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VIANELLI, G.
2001

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MATHEMATICS

Articles

A NEWLY DISCOVERED TEXT BY RUSSELL ON PYTHAGORAS AND THE HISTORY OF MATHEMATICS

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This paper presents a lengthy unpublished passage by Russell on the attempts by Pythagoras and subsequent mathematicians to deal with continuity and the logical paradoxes, recently discovered in the manuscript of the *History of Western Philosophy*. In the first part, I provide a short introduction to the new material. In the second, I analyze its philosophical content. In the third, I develop some considerations, mainly in the attempt to solve the following problems: can we determine when and for what purpose Russell wrote these leaves? why they never came to be published? where do we find similar subjects expounded in the *History*? where does the significance of the discovery really lie?

INTRODUCTION

The third chapter of *A History of Western Philosophy* provides a critical exposition of the philosophy of Pythagoras, considered by Russell to be "intellectually one of the most important men

* The work presented here could not have been completed without the valuable help provided by the Russell Archives and the Russell Research Centre. I'd like to thank the Editor; Nicholas Griffin, whose teaching repeatedly helped me to understand the weak points of my analysis and to improve them; Andrew Bone, who gave freely of his time revising my English; and the Bertrand Russell Society, who gave me the opportunity of reading a draft to the Society's annual meeting at McMaster on 25-27 May 2001.

that ever lived, both when he was wise and when he was unwise."¹ Following the organizational scheme employed frequently throughout the *History*, Russell begins by connecting the main episodes in the philosopher's life to the cultural environment from which he emerged. For the remainder of the chapter he turns to the different aspects of Pythagorean thought, considering in turn its mystical components, the underlying metaphysical and religious assumptions, and his ethico-political project. This exposition allows Russell on several occasions to make some agreeable theoretical digressions. For example, he etymologizes the word "theory" and, on establishing its association with the notion of "contemplation", considers briefly the relationship between the contemplative, the practical and the useful (pp. 33, 34).

Russell regards Pythagorean wisdom as the product of two aspects that are "not so separate as they seem to a modern mind" (p. 34): the religious and the purely mathematical. It is to the latter side of Pythagoras that Russell directs his attention in the final pages of Chapter 3, emphasizing that he was the first philosopher to perceive not only the "importance of numbers" (p. 35) but also the overall significance of mathematics. Indeed, "Mathematics, in the sense of demonstrative deductive argument, begins with him" (p. 29).

In this section Russell does not restrict himself to a formal presentation of Pythagoras' Theorem. He is more inclined to ponder the profound consequences of its discovery. Not the least important effect of the theorem was that it revealed a new and important problem. "Unfortunately for Pythagoras," Russell writes, "his theorem led at once to the discovery of incommensurables, which appeared to disprove his whole philosophy" (p. 35). In turn, the failure of Greek arithmetic to confront the puzzle of incommensurables provided a basis for the distinction between geometry and arithmetic, which characterized the history of all subsequent mathematical thought, at least until Descartes. For Russell, a clear illustration of this distinction was contained in Euclid's deductive system. "Euclid, in Book II, proves geometrically many things which we should naturally prove by algebra, such as $(a + b)^2 = a^2 + 2ab + b^2$. It was because of the difficulty about incommensurables that he considered this

¹ *HWP*, p. 29. This reference is to the first and only American edition, first published in 1945. The corresponding page in the current British edition, *HWP*, first published in 1961, is 49.

course necessary" (p. 36).

These reflections prefigure a further theoretical excursus, of astounding density and clarity. It opens with the following statement: "The influence of geometry upon philosophy and scientific method has been profound" (*ibid.*), and it continues through the last three paragraphs of the chapter.

The manuscript of the *History* preserved at the Russell Archives,² however, shows some extraordinarily interesting and important variations from the printed version. Specifically, instead of the last paragraph of Chapter 3 as it appears in the book, we find nine other paragraphs (i.e. six leaves) in which Russell explores further the "Influence of geometry on philosophy"—his heading for the above-mentioned excursus, written and underlined at the top in customary fashion two leaves before.³ Nowhere in this hitherto unpublished holograph material is there any textual evidence of the final paragraph of the Pythagoras chapter. The next manuscript leaf contains the opening passages of the Heraclitus chapter.

What follows is a transcription of the six manuscript leaves. Editorial comments and later archival foliation are enclosed in angle brackets.⁴

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From these general reflections it is time to return to the history of mathematics. Geometry, which had been one study, at once *a priori* and applicable to actual space, was torn in two: its *a priori* aspect was swallowed up in arithmetic, and its factual aspect became absorbed into physics. This process began with Lobatchevsky's (Lobatchevsky's *printed in pencil above*) invention of non-Euclidean geometry in 1829, from which it appeared that a geometry different from Euclid's could be developed self-consistently from axioms (axioms *after deleted* different) which contradict Euclid's (which contradict Euclid's *inserted*). Pythagoras had proved his theorem, but he had proved it by means of the ax-

² RAI 210.006657; the original title is *Western Philosophical Thought: A History*.

³ This heading is not to be found in the published version.

⁴ Russell underlined both upper corner notations on each leaf.

iom of parallels, which there was no *a priori* reason to believe. The only way to be sure of having a space obeying Euclid's axioms is to replace points by triads of numbers. When this is done, geometry becomes merely a development of arithmetic. This sort of geometry is independent of observation, but tells us nothing whatever about actual space, any more than pure (pure *inserted*) arithmetic tells us how many inhabitants there are in Chicago.

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II (fol. 55)

Geometry as an empirical investigation of actual space, on the other hand, has ceased to be independent of physics. The process of measurement is a physical process, and if measuring instruments undergo changes during the process, the apparent results may be misleading. We must therefore consider physical and geometrical laws together. The conclusions reached by physicists have been surprising. So far as the large-scale phenomena studied by astronomy are concerned, the general theory of relativity has led to the belief that Euclidean geometry would only be strictly true in a space devoid of matter: where there is matter, there are deviations from Euclid which—odd as it may seem—account for the law of gravitation. As regards the minute occurrences studied in quantum theory, the innovations are even more serious, so serious that, in atomic phenomena, there is no longer any reason to believe in anything at all closely analogous to space as we ordinarily conceive it.

Thus geometry has completely lost its old position as an *a priori* deductive science dealing with something that actually exists.

