

A Treatise on Number Theory from a Tenth Century Arabic Source

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The Ismā'īlī sect which arose in the ninth century was responsible for the composition of a number of encyclopedic works in Arabic. One of the most important was the encyclopedia composed by the Ikhwān al-Ṣafā ca. 960, a sect from Basra closely related or perhaps identical with the Ismā'īliyya¹. Five of the collaborators on this encyclopedia are known: Abū Sulaiman Muḥammad b. Muṣṣir al-Bustī; Abū'l-Ḥasan 'Alī b. Hārūn al-Zanjānī; Muḥammad b. Nahrājūrī; al-'Awfī; Zaid b. Rifā'a. The central theme of the Ismā'īlī doctrine was that God is far away from the world and works in the world only through his attributes. According to the Ikhwān al-Ṣafā the basic attribute of God is His unity, which is compared to the unity of the number one. The relation of God to the universe is compared to the relation of the number one to the rest of the numbers. Creation is abstract and impersonal and proceeds from the light of God's unity, a kind of theory of emanation. The universe is composed of spiritual things and material things on different levels. The spiritual things are God, Active Universal Reason, Universal Soul, and Primary Matter, and material things are similarly arranged in four ranks (see page 27 of Arabic text). The final goal of philosophy is the knowledge of the soul and of God, but to achieve this one must begin with the science of numbers and then study all the other sciences in turn.

The entire encyclopedia is known as the *Rasā'il Ikhwān al-Ṣafā* and contains fifty-one (sometimes given as fifty-two) treatises. They are divided into four major sections: Abstract Sciences; Logical Sciences; Natural Sciences; and Theological Sciences. The treatise on Numbers is both the first of the Abstract Sciences and the first treatise of the entire encyclopedia. The fundamental importance of arithmetic for the study of philosophy is found in Plato's *Republic* as well as in the writings of many Arabic philosophers.

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Nicomachus Gerasenus. — GOLDSTEIN B. R., *A treatise on number theory from a tenth century Arabic source* : *Centaurus* X 1964 129-160. | A propos du *Rasā'il Ikhwān al-Ṣafā* dans lequel on trouve de nombreuses références à l'Arithmétique de Nicomaque.

TEXT

The First Treatise of the Division of Abstract Sciences

Numbers

Praise to God and peace upon His worshippers whom He has chosen! (23)

Know that it is the method of our noble brothers to study all the sciences of the things which are in this world, be they substances or accidents, tangible or abstract, simple or compound; and to inquire into their principles and the numbers of their species, kinds, and properties, and into their arrangement and order as well as into the process of their originating and growing out of one cause and one origin by one Creator; and to rely, in demonstrating them, on numerical analogies and geometric proofs, similar to what the Pythagoreans used to do. Therefore, we had to put this treatise before all the others, and in it we will mention interesting things belonging to the science of numbers and their properties which is called *Arithmetic*, by way of preface or introduction so that the way may be easier for students to acquire the wisdom which is called philosophy, and its acquisition may be simpler for novices in the study of abstract sciences.

So we say: the beginning of philosophy is the love of the sciences and the middle of it is the knowledge of the true nature of the universe by virtue of human ability, and its end is speech and action which is in accord with knowledge.

The philosophical sciences are of four kinds: the first kind is the abstract sciences, the second is the logical sciences, the third is the natural sciences, and the fourth is the theological sciences. The abstract sciences are of four kinds: the first kind is *Arithmetic*, the second is *Geometry*, the third is *Astronomy*, and the fourth is *Music*¹. *Music* is the knowledge of the composition of sounds and the principles of melodies are derived from it. *Astronomy* is the science of the stars by means of proofs which are recorded in the book, *Almagest*. *Geometry* is the science of mensuration by means of proofs which are recorded in the book of *Euclid*. (24)

Arithmetic is the study of the properties of numbers and the qualities of the universe which conform to it, which Pythagoras and Nicomachus recorded. One begins the study of the philosophical sciences with the abstract sciences, and the first of the abstract sciences is the study of the properties of numbers because it is the easiest science to acquire²; then mensuration, (musical) composition, astronomy, the logical sciences, the natural sciences, and finally the theological sciences.

The first thing about which we will speak in the science of numbers is in the nature of an introduction or a preface.

Expressions point to certain meanings, the meanings are the objects of names and the expressions are the names. The most general expression or name is "thing", and a "thing" may be one or more than one. *One* is used in two ways: in its proper usage, and in metaphor. In its proper usage it is a thing which can not be partitioned or divided, and everything which can not be divided is one when looked upon from the aspect by which it can not be divided. If you wish, you may say: one is that in which there is nothing but itself, by which it is one.

