meaning of mathematics, Pythagoras was developing the language that would enable him and others to describe the nature of the universe. Henceforth each breakthrough in mathematics would give scientists the vocabulary they needed to better explain the phenomena around them. In fact developments in mathematics would inspire revolutions in science.

Of all the links between numbers and nature studied by the Brotherhood, the most important was the relationship that bears their founder's name. Pythagoras's theorem provides us with an equation that is true of all right-angled triangles and that therefore also defines the right angle itself. In turn, the right angle defines the perpendicular and the perpendicular defines the dimensions—length, width, and height—of the space in which we live. Ultimately mathematics, via the right-angled triangle, defines the very structure of our three-dimensional world.

It is a profound realization, and yet the mathematics required to grasp Pythagoras's theorem is relatively simple. To understand it, simply begin by measuring the length of the two short sides of a right-angled triangle (x and y), and then square each one (x^2 , y^2). Then add the two squared numbers ($x^2 + y^2$) to give you a final number. If you work out this number for the triangle shown in Figure 2, then the answer is 25.

You can now measure the longest side z the so-called hypotenuse, and square this length. The remarkable result is that this number z^2 is identical to the one you just calculated, i.e., $z^2 = 25$.

That is to say,

In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Or in other words (or rather symbols):

$$x^2 + y^2 = z^2.$$

This is clearly true for the triangle in Figure 2, but what is remarkable is that Pythagoras's theorem is true for every right-angled triangle you can possibly imagine. It is a universal law of mathematics, and you can rely on it whenever you come across any triangle with a right angle. Conversely if you have a triangle that obeys Pythagoras's theorem, then you can be absolutely confident that it is a right-angled triangle.

At this point it is important to note that, although this theorem will forever be associated with Pythagoras, it was actually used by the Chinese and the Babylonians one thousand years before. However, these cultures did not know that the theorem was true for every right-angled triangle. It was certainly true for the triangles they tested, but they had no way of showing that it was true for all the right-angled triangles that they had not tested. The reason for Pythagoras's claim to the theorem is that it was he who first demonstrated its universal truth.

But how did Pythagoras know that his theorem is true for every right-angled triangle? He could not hope to test the infinite variety of right-angled triangles, and yet he could still be one hundred percent sure of the theorem's absolute truth. The reason for his confidence lies in the concept of mathematical proof. The search for a mathematical proof is the search for a knowledge that is more absolute than the knowledge accumulated by any other discipline. The desire for ultimate truth via the method of proof is what has driven mathematicians for the last two and a half thousand years.

Absolute Proof

The story of Fermat's Last Theorem revolves around the search for a missing proof. Mathematical proof is far more powerful and rigorous than the concept of proof we casually use in our everyday language, or even the concept of proof as understood by physicists or chemists. The difference between scientific and mathematical proof is both subtle and profound, and is crucial to understanding the work of every mathematician since Pythagoras.

The idea of a classic mathematical proof is to begin with a series of axioms, statements that can be assumed to be true or that are self-evidently true. Then by arguing logically, step by step, it is possible to arrive at a conclusion. If the axioms are correct and the logic is flawless, then the conclusion will be undeniable. This conclusion is the theorem.

Mathematical theorems rely on this logical process and once proven are true until the end of time. Mathematical proofs are absolute. To appreciate the value of such proofs they should be compared with their poor relation, the scientific proof. In science a hypothesis is put forward to explain a physical phenomenon. If observations of the phenomenon compare well with the hypothesis, this becomes evidence in favor of it. Furthermore, the hypothesis should not merely describe a known phenomenon, but predict the results of other phenomena. Experiments may be performed to test the predictive power of the hypothesis, and if it continues to be successful then this is even more evidence to back the hypothesis. Eventually the amount of evidence may be overwhelming and the hypothesis becomes accepted as a scientific theory.

However, the scientific theory can never be proved to the same absolute level of a mathematical theorem: It is merely considered highly likely based on the evidence available. So-called scientific proof relies on observation and perception, both of which are fallible and provide only approximations to the truth. As Bertrand Russell pointed out: "Although this may seem a paradox, all exact science is dominated by the idea of approximation." Even the most widely accepted scientific "proofs" always have a small element of doubt in them. Sometimes this doubt diminishes, although it never disappears completely, while on other occasions the proof is ultimately shown to be wrong. This weakness in scientific proof leads to scientific revolutions in which one theory that was assumed to be correct is replaced with another theory, which may be merely a refinement of the original theory, or which may be a complete contradiction.

For example, the search for the fundamental particles of matter involved each generation of physicists overturning or, at the very least, refining the theory of their predecessors. The modern quest for the building blocks of the universe started at the beginning of the nineteenth century when a series of experiments led John Dalton to suggest that everything was composed of discrete atoms, and that atoms were fundamental. At the end of the century J.J. Thomson discovered the electron, the first known subatomic particle, and therefore the atom was no longer fundamental.