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12 (fol. 56)

Another change, which begins with Galileo, has profoundly affected the philosophic mood induced by mathematics. Everything considered by the Greeks was static, or, at most, a steady motion such as the (the *inserted*) uniform diurnal revolution of the heavens. (diurnal ... heavens. *inserted above deleted* circular movement.) They, and the scholastics after them, possessed no technique for dealing with change, and this led them into grotesque errors. The scholastics believed, for instance, that a projectile fired horizontally will move horizontally for a certain time, and then suddenly begin to fall vertically. Galileo proved, by means of obser-

vations on falling bodies, that this is a mistake, and that projectiles move in parabolas, in which the inclination to the horizontal changes continuously, not suddenly. The idea of acceleration, which was fundamental in Galileo's system as in all subsequent dynamics, was totally alien to Greek modes of thought. So, also, was the method of using observation to establish a mathematical formula. The fruitfulness of this method made it clear that mathematical reasoning can start from empirical premisses just as well as from such as are *a priori*, and that it can deal with a changing world as easily as with one that is static.

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13 (fol. 57)

The puzzle about incommensurables, which led to the separation of geometry from arithmetic in Greek mathematics, was part of a larger problem, that of the arithmetical treatment of continuity. The integers are essentially discrete, and counting requires discrete units; even the series of rational fractions, as Pythagoras discovered, does not suffice for the accurate measurement of lengths in terms of a fixed unit. It seemed impossible that arithmetic should ever prove adequate to the treatment of continuity. Accordingly, while rationalistic theologians and philosophers took refuge in a super-sensible world exempt from the irrationality of sense, others, of a more obscurantist tendency, used continuity to condemn the intellect, and to laud a more intuitive, less discursive kind of knowledge than that of the logician and mathematician. Both points of view still exist in our own day, the former in Hegelianism, the latter in such philosophies as that of Bergson.

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14 (fol. 58)

But the ingenuity of the mathematicians proved to be greater than either of these schools of philosophers had supposed possible. The conquest of continuity began with the infinitesimal calculus, invented independently by Leibniz and Newton. At first, however, this method, in spite of its practical success, was not logically unassailable; it was only in the latter half of the nineteenth century that its foundations were rendered comparatively secure. Weierstrass showed that, by the method of limits, all its theorems can be stated without assuming infinitesimals. It was found that irrational numbers such as $\sqrt{2}$ could be defined arith-

metically, and Georg Cantor achieved a purely arithmetical definition of continuity, which proved adequate for all the purposes of geometry and physics. It seemed, at this stage, as if Pythagoras had been completely avenged, and number had been reinstated in the supreme position from which his own discovery of irrationals had ousted it.

But at this moment history repeated itself in an astonishing way. The new Pythagoreans suddenly found themselves confronted with a new difficulty, as formidable, in our own day, as incommensurables were in the time of Pythagoras. I will explain as shortly as I can how this came about.

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15 (fol. 59)

At the same time that the rest of traditional (traditional *inserted*) mathematics was being reduced to arithmetic, arithmetic itself was being reduced to logic, and new branches of mathematics, more or less independent of number, were being invented, such as topology, mathematical logic, and the theory of groups. It thus appeared that number is not essential to mathematics, but is merely one of many possible mathematical developments of logic. And logic itself, in the form in which the mathematician seems to need it, was found to lead to paradoxes which showed it to be as imperfect as the arithmetic of Pythagoras. *(the following line was deleted after the later addition of its final example:)*

[Explain Epimenides, heterological, $\hat{x} (x \sim \varepsilon x)$, max Nc]⁵

The paradoxes can be resolved, but unfortunately only by methods which throw doubt on things which have been accepted in mathematics since the seventeenth century. And among the things upon which they throw doubt are those that had seemed to overcome the difficulties of Pythagoras. The solution of a problem, it should seem, consists only in reducing it to another problem more difficult than the first.

ANALYSIS

In order to provide a clear analysis of Russell's unpublished text, it is necessary to reconstruct his arguments from the top of folio 52.

⁵ "max Nc]" was added after a deleted "[)".

If we were to label these passages without using Russell's heading (i.e. "Influence of geometry on philosophy"), we would face something of a dilemma, for Russell is here concerned with arithmetic, logic, philosophy, mystics, theology, law and physics, as well as geometry. And he considers all these subjects from both a synchronic and diachronic perspective. Thus we are left pondering whether his thought was extremely chaotic or whether our urge to classify is the real source of chaos. I lean towards the second hypothesis, but do not wholly deny that in Russell's analysis many different topics merge into one. The purpose of the following account is to unravel this particular skein.

After asserting that "the influence of geometry upon philosophy and scientific method has been profound", Russell continues:

Geometry, as established by the Greeks, starts with axioms which are (or are deemed to be) self-evident, and proceeds, by deductive reasoning, to arrive at theorems that are very far from self-evident. The axioms and theorems are held to be true of actual space, which is something given in experience. (HWP, p. 36)

By invoking the well-known Kantian formula, we can say that, according to Russell, geometry, from the Greeks on, has been regarded as a *synthetic a priori* science. As such, it provided an exemplary model to be emulated by:

- (1) Philosophy, from Plato to Kant: "what appears as Platonism is, when analysed, found to be in essence Pythagoreanism" (p. 37);
- (2) Rationalistic theology, not only Scholasticism: "Hence Plato's doctrine that God is a geometer, and Sir James Jeans' belief that He is addicted to arithmetic" (p. 37);
- (3) Science, peculiarly Newton's *Principia*: "The form of Newton's *Principia*, in spite of its admittedly empirical material, is entirely dominated by Euclid" (pp. 36-7);
- (4) Law, as revealed by the notion of "self-evident" in the eighteenth-century doctrine of natural rights and one of its most celebrated expressions, the Declaration of Independence (p. 36 and n.).

Broadly speaking, "mathematics is ... the chief source of the belief in eternal and exact truth, as well as in a super-sensible intelligible world" (p. 37). From the intimate connection of eternity with truth and intelli-

gibility, a general devaluation of sensory and empirical knowledge follows as a result. Besides, the mystical aspect of the Pythagorean doctrine affects what Russell calls "personal religion" (*ibid.*), as opposed to theology. Thus Pythagoras' alliance between mathematics and mysticism maintains itself over time.

So much for the influence of mathematics and geometry on western thought. But at the beginning of the omitted paragraphs Russell goes further, describing the vicissitudes of Greek-based geometry:

Geometry, which had been one study, at once *a priori* and applicable to actual space, was torn in two: its *a priori* aspect was swallowed up in arithmetic, and its factual aspect became absorbed into physics. (Fol. 54, p. 7 above)

The *a priori* side of geometry has undergone wholesale changes:

- (a) Euclid's geometry has been reduced to arithmetic;
- (b) In 1829 Lobachevsky showed "that a geometry different from Euclid's could be developed self-consistently from axioms which contradict Euclid's" (fol. 54).⁶

From (b) it follows that Euclid's geometry, unfortunately, is not *a priori*, in the sense that Euclid's axioms can no longer be considered as self-evident truths; from (a) it follows that geometry and arithmetic are not truly separate. We will be sent back to (a) presently.