During the early years of the twentieth century, physicists developed a "complete" picture of the atom—a nucleus consisting of protons and neutrons, orbited by electrons. Protons, neutrons, and electrons were proudly held up as the complete ingredients for the universe. Then cosmic ray experiments revealed the existence of other fundamental particles—pions and muons. An even greater revolution came with the discovery in 1932 of antimatter—the existence of antiprotons, antineutrons, antielectrons, etc. By this time particle physicists could not be sure how many different particles existed, but at least they could be confident that these entities were indeed fundamental. That was until the 1960s when the concept of the quark was born. The proton itself is apparently built from fractionally charged quarks, as is the neutron and the pion. In the next decade the very concept of a particle as a pointlike object may even be replaced by the idea of particles as strings. The theory is that strings a billionth of a billionth of a billionth of a meter in length (so small that they appear pointlike) can vibrate in different ways, and each vibration gives rise to a different particle. This is analogous to Pythagoras's discovery that one string on a lyre can give rise to different notes depending on how it vibrates. The moral of the story is that physicists are continually altering their picture of the universe, if not rubbing it out and starting all over again.

The science fiction writer and futurologist Arthur C. Clarke wrote that if an eminent professor states that something is undoubtedly true, then it is likely to be proved false the next day. Scientific proof is inevitably fickle and shoddy. On the other hand, mathematical proof is absolute and devoid of doubt. Pythagoras died confident in the knowledge that his theorem, which was true in 500 B.C., would remain true for eternity.

Science is operated according to the judicial system. A theory is assumed to be true if there is enough evidence to prove it "beyond all reasonable doubt." On the other hand, mathematics does not rely on evidence from fallible experimentation, but it is built on infallible logic. This is

demonstrated by the problem of the "mutilated chessboard," illustrated in Figure 3.

We have a chessboard with the two opposing corners removed, so that there are only 62 squares remaining. Now we take 31 dominoes shaped such that each domino covers exactly two squares. The question is: Is it possible to arrange the 31 dominoes so that they cover all the 62 squares on the chessboard?

There are two approaches to the problem:

(1) The scientific approach

The scientist would try to solve the problem by experimenting, and after trying out a few dozen arrangements would discover that they all fail. Eventually the scientist believes that there is enough evidence to say that the board cannot be covered. However, the scientist can never be sure that this is truly the case, because there might be some arrangement that has not been tried that might do the trick. There are millions of different arrangements, and it is possible to explore only a small fraction of them. The conclusion that the task is impossible is a theory based on experiment, but the scientist will have to live with the prospect that one day the theory may be overturned.

(2) The mathematical approach

The mathematician tries to answer the question by developing a logical argument that will derive a conclusion that is undoubtedly correct and that will remain unchallenged forever. One such argument is the following:

- * The corners that were removed from the chessboard were both white. Therefore there are now 32 black squares and only 30 white squares.
- * Each domino covers two neighboring squares, and neighboring squares are always different in color, i.e., one black and one white.
- * Therefore, no matter how they are arranged, the first 30 dominoes laid on the board must cover 30 white squares and 30 black squares.
- * Consequently, this will always leave you with one domino and two black squares remaining.
- * But remember, all dominoes cover two neighboring squares, and neighboring squares are opposite in color. However, the two squares remaining are the same color and so they cannot both be covered by the one remaining domino. Therefore, covering the board is impossible!

This proof shows that every possible arrangement of dominoes will fail to cover the mutilated chessboard. Similarly Pythagoras constructed a proof that shows that every possible right-angled triangle will obey his theorem. For Pythagoras the concept of mathematical proof was sacred, and it was proof that enabled the Brotherhood to discover so much. Most modern proofs are incredibly complicated and following the logic would be impossible for the layperson, but fortunately in the case of Pythagoras's theorem the argument is relatively straightforward and relies on only high-school mathematics. The proof is outlined in Appendix 1.

Pythagoras's proof is irrefutable. It shows that his theorem holds true for every right-angled triangle in the universe. The discovery was so momentous that one hundred oxen were sacrificed as an act of gratitude to the gods. The discovery was a milestone in mathematics and one of the most important breakthroughs in the history of civilization. Its significance was twofold. First, it developed the idea of proof. A proven mathematical result has a deeper truth than any other truth because it is the result of step-by-step logic. Although the philosopher Thales had already invented some primitive geometrical proofs, Pythagoras took the idea much further and was able to prove far more ingenious mathematical statements. The second consequence of Pythagoras's theorem is that it ties the abstract mathematical method to something tangible. Pythagoras showed that the truth of mathematics could be applied to the scientific world and provide it with a logical foundation. Mathematics gives science a rigorous beginning, and upon this infallible foundation scientists add inaccurate measurements and imperfect observations.

An Infinity of Triples

The Pythagorean Brotherhood invigorated mathematics with its zealous search for truth via proof.

News of their success spread and yet the details of their discoveries remained a closely guarded secret. Many requested admission to the inner sanctum of knowledge, but only the most brilliant minds were accepted. One of those who was blackballed was a candidate by the name of Cylon. Cylon took exception to his humiliating rejection and twenty years later he took his revenge.

During the sixty-seventh Olympiad (510 B.C.) there was a revolt in the nearby city of Sybaris. Telys, the victorious leader of the revolt, began a barbaric campaign of persecution against the supporters of the former government, which drove many of them to seek sanctuary in Croton. Telys demanded that the traitors be returned to Sybaris to suffer their due punishment, but Milo and Pythagoras persuaded the citizens of Croton to stand up to the tyrant and protect the refugees. Telys was furious and immediately gathered an army of 300,000 men and marched on Croton, where Milo defended the city with 100,000 armed citizens. After seventy days of war Milo's supreme generalship led him to victory, and as an act of retribution he turned the course of the river Crathis upon Sybaris to flood and destroy the city.

Despite the end of the war, the city of Croton was still in turmoil because of arguments over what should be done with the spoils of war. Fearful that the lands would be given to the Pythagorean elite, the ordinary folk of Croton began to grumble. There had already been growing resentment among the masses because the secretive Brotherhood continued to withhold their discoveries, but nothing came of it until Cylon emerged as the voice of the people. Cylon preyed on the fear, paranoia, and envy of the mob and led them on a mission to destroy the most brilliant school of mathematics the world had ever seen. Milo's house and the adjoining school were surrounded, all the doors were locked and barred to prevent escape, and then the burning began. Milo fought his way out of the inferno and fled, but Pythagoras, along with many of his disciples, was killed.

Mathematics had lost its first great hero, but the Pythagorean spirit lived on. The numbers and their truths were immortal. Pythagoras had demonstrated that more than any other discipline mathematics is a subject that is not subjective. His disciples did not need their master to decide on the validity of a particular theory. A theory's truth was independent of opinion. Instead the construction of mathematical logic had become the arbiter of truth. This was the Pythagoreans' greatest contribution to civilization--a way of achieving truth that is beyond the fallibility of human judgment.

Following the death of their founder and the attack by Cylon, the Brotherhood left Croton for other cities in Magna Graecia, but the persecution continued and eventually many of them had to settle in foreign lands. This enforced migration encouraged the Pythagoreans to spread their mathematical gospel throughout the ancient world. Pythagoras's disciples set up new schools and taught their students the method of logical proof. In addition to their proof of Pythagoras's theorem, they also explained to the world the secret of finding so-called Pythagorean triples.

Pythagorean triples are combinations of three whole numbers that perfectly fit Pythagoras's equation: $[x.\text{sup.}2] + [y.\text{sup.}2] = [z.\text{sup.}2]$. For example, Pythagoras's equation holds true if $x = 3$, $y = 4$, and $z = 5$:

$$[3.\text{sup.}2] + [4.\text{sup.}2] = [5.\text{sup.}2], 9 + 16 = 25.$$

Another way to think of Pythagorean triples is in terms of rearranging squares. If one has a 3×3 square made of 9 tiles, and a 4×4 square made of 16 tiles, then all the tiles can be rearranged to form a 5×5 square made of 25 tiles, as shown in Figure 4.

The Pythagorean wanted to find other Pythagorean triples, other squares that could be added to form a third, larger square. Another Pythagorean triple is $x = 5$, $y = 12$, and $z = 13$:

$$[5.\text{sup.}2] + [12.\text{sup.}2] = [13.\text{sup.}2], 25 + 144 = 169.$$

A larger Pythagorean triple is $x = 99$, $y = 4,900$, and $z = 4,901$. Pythagorean triples become rarer as the numbers increase, and finding them becomes harder and harder. To discover as many triples as possible the Pythagoreans invented a methodical way of finding them, and in so doing they also demonstrated that there are an infinite number of Pythagorean triples.

From Pythagoras's Theorem to Fermat's Last Theorem

Pythagoras's theorem and its infinity of triples was discussed in E. T. Bell's *The Last Problem*, the library book that caught the attention of the young Andrew Wiles. Although the Brotherhood had achieved an almost complete understanding of Pythagorean triples, Wiles soon discovered that this apparently innocent equation, $[x.\text{sup.}2] + [y.\text{sup.}2] = [z.\text{sup.}2]$, has a darker side--Bell's book

described the existence of a mathematical monster.