The *synthetic* side of geometry, "geometry as an empirical investigation of actual space, on the other hand, has ceased to be independent of physics" (fol. 55). Progress in twentieth-century physics, in the fields of general relativity and quantum mechanics, has once again exposed the shortcomings of the Euclidean paradigm.

So much for the history of geometry. In order to understand its subsequent development it is necessary to take into consideration, first, the history of arithmetic. As previously stated, the discovery of incommen-

⁶ See Russell, "Logical Positivism", *Revue internationale de philosophie*, 4 (Jan. 1950): 3-19, *Papers* II: 157: "... Lobachevsky's non-Euclidean geometry showed that only empirical observation can decide whether Euclidean geometry is true of actual space, and that geometry as a part of *pure* mathematics throws no more light upon actual space than the multiplication table throws on the population of an actual town."

surables caused Greek deductive thought to regard geometry and arithmetic as heterogeneous studies. "The puzzle about incommensurables, which led to the separation of geometry from arithmetic in Greek mathematics, was part of a larger problem, that of the arithmetical treatment of continuity" (fol. 57). The obstacles encountered by arithmetic confronting the incommensurables and the continuum in general, have some bearing on anti-scientific philosophical attitudes which persist to this day. All such attitudes are based on a doctrinal mistrust of experimental knowledge and deductive reasoning. The anti-scientific philosophies can be divided into two broad categories:

- (1) Philosophies which condemn intellectual knowledge by advocating a different rational pattern, such as that of Hegel;
- (2) Philosophies which condemn intellectual knowledge by advocating "a more intuitive, less discursive kind of knowledge" (*ibid.*), such as that of Bergson.

But "... another change, which begins with Galileo, has profoundly affected the philosophical mood induced by mathematics" (fol. 56): Galileo's method and the achievements associated with it led to a reassessment of mathematics as a cognitive *ὑπόθεσις*. "The fruitfulness of this method", says Russell, "made it clear that mathematical reasoning can start from empirical premisses just as well as from such as are *a priori*, and that it can deal with a changing world as easily as with one that is static" (*ibid.*).

Hence Galilean science rekindled hope in the possibility of a "conquest of continuity" (fol. 58) by the mathematical sciences. Such a conquest, which began with the formulation of "the infinitesimal calculus, invented independently by Leibniz and Newton" (*ibid.*), has gradually been achieved.

An adequate logical foundation of the calculus, however, was not obtained until the second half of the nineteenth century, in the first instance by Weierstrass⁷ and Cantor. The former, by the method of the

⁷ See Russell, "Addendum to My 'Reply to Criticisms'", in Schilpp (1971), *Papers* II: 65: "The axiom of parallels troubled people from the time of Euclid to the time of Lobachevsky, a period of some two thousand, three hundred years. Incommensurables troubled people from Pythagoras to Weierstrass." Russell wrote the Addendum in 1965.

limits, showed that all the theorems of the calculus "can be stated without assuming infinitesimals",⁸ while the latter "achieved a purely arithmetical definition of continuity, which proved adequate for all purposes of geometry and physics" (*ibid.*).

At this stage, the circuitous paths taken by the deductive sciences crossed. They now seemed to be progressing towards an apparently wonderful synthesis, able to avenge once and forever the excellence of sort of knowledge praised by Pythagoras. This synthesis was made possible by two factors:

- (1) Geometry (Euclid-based geometry, first of all) had already been reduced to arithmetic;
- (2) The elaboration of new branches of mathematics (such as topology, mathematical logic,⁹ theory of groups) showed that the notion of number was no longer essential.

Thus did mathematics become definitely independent of number, and arithmetic lost its hitherto central role. Meanwhile, "arithmetic itself was being reduced to logic" (fol. 59), which suggested the possibility of reducing *all* mathematics to logic and ultimately of achieving a supreme synthesis. No commentary can replace Russell's own account of what happened next:

But at this moment history repeated itself in an astonishing way. The new Pythagoreans suddenly found themselves confronted with a new difficulty, as formidable, in our own day, as incommensurables were in the time of Pythagoras. (Fol. 58)

... logic itself, in the form in which the mathematician seems to need it, was

See also I. Grattan-Guinness, *The Development of the Foundations of Mathematical Analysis from Euler to Riemann* (Cambridge, Mass.: MIT P., 1970).

⁸ *Ibid.* See Russell, "Logical Positivism", *Polemic*, London, no. 1 (c. Nov. 1945): 6–13, *Papers* 11: 149: "Weierstrass showed in detail how to develop the differential and integral calculus without the use of infinitesimals, thereby sweeping away great masses of metaphysical lumber, of which the accumulation began with Leibniz."

⁹ See Russell, "My Own Philosophy" (1946), *Papers* 11: 71: "Peano's three concepts, one of which is 'number', could, I found, themselves be defined in terms derived from logic, but this discovery, which I had at first supposed new, had, as I soon learnt, been already made by Frege in the year 1884."

found to lead to paradoxes which showed it to be as imperfect as the arithmetic of Pythagoras. (Fol. 59)

The paradoxes can be resolved, but unfortunately only by methods which throw doubt on things which have been accepted in mathematics since the seventeenth century. And among the things upon which they throw doubt are those that had seemed to overcome the difficulties of Pythagoras. The solution of a problem, it should seem, consists only in reducing it to another problem more difficult than the first. (*Ibid.*)

With the last-quoted paragraph the six leaves of holograph material conclude. Between its final two paragraphs we find a line, later deleted, whose purpose was probably to provide Russell with some examples of the paradoxes for his lecture. Russell lists four paradoxes: Epimenides', Grelling's, his own and Cantor's.¹⁰

I will now provide a short non-technical exposition of the four paradoxes mentioned above. I will use Russell's own account of one possible version of Epimenides' paradox, also known as the Paradox of the Liar:

This paradox is seen in its simplest form if a man says, "I am lying". If he is lying, it is a lie that he is lying, and therefore he is speaking the truth; but if he is speaking the truth, he is lying, for that is what he says he is doing. Contradiction is thus inevitable. (*MPD*, p. 77)

The heterological paradox of Kurt Grelling¹¹ is explained by Ramsey as follows:

The word "short" is a short word, but the word "long" is not a long word. This suggests the division of adjective words according as they do or do not have the property which they connote. Words like "short", which apply to themselves, let us call *autological*; and words like "long", which do not apply to themselves, let us call *heterological*. Now suppose we put the question, "Is the word 'hetero-

¹⁰ Some later expositions by him of the logical paradoxes, and their implications, are to be found in "The Relevance of Psychology to Logic", *Aristotelian Society, Supplementary Volume: Action, Perception and Measurement*, 17 (1938): 42–53, *Papers* 10: 363–4; "My Own Philosophy", *Papers* 11: 71–2; "Logical Positivism", *Papers* 11: 149–52.

¹¹ Of the above mentioned paradoxes, the heterological paradox is the only one that Russell did not take into consideration while writing *Principia Mathematica*. See Grattan-Guinness, *The Search for Mathematical Roots 1870–1940* (Princeton: Princeton U.P., 2000), pp. 336–8.

logical' a heterological word?" Then we at once obtain contradictory answers. For if it is heterological, that means that it does not apply to itself, i.e., that it is not heterological; but if it not heterological, then it does apply to itself, i.e., it is heterological. How can these conclusions be reconciled with the Law of Contradiction?¹²

At this point we should consider Russell's paradox. Before doing so, however, it is interesting to note that Russell did not at first include the "max N_c " paradox in the list of examples on his manuscript. He added it later, as a fourth example. This fact is striking, because it was in trying to solve a contradiction arising from Cantor's power-set theorem—from which the paradox of the largest cardinal number arises—that Russell discovered the paradox that bears his own name. For this reason, I will start with the max N_c paradox, and come to Russell's paradox last.

In 1891 Cantor published an article in which he proved that there is no greatest cardinal number: he reached this conclusion by developing set theory.¹³ Given that a class with n members has 2^n sub-classes, Cantor proved that 2^n is always greater than n , even when n is infinite.¹⁴ It follows that for every given cardinal number n , there is a greater one (i.e. 2^n): thus, there is no greatest cardinal number.

At the end of 1900 Russell advanced the hypothesis that there *is* a greatest cardinal number¹⁵:

... if we could ... add together into one class the individuals, classes of individuals, classes of classes of individuals, etc., we should obtain a class of which its own sub-classes would be members. The class consisting of all objects that can be counted, of whatever sort, must, if there be such a class, have a cardinal number which is the greatest possible. (*IMP*, p. 75)

¹² Frank P. Ramsey, "Achilles and the Tortoise", *The Forum*, 78 (July-Dec. 1927): 430-4; reprinted in *Papers* 10: 588. See also Ramsey's "The 'Long' and 'Short' of It or a Failure of Logic", *American Philosophical Quarterly*, 24 (1987): 357-9.

¹³ Georg Cantor, "Ueber eine elementare Frage der Mannigfaltigkeitslehre", *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 1 (1891): 75-8.

¹⁴ Cantor's proof is now called the power-set theorem (the power-set of a set is the set of all its subsets). See *SLBR*, 1: 209-10.

¹⁵ See Russell's letter to Couturat, 8 December 1900, in G. H. Moore and A. Garciadiego, "Burali-Forti's Paradox: a Reappraisal of Its Origins", *Historia Mathematica*, 8 (1981): 325-7. See also *MPLD*, p. 75.

Applying Cantor's theorem to the class of all classes (i.e. the greatest possible class), the paradox clearly emerges: according to Cantor's proof, the class of all classes has fewer members than the class of all its possible sub-classes (i.e. the power-set of the class of all classes). But the power-set is made by classes; thus each of them should be a member of the class of all classes. Hence we arrive at what is called Cantor's paradox. Is there a greatest class or not? Is there a greatest cardinal number or not?

Russell did not at first see the paradox; he merely thought there was a flaw in Cantor's proof. In May 1901, as he was trying to apply the proof to the class of all classes, he saw clearly not a mistaken proof but a contradiction, albeit a different one.¹⁶ This was Russell's paradox:

The comprehensive class we are considering, which is to embrace everything, must embrace itself as one of its members. In other words, if there is such a thing as "everything", then "everything" is something, and is a member of the class "everything". But normally a class is not a member of itself.... Form now the assemblage of all classes which are not members of themselves. This is a class: is it a member of itself or not? If it is, it is one of those classes that are not members of themselves, i.e. it is not a member of itself. If it is not, it is not one of those classes that are not members of themselves, i.e. it is a member of itself. Thus of the two hypotheses—that it is, and that it is not, a member of itself—each implies its contradictory. This is a contradiction. (*IMP*, p. 136)

In the last paragraph of the manuscript under discussion, Russell, through use of the word "methods", makes an implicit reference to his own solution of the paradoxes, the Theory of Types, or perhaps to revised versions of this theory. These methods "throw doubt on things which have been accepted in mathematics since the seventeenth century", and "among the things upon which they throw doubt are those that had seemed to overcome the difficulties of Pythagoras." He does not provide more specific information. It is therefore quite difficult to determine, on the one hand, to which version of the theory of types he is referring and, on the other, what he precisely means by "things which have been accepted in mathematics since the seventeenth century." As I will try to show below, the only indisputable fact is that Russell became

¹⁶ For a detailed account of Russell's discovery see Grattan-Guinness, "How Did Russell Write *The Principles of Mathematics*?", *Russell*, n.s. 16 (1996): 101-27; also Moore, "The Roots of Russell's Paradox", *Russell*, n.s. 8 (1988): 46-56.

dissatisfied with all the solutions he had considered plausible.

The theory of types "enables us easily to avoid the contradictions",¹⁷ but at the same time

has also the unfortunate consequence of invalidating an ordinary and important type of mathematical argument, the sort of argument by which we ultimately establish the existence of the upper bound of an aggregate, or the existence of the limit of a bounded monotonic sequence. It is usual to deduce these propositions from the principle of Dedekindian section.... This in turn is proved by regarding real numbers as sections of rationals; sections of rationals are a particular kind of classes of rationals, and hence a statement about real numbers will be a statement about a kind of classes of rationals, that is about a kind of characteristics of rationals, and the characteristics in question will have to be limited to be of a certain order.... This means that analysis as ordinarily understood is entirely grounded on a fallacious kind of argument, which when applied in other fields leads to self-contradictory results.¹⁸

With the collapse of analysis the differential and integral calculus lose their logical foundation. But it was with the infinitesimal calculus that mathematics handled the continuum: thus, if the calculus is not logically grounded, it cannot be claimed that we have overcome the difficulties of Pythagoras (i.e. the puzzle of incommensurables), which are part of the

¹⁷ Ramsey, "Mathematical Logic", *Mathematical Gazette*, 13 (1926): 185-94, in A. D. Irvine, ed., *Bertrand Russell: Critical Assessments*, 4 vols. (London: Routledge, 1999), 2: 55-6.