In Pythagoras's equation the three numbers, x , y , and z , are all squared (i.e., $x^2 = xxx$):

$$x^2 + y^2 = z^2.$$

However, the book described a sister equation in which x , y , and z are all cubed (i.e., $x^3 = xxxx$). The so-called power of x in this equation is no longer 2, but rather 3:

$$x^3 + y^3 = z^3.$$

Finding whole number solutions, i.e., Pythagorean triples, to the original equation was relatively easy, but changing the power from 2 to 3 (from a square to a cube) and finding whole number solutions to the sister equation appears to be impossible. Generations of mathematicians have failed to find numbers that fit the cubed equation perfectly.

With the original "squared" equation, the challenge was to rearrange the tiles in two squares to form a third, larger square. In the "cubed" version the challenge is to rearrange two cubes made of building blocks, to form a third, larger cube. Apparently, no matter what cubes are chosen to begin with, when they are combined the result is either a complete cube with some extra blocks left over, or an incomplete cube. The nearest that anyone has come to a perfect rearrangement is one in which there is one building block too many or too few. For example, if we begin with the cubes 6^3 (x^3) and 8^3 (y^3) and rearrange the building blocks, then we are only one short of making a complete $9 \times 9 \times 9$ cube, as shown in Figure 5.

Finding three numbers that fit the cubed equation perfectly seems to be impossible. That is to say, there appear to be no whole number solutions to the equation

$$x^3 + y^3 = z^3.$$

Furthermore, if the power is changed from 3 (cubed) to any higher number n (i.e., 4, 5, 6, ...), then finding a solution seems equally impossible. There appear to be no whole number solutions to the more general equation

$$x^n + y^n = z^n \text{ for } n \text{ greater than } 2.$$

By merely changing the 2 in Pythagoras's equation to any higher number, finding whole number solutions turns from being relatively simple to being mind-bogglingly difficult. In fact, the great seventeenth-century Frenchman Pierre de Fermat made the astonishing claim that the reason why nobody could find any solutions was that no solutions existed.

Fermat was one of the most brilliant and intriguing mathematicians in history. He could not have checked the infinity of numbers, but he was absolutely sure that no combination existed that would fit the equation perfectly because his claim was based on proof. Like Pythagoras, who did not have to check every triangle to demonstrate the validity of his theorem. Fermat did not have to check every number to show the validity of his theorem. Fermat's Last Theorem, as it is known, stated that

$$x^n + y^n = z^n$$

has no whole number solutions for n greater than 2.

As Wiles read each chapter of Bell's book, he learned how Fermat had become fascinated by Pythagoras's work and had eventually come to study the perverted form of Pythagoras's equation. He then read how Fermat had claimed that even if all the mathematicians in the world spent eternity looking for a solution to the equation they would fail to find one. He must have eagerly turned the pages, relishing the thought of examining the proof of Fermat's Last Theorem. However, the proof was not there. It was not anywhere. Bell ended the book by stating that the proof had been lost long ago. There was no hint of what it might have been, no clues as to the proof's construction or derivation. Wiles found himself puzzled, infuriated, and intrigued. He was in good company.

For over 300 years many of the greatest mathematicians had tried to rediscover Fermat's lost proof and failed. As each generation failed, the next became even more frustrated and determined. In 1742, almost a century after Fermat's death, the Swiss mathematician Leonhard Euler asked his friend Clerot to search Fermat's house in case some vital scrap of paper still remained. No clues were ever found as to what Fermat's proof might have been.

Fermat's Last Theorem, a problem that had captivated mathematicians for centuries, captured the imagination of the young Andrew Wiles. In Milton Road Library, ten-year-old Wiles stared at the most infamous problem in mathematics, undaunted by the knowledge that the most brilliant minds on the planet had failed to rediscover the proof. Young Wiles immediately set to work using all his textbook techniques to try to recreate the proof. Perhaps he could find something that everyone else, except Fermat, had overlooked. He dreamed he could shock the world.

Thirty years later Andrew Wiles stood in the auditorium of the Isaac Newton Institute. He scribbled on the board and then, struggling to contain his glee, stared at his audience. The lecture was reaching its climax and the audience knew it. One or two of them had smuggled cameras into the lecture room and flashes peppered his concluding remarks

With the chalk in his hand he fumed to the board for the last time. The final few lines of logic completed the proof. For the first time in over three centuries Fermat's challenge had been met. A few more cameras flashed to capture the historic moment. Wiles wrote up the statement of Fermat's Last Theorem, turned toward the audience, and said modestly: "I think I'll stop here."

Two hundred mathematicians clapped and cheered in celebration. Even those who had anticipated the result grinned in disbelief. After three decades Andrew Wiles believed he had achieved his dream, and after seven years of isolation he had revealed his secret calculation. While a general mood of euphoria filled the Newton Institute, everybody realized that the proof still had to be rigorously checked by a team of independent referees. However, as Wiles enjoyed the moment, nobody could have predicted the controversy that would evolve in the months ahead.

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