¹⁸ *Ibid.* See also Ramsey, "Achilles and the Tortoise", in *Papers* 10: 589-91. Of Russell's theory of types, Ramsey concludes: "Such is Mr. Russell's solution, but it can not be regarded as completely satisfactory, because it has the unfortunate consequence of invalidating an important type of mathematical argument, on which a good deal of modern mathematics rests." See Russell, "Bertrand Russell Catches the Tortoise", *The Forum*, 79 (Feb. 1928): 262-3, *Papers* 10: 91, under the original manuscript title, "Mr. F. P. Ramsey on Logical Paradoxes". Even if not satisfied with the theory of types, Russell did not accept Ramsey's critique, and wrote in his response: "I cannot ... admit as conclusive the argument that my solution, as it stands, invalidates certain parts of mathematics. This is true in a sense: certain kinds of mathematical argument, hitherto accepted as valid, become invalid. But it by no means follows that the conclusions reached by arguments of this kind are false." According to Russell, the invalidation was temporary: "... when logical methods are so amended as to avoid the paradoxes, proofs will become more cumbersome, and will probably prove rather less sweeping theorems, but the practical difference will be small. I cannot therefore give much force to the pragmatic argument that mathematics must not be invalidated."

problem of the arithmetical treatment of continuity.

Russell and Whitehead tried to avoid this consequence by introducing *ad hoc* the Axiom of Reducibility. But this axiom is not self-evident: it is true in the actual world, but there is no reason to suppose it is true in all possible worlds. Thus this axiom is not logically necessary.¹⁹ In *My Philosophical Development*'s discussion of the axiom of reducibility, Russell writes: "This axiom ... seemed necessary if we are, on the one hand, to avoid contradictions and, on the other hand, to preserve all of mathematics that is usually considered indisputable. But it was an objectionable axiom because its truth might be doubted and because (what is more important) its truth, if it is true, seems to be empirical and not logical" (p. 120).²⁰ The axiom of reducibility, according to Russell, forces us into an undesirable choice: if we accept the axiom we can preserve all that is usually considered indisputable in mathematics; but if we want to preserve mathematics from relying on an axiom whose nature is not *a priori*, a large part of mathematics cannot be preserved.

Since 1910 Russell himself had suggested that an alternative solution was required. This was to be found either in a new theory, or in a revised version of the Theory of Types (one making no use of the axiom of reducibility, or one deriving it from a self-evident axiom).

Russell always considered that some sort of hierarchic theory would be indispensable in any solution of the paradoxes. As late as 1959 he wrote: "I lay no stress upon the particular form of that doctrine which is embodied in *Principia Mathematica*, but I remain wholly convinced that without *some* form of the doctrine the paradoxes cannot be resolved."²¹

¹⁹ See *IMP*, p. 193. Russell states the axiom as follows: "There is a type π such that if ϕ is a function which can take a given object a as argument, then there is a function ψ of the type π which is formally equivalent to ϕ ." The Axiom of Reducibility enables us to make propositions about *all* the classes that are composed of objects of any one logical "type".

²⁰ Quoting from the first edition of the *Principia* (Introduction, Chap. 11, Sec. VII, p. 62), Russell specifies the inductive nature of the axiom: "And if the axiom itself is nearly indubitable, that merely adds to the inductive evidence derived from the fact that its consequences are nearly indubitable: it does not provide new evidence of a radically different kind" (*IMP*, p. 121; unchanged in *PM*, 1: 59). See also: Bernard Linsky, "Was the Axiom of Reducibility a Principle of Logic?", *Russell*, n.s. 10 (1990): 125-40.

²¹ *MPD*, p. 79. See also "Reply to Criticisms", *Papers* 11: 26-7: "I have never been satisfied that the theory of types, as I have presented it, is final. I am convinced that some sort of hierarchy is necessary, and I am not sure that a purely extensional hierarchy

In the second edition of *Principia Mathematica* he tried to follow Wittgenstein's purely extensional approach, in order to eliminate the Axiom of Reducibility.²² Nevertheless, he seemed aware that every method of solving the paradoxes was likely to raise serious problems.²³ In 1946, Russell wrote: "For me, the net outcome of the years that I devoted to the principles of mathematics was that the solution of old problems consists in the raising of new ones."²⁴ In the *History* manuscript he reaches the same pessimistic conclusion.

In the published version of the *History*, as mentioned already, Russell replaced most of the excursus discussed above with a single concluding paragraph to the chapter in question. The main thrust of the text that was introduced later was simply to reiterate the influence of the Pythagorean disposition on later metaphysics and religion. There is but one implicit reference to epistemological matters, and this is contained in a

suffices. But I hope that, in time, some theory will be developed which will be simple and adequate, and at the same time be satisfactory from the point of view of what might be called logical common sense."

²² See Grattan-Guinness, *The Search for Mathematical Roots 1870-1940*, pp. 440-3.

²³ *MPD*, p. 122: quoting from the second edition of the *Principia* (Introduction, p. xiv), Russell writes: "Dr Leon Chwistek took the heroic course of dispensing with the axiom without adopting any substitute; from his work, it is clear that this course compels us to sacrifice a great deal of ordinary mathematics." Then he takes into consideration Wittgenstein and Ramsey's attempts to abolish the intensional hierarchy (i.e. the one that necessitates the axiom of reducibility). The consequences of Wittgenstein's proposal, even if worth following, lead to the collapse of many fields of mathematics, as Russell points out, still quoting from the second edition of the *Principia*, 1: 123: "It appears that everything in Vol. 1 remains true (though often new proofs are required); the theory of inductive cardinals and ordinals survives; but it seems that the theory of infinite Dedekindian and well-ordered series largely collapses, so that irrationals, and real numbers generally, can no longer be adequately dealt with. Also Cantor's proof that $2^n > n$ breaks down unless n is finite."

Nor does Russell seem fully satisfied with Ramsey's solution, even if it is the kind of solution to which he gives more attention: "Ramsey abolishes this hierarchy by means of his new interpretation of the concept 'propositional function' [i.e. Ramsey's propositional functions in extension], and is thus left with only the extensional hierarchy. I hope his theories are valid" (p. 126). But on the preceding page, he writes: "I find it very difficult to make up my mind as to the validity of Ramsey's new interpretation of the concept 'propositional function'." According to Russell, if Ramsey's solution is valid, it is able to preserve a great part of mathematics without the axiom of reducibility being necessary. See Russell, "Review of Ramsey, *The Foundations of Mathematics*", *Mind*, 40 (Oct. 1931): 476-82, *Papers* 10: 107-14.

²⁴ "My Own Philosophy", *Papers* 11: 72.

rather vague statement: "It is only in quite recent times that it has been possible to say clearly where Pythagoras was wrong" (p. 37). Russell, it seems, didn't make any further use of the text that was cut; he simply replaced the nine paragraphs in question with a single sentence.

CONSIDERATIONS

Faced by these newly discovered passages of Russell, we must confront at least five questions:

- (1) Can we determine when and for what purpose Russell wrote these leaves?
- (2) What is meant by my suggestion that Russell did not use this material elsewhere?
- (3) Why did Russell omit these nine paragraphs prior to publication of the first edition of *A History of Western Philosophy*?
- (4) Where do we find similar ideas expounded in the *History*?
- (5) Where does the significance of the discovery really lie?

For the remainder of this paper I will try to answer the above five questions, referring to my answers as numbered points.

1. Can we determine when and for what purpose Russell wrote these leaves?

The *History of Western Philosophy* did not emerge from a linear and uniform process of composition. Analysis of the extant manuscript in the Russell Archives reveals that its final form was shaped by interpolation, cutting and revision, not to mention the incorporation of previously written material.²⁵ The Pythagoras chapter, in particular, illustrates clearly this multi-layered process.²⁶ Accordingly, it cannot be assumed that the six leaves were written around the same time as Russell's lectures for the Barnes Foundation, which took place in 1941 and 1942. On the contrary, certain aspects of the manuscript suggest that an initial draft was made prior to this period.

One intimation of the above possibility is that Russell wrote the text

²⁵ See, for example, the structure of the Locke chapter in the *History's* manuscript (RA1 210.006657-F18, -F19).

²⁶ See the top-right numeration, the revisions and the different inks.

in ink that does not match the one used for its revision. Another extremely interesting feature is the foliation. While cardinal numbers were placed in the top-right corner of each folio throughout the entire *History* manuscript, additional foliation in the top left was used only occasionally. The notation "*Pp XII*" can be found here on each of the six leaves, but only on eight other leaves of the manuscript—all of them in the Pythagoras chapter.²⁷ On the one hand this evidence proves that the six leaves constitute an homogeneous block; on the other hand, it invites further inquiry.

The identifier "*Pp*" is a typically Russellian device, probably an abbreviation for a title: it could refer to "Problems of Philosophy", a course given by Russell at the University of Chicago during the 1938–39 academic year, or even to "Philosophic Ideas in Practice", which he presented at UCLA in 1939–40. By moving from the physical attributes of the manuscript to the text, we find some additional information. The first leaf ends with the following statement: "This sort of geometry is independent of observation, but tells us nothing whatever about actual space, any more than pure arithmetic tells us how many inhabitants there are in Chicago" (fol. 54). It is likely that Russell drew this "winking" parallel in order to strike a chord with his intended audience.

I propose, therefore, that these six manuscript leaves were conceived as part of the twelfth lecture of the course "Problems of Philosophy", held by Russell at the University of Chicago during the term 1938–39.²⁸

Unfortunately, though, it is very difficult to determine whether or not the final paragraph of the manuscript also ended Russell's lecture. On the one hand it reads like a conclusion, and by turning to the next leaf we indeed find the opening of the Heraclitus chapter. Yet, on the other hand, there are many gaps in the manuscript and the closing comments leave the reader in a state of suspense. Thus a definitive answer cannot easily be provided.

²⁷ RA1 210.006657-F2, fos. 44, 46, 47, 49–59: all fourteen leaves seem to constitute an homogeneous block, whose original disposition follows the same order we find in the manuscript. Folios 45 and 48 were added to the sequence later, and provide more non-technical particulars of Pythagoras' life and work.

²⁸ For more on the "Problems of Philosophy" course, see Gary Slezak and Donald W. Jackanicz, "'The Town Is Beastly and the Weather Was Vile': Bertrand Russell in Chicago, 1938–1939", *Russell*, n.s. 25–8 (1977): 5–20.

2. *What is meant by my suggestion that Russell did not use this material elsewhere?*

From a survey of Russell's books and the *Collected Papers*, it seems that he had made no prior use in print of the material in question. Nor did it surface in any work published after its initial composition in the late 1930s. This is not to imply that most of the subjects featured therein cannot be encountered elsewhere;²⁹ but both Russell's overall argument and the particulars of its disposition are unique.

The import of the excursus is such that further, detailed research is required. A catalogue of references to Russell's settled or changing views on the issues addressed should definitely be constructed. Point 4, below, attempts to create such a list, at least as far as the *History* is concerned.

3. *Why did Russell omit these nine paragraphs prior to publication of the first edition of A History of Western Philosophy?*

Here I will consider some probable hypotheses and try to establish which is the most plausible.

A preliminary hypothesis might go as follows: Russell resolved to cut this text because he was not satisfied either with its content or style. But this hypothesis is weak because, throughout the manuscript, *whenever* Russell was dissatisfied with a sentence or paragraph, he would simply rewrite it.³⁰ These revisions never involved such drastic changes as were effected on this occasion. Even the most extensively emended pages of text, namely in the first chapter—where both the details and the overall structure underwent massive revisions—the changes were made with a view to clarifying the individual arguments, not removing them.³¹ Moreover, the minor stylistic corrections that Russell made to the six leaves suggest that he was at one stage broadly satisfied with the text as a whole.

Alternatively, we might conjecture that Russell omitted these paragraphs because he intended to use them elsewhere. But as observed above in point (2), it seems more likely that he had no such intention in mind.

We could even speculate that Russell decided that such a long and

²⁹ See above, n. 11; see also point (4).

³⁰ See, for example, RA1 210.006657-F5, fol. 166, on Aristotle's logic.

³¹ See RA1 210.006657-F1, fos. 3–30.

complex digression was not appropriate to the framework of a book whose purpose was to recount the history of philosophy. But if so, how then do we explain the recurrence of theoretical digressions throughout the *History*, some of which are even broader than the material which concerns us here, and at least equally as technical?³²

It is obvious, by the way, that the text was not excluded by mistake, for if such an error had been made we could hardly expect to find a paragraph replacing those that had been "lost". I reject also the marginal hypothesis according to which Russell cut out the digression because he did not, on reflection, deem its content sufficiently important. If this hypothesis held it would seem as if Russell regarded the reduction of mathematics to logic as unimportant.

From this Ockhamistic evaluation of various conjectures, I conclude that the two most credible hypothesis are as follows: either Russell thought that his subject was far too broad to be so compressed, or he judged it too difficult and technical for the kind of *public* he was addressing.³³ (These two hypotheses are not mutually exclusive.) It is not certain exactly what is meant by the word "public" in this context, for it cannot be ascertained whether Russell actually delivered these passages at the Barnes Foundation. There is some evidence to suggest that he did so in Chicago and, from the Pythagoras chapter-title on the manuscript (where the word "Chapter" is inserted after a deleted "Lecture"), we can infer that the whole chapter originated as a lecture. We do not know if, on the original occasion, it was presented by Russell in its entirety, but there is no good reason to suppose it was not: he was talking to undergraduate students in philosophy, which he would consider sufficiently suitable an audience. Thus, when I say he judged the subject too difficult for the public, I propose using the word "public" to mean not only a prospective audience at the Barnes Foundation (mainly fine arts students), but potential readers of his *History* as well.

³² See below, point (4). It is important to notice that technical diachronic digressions took place since the first chapters of the book; see, for example, the Heraclitus chapter and the Atomists chapter.

³³ In the Bergson chapter, p. 804, before illustrating the problem of the continuum, Russell writes: "The question is important, and in spite of its difficulty we cannot pass by it." The same point appears in the original 1912 article and in the shortened chapter of the second British edition of the *History*. Russell, it seems, is reluctant to present the problem of the continuum in a non-technical form.

An alternative version of the second hypothesis might go as follows: Russell may have omitted this material because, as far as Chapter 3 of the book, the groundwork had not sufficiently been laid for this type of digression into twentieth-century logic and mathematics. In other words, without suitable preparation, the general reader may not yet be ready for these points to be raised.

It is also possible to put forward one more hypothesis. The fact that the six leaves survived the manuscript stage suggests that they must have reached the typescript or typesetting stage where they would have been read by people working for the publisher, such as an editor or a publisher's reader. Perhaps it was one of them who persuaded Russell that the lengthy passage was too difficult for a general audience. Russell would then have resolved to substitute the digression with the actual final paragraph of the chapter, which simply and seamlessly reiterates ideas explained before. He did make many other changes to this chapter at a stage following the manuscript stage.

4. *Where do we find similar ideas expounded in the History?*

In this section I shall provide a catalogue of references to digressions on kindred topics in the *History*; I will then try to enhance our evaluation of the six leaves through a process of textual comparison.

Russell explores the relationship of the philosophies of Pythagoras and Plato to Orphism and mysticism on several occasions: in the first chapter (*HWP*, pp. 16, 19), in the Pythagoras chapter (as we have seen), and in the first and third Plato chapters (pp. 106, 119–20, 126). This subject is part of a larger problem, namely the connection of rationalism to mysticism in Greek philosophy (pp. 21–3).

Russell returns to the problem of incommensurables and its impact on Greek mathematics in Chapter 24, "Early Greek Mathematics and Astronomy" (pp. 208–12).

Several comparisons between Greek and modern science can be found in the following chapters: the Heraclitus chapter (pp. 39, 46–7), the Atomists chapter (pp. 65–71), the Socrates chapter (pp. 92–3), the third Plato chapter (pp. 131–2), the fourth and fifth Aristotle chapters (pp. 199, 203–7), Chapter 24 (pp. 214–17) and the Epicureans chapter (p. 246 and n.).

In the fourth chapter on Plato Russell examines in depth the problem of intelligible knowledge being favoured over empirical knowledge (pp.

134–43). In the last chapter on Plato we find a digression about mathematics and the nature of number (pp. 155–7); Russell returns to these topics in the Bergson chapter, where he shows how dangerous mistakes can follow from a wrong definition of the notion of number (pp. 801–2).

Epistemological digressions become more frequent beginning with the chapter on the rise of modern science. In this chapter Russell analyses the persistence of Pythagorism in the work of Copernicus and Kepler (pp. 526, 529–30), and he discusses the notion of acceleration in Galileo's dynamics (pp. 531–3). He also refers to the discovery of the infinitesimal calculus by Newton and Leibniz (p. 536) and compares Newtonian mechanics, on the one hand, to quantum mechanics and, on the other, to the theory of relativity (p. 539–40).

From the Leibniz chapter onwards, Russell amplifies his reflections about the importance of logic and its connection with mathematics and philosophy.³⁴ In the Kant chapter he dwells once more on the dual-sided nature of geometry (p. 716): indeed, it is quite easy to connect these passages to the corresponding section of the manuscript leaves. In the Bergson chapter he returns again to the problem of the continuum: it should perhaps be stressed that here Russell's inquiry is directed to the logical side of the problem (pp. 803–6).

But we must wait until the last chapter, entitled "The Philosophy of Logical Analysis", before we find reflections that are closely linked to the most significant ones of the "*Pp XII*" manuscript. The very first sentence of this chapter, in fact, contains an explicit reference to Pythagoras (p. 828), and the names of Weierstrass and Cantor finally appear in the paragraphs which follow (p. 829). Moreover, in these pages Russell talks about mathematical paradoxes (pp. 829, 830), about Frege's work (p. 830), and he gives a brief explanation of his theory of descriptions (p. 831).

A careful comparison of these passages with the rejected sections of text reveals some notable differences. In the last chapter Russell declares that mathematics has been dethroned "from the lofty place that it has occupied since Pythagoras ..." (*ibid.*); mathematical knowledge "... is, in fact, merely verbal knowledge" (p. 832); and "... it is ... not *a priori* knowledge about the world" (*ibid.*). This judgment does not so much

³⁴ See the Leibniz chapter, *passim*.

represent the defeat of mathematics as it does the possibility of evaluating empiricism anew.³⁵ In the same pages Russell discusses a paradox of infinite number and concludes that Cantor had surmounted it.³⁶

At this stage the differences between the two expositions emerge. In the last chapter we find *not a single word* about the destructive power of the paradoxes, and no reference to the fact that "The paradoxes can be resolved, but unfortunately only by methods which throw doubt on things which have been accepted in mathematics since the seventeenth century" (fol. 59; p. 6 above). Why?

The search for an answer opens up a new hypothesis for Russell's omission of the six manuscript leaves. The key to a solution is buried in the rejected text, but it can only be unearthed by a reading of the *History* as a whole.

The deep meaning of the work can hardly be understood without considering the dramatic historical circumstances in which it was conceived and developed. Russell regarded it as both his intellectual and ethical duty to search for the historical and cultural roots of the present tragedy of World War II. He relates the cultural background of contemporary conflict to four primary ideological attitudes: dogmatism, nationalism, subjectivism and irrationalism. In philosophy, these attitudes can be traced back to some traditions of thought: the dogmatic metaphysical tradition, beginning with Plato and peaking with Hegel;³⁷ and the irra-

³⁵ *HWP*, pp. 831, 832; see also *MPD*, pp. 208–13.

³⁶ P. 830. Russell is referring to one of the paradoxes related with infinite progressions. "Leibniz ... thought it a contradiction, and concluded that, though there are infinite collections, there are no infinite numbers. Georg Cantor, on the contrary, boldly denied that it is a contradiction. He was right; it is an oddity." On the other hand, in these pages Russell does not mention the contradiction of the greatest cardinal, arising from Cantor's power-set theorem (see fol. 59, the "max N_c " paradox).

For an extended discussion by Russell of Cantor's work, see *PoM*, Chaps. 35, 37–8. For some statements by the "later" Russell about the historical role of Cantor and about the role of Cantor's theories in the development of his own mathematical logic, see "My Mental Development", *Papers* 11: 12; "My Own Philosophy", *Papers* 11: 70; "Logical Positivism", *Revue internationale de philosophie*, 4 (Jan. 1950): 3–19, *Papers* 11: 157; "Some Philosophical Contacts", BBC broadcast, 8 January 1955, *Papers* 11: 102–3; "My Debt to German Learning" (1955), *Papers* 11: 106–7.

³⁷ See, for example, *HWP*, pp. 115–16, 742; see also Grattan-Guinness, "Russell and Karl Popper", *Russell*, n.s. 12 (1992): 10–11 (Doc. 2b), for Russell's appraisal of Popper's *The Open Society and Its Enemies*.

rationalistic tradition, beginning with Rousseau and developed further by Schopenhauer, Bergson and Nietzsche.³⁸

As an antidote to this sort of philosophy and to avert a possibly horrific future, Russell advocates an opposite model of wisdom based on analysis and experience. The strength of such a philosophy is its reliance on scientific method and trust in scientific knowledge. On the last page of the *History* Russell writes:

In the welter of conflicting fanaticisms, one of the few unifying forces is scientific truthfulness, by which I mean the habit of basing our beliefs upon observations and inferences as impersonal, and as much divested of local and temperamental bias, as is possible for human beings. To have insisted upon the introduction of this virtue into philosophy, and to have invented a powerful method by which it can be rendered fruitful, are the chief merits of the philosophical school of which I am a member. The habit of careful veracity acquired in the practice of this philosophical method can be extended to the whole sphere of human activity, producing, wherever it exists, a lessening of fanaticism with an increasing capacity of sympathy and mutual understanding. In abandoning a part of its dogmatic pretensions, philosophy does not cease to suggest and inspire a way of life. (P. 836)

The defence of this Enlightenment ideal is the leitmotif of the entire book and, as in a symphony, this touching appeal constitutes its *finale appassionato*.

The six-leaf composition, by contrast, seems to follow a somewhat divergent scheme. In what we should call its *finale addolorato*, Russell suggests that the problem of incommensurables was ultimately reduced to the still more complex problem of the paradoxes. According to Russell, the paradoxes shake the very foundation of the deductive sciences.

If we consider that Russell wanted his book to be read not only by professional philosophers, but also by a much wider public, it is reasonable to speculate that he feared that the six leaves of his excursus might easily be misunderstood, perhaps even purposefully. To the enemies of analysis,³⁹ the puzzle of paradoxes could be used as the puzzle of incommensurables has been used in the philosophy of Hegel or Bergson:

³⁸ See, for example, *HWP*, pp. 701, 759, 770; see also "The Thinkers behind Germany's Sins", *Leader Magazine*, 2, no. 5 (18 Nov. 1944): 6, *Papers* 11: 368–70.

³⁹ For this expression, see the Hegel chapter, p. 744.

a pretext for the downgrading of intellect.⁴⁰ Like Pythagoras before him,⁴¹ perhaps Russell preferred to stay under cover.

However, there are two arguments against this last hypothesis: first, during the 1940s and the 1950s Russell mentioned the puzzle of the paradoxes and its implications on several occasions;⁴² second, he often criticized any kind of obscurantism on the part of other philosophers.⁴³ Thus the preceding hypothesis must be weighed against the larger context of his life's work. Even if the hypothesis could be proved somehow, it would still be difficult to contend that Russell was disinclined to discuss openly the many problems arising from his philosophical proposal, for he was, after all, dedicated to the pursuit of knowledge throughout his life.

5. *Where does the significance of the discovery really lie?*

The central concern of this paper has not been to provide a final assessment of the value of this hitherto unpublished section of manuscript. Rather its chief purpose has been to provide an exposition of the new material. Thus, I will confine myself to a few brief reflections.

As asserted in point (4), the significance of these passages lies principally on their culminating exposition of a subject that is literally *crucial* to almost all of Russell's different theoretical contributions, namely the puzzle of the paradoxes. The passages' intrinsic worth is not to be found in the topic itself, which was dissected by Russell in greater depth and with a more abundant supply of detail on several illustrious occasions. Rather, its value derives from the fact that the aforementioned puzzle is not described, much less clarified, at any point in the *History*.

Moreover, the thematic scope of his argument in the six leaves is a

⁴⁰ See fol. 57; see also the Bergson chapter, pp. 793–5.

⁴¹ In *OKEW*, p. 163, Russell writes: "The Pythagoreans, it is said, resolved to keep the existence of incommensurables a profound secret, revealed only to a few of the supreme heads of the sect; and one of their number, Hippasos of Metapontion, is even said to have been shipwrecked at sea for impiously disclosing the terrible discovery to their enemies." See also his "Atoms in Modern Physics" in *The Nation and the Athenaeum*, 35 (27 Sept. 1924): 780, *Papers* 9: 252; *HWP*, pp. 32, 33; *Wisdom of the West* (London: Macdonald, 1959), p. 22. It seems that Russell had been aware of this legend at least since 1914.

⁴² See, for example, "My Own Philosophy", *Papers* 11: 72; and *MPD*, pp. 208–13.

⁴³ See, for example, *PL*, pp. 1–2, and *HWP*, pp. 834–5.

praeclarum exemplum of that broad perspective by which Russell, in the years before his seventies, connected individual epistemological problems. The holistic nature of his excursus may enable us to confirm that Russell's later philosophy cannot be fully understood unless we embrace the wide horizon that he favoured during this period.

The theoretical character of these paragraphs calls for a deeper understanding of *A History of Western Philosophy* as a whole: it is somewhat naïve to assess its significance only in historiographical terms. As Russell himself suggested (*Auto.*, 2: 223), the book transcends the historico-cultural framework; I will say that it achieves a *philosophical* importance in a very wide sense.

On the other hand, if our hypothesis about why Russell omitted these leaves from the printed work proves acceptable, a final reflection is appropriate. From Russell's editing of the text, we can see that the ethical and political intent gradually assumes precedence over the purely epistemological; it follows, therefore, that the *History*, in accordance with "its connection with political and social circumstances", foreshadows the shift in Russell's priorities, the outcome of which became clearly recognizable in the years to come.