As for one in metaphor, it is every aggregate which is considered a unity, so for example, ten is called a unit, and a hundred is called a unit, and a thousand is called a unit. One is the epitome of oneness as black is the epitome of blackness; and oneness is the quality of being one as blackness is the quality of being black. Plurality is an aggregate of ones, and the first of the plural numbers is two, then three, four, five, and so on, ad infinitum. Plurality is of two kinds, numbers and that which is numbered. The difference between them is that a number is the quantity of forms (*suwar*) of things in the mind of the counter, while that which is numbered are the things themselves³.

Reckoning is the putting of numbers together and their separation. Numbers are of two kinds, whole numbers and fractions. One, which precedes two, is the source and principle of all numbers, and from it all the numbers are generated both whole and fractional, and they may be reduced to it again.

The whole numbers are generated by augmentation and the fractional numbers by division as follows: when another one is adjoined to one, it is said that they are two; and when another one is adjoined to the two of them, the aggregate is called three; and when another one is adjoined to them, it is called four; and when one is adjoined to them, it is called five.

Similarly the whole numbers are generated by increasing them one by one ad infinitum, and this is their table: (25)

1 2 3 4 5 6 7 8 9.

Numbers are reduced to one as follows: if one is taken from ten, nine remains; and if one is taken from nine, eight remains; and when one is taken from eight, seven remains; and similarly ones are taken away until only one remains. But nothing can be removed from one because (by definition) a part can not be taken from it. So now you understand how the whole numbers are generated from one and how they are reduced to it.

Fractional numbers are obtained from one as follows: the whole numbers are put in their natural order, namely, one, two, three, four, five, six, seven, eight, nine, ten; and one is pointed out from every aggregate. It will be clear how fractions are obtained from one. If one is pointed out from two, it is called a half; and if one is pointed out from three, it is called a third; and if one is pointed out from four, it is called a fourth; and if one is pointed out from five, it is called a fifth; and similarly for a sixth, seventh, eighth, ninth, and tenth. Moreover, if one is pointed out from eleven, it is called one part in eleven⁴; and from twelve, a half of a sixth; and from thirteen, one part in thirteen; and from fourteen, a half of a seventh; and from fifteen, a third of a fifth; and according to this pattern one may regard the rest of the fractions. So now you understand how the fractional numbers as well as the whole numbers are generated from one and how one is the origin of both of them, and this is their table:

2	3	4	5	6	7	8	9
half	third	fourth	fifth	sixth	seventh	eighth	ninth
10	11	12	13	14	15		
tenth	eleventh	twelfth	thirteenth	fourteenth	fifteenth		

Whole numbers are fixed in four ranks: units, tens, hundreds, and thousands. The units are the numbers from one to nine, the tens from ten to ninety, the hundreds from one hundred to nine hundred, and the thousands from one thousand to nine thousand. Twelve single words (in Arabic) suffice for all the numbers, namely, (the numbers) from one to ten, ten words; and one word, hundred; and one word, thousand; so (26)

there are twelve single words in all. The other words are derived from these or combined from them or they are a repetition of them. For example, twenty is derived from ten, thirty from three, forty from four, and so on. Combinations such as two hundred, three hundred, four hundred, five hundred, are combinations of a hundred with the unit numbers. And similarly, two thousand, three thousand, and four thousand are combinations of the word thousand with the words for the unit numbers, the tens and the hundreds, so one says: five thousand, seven thousand, twenty thousand, a hundred thousand, etc., and this is their table:

	1	2	3	4	5	6	7	8	9	10	
	20	30	40	50	60	70	80	90			
	100	200	300	400	500	600	700	800	900	1,000	
	2,000	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000		
(27)	20,000	30,000	40,000	50,000	60,000	70,000	80,000	90,000	100,000		
	200,000	300,000	400,000	500,000	600,000	700,000	800,000	900,000			

The units are 1 2 3 4 5 6 7 8 9 10; the tens are 20 30 40 50 60 70 80 90; the hundreds are 100 200 300 400 500 600 700 800 900; the thousands are 1,000 2,000 3,000 4,000 5,000 6,000 7,000 8,000 9,000 10,000.

The existence of numbers on four ranks, i.e., units, tens, hundreds, and thousands, is not a thing which follows necessarily from the nature of numbers as the existence of even and odd numbers, whole and fractional numbers. But it is a conventional matter which the philosophers have laid down by their own will. They did this so that numbers would conform to the arrangement of natural things, for most natural things were established by the Creator in four orders. For example, there are four natures, heat, cold, dampness, and dryness; four elements, fire, air, water, and earth⁵; four humours, blood, phlegm, and the two biles, yellow bile and black bile; four seasons, spring, summer, autumn, and winter; four directions and four winds, the east wind, the west wind, the south wind, and the north wind; four cardines, the first house, the seventh house, the tenth house, and the fourth house⁶; and four sublunar existents, metals, vegetables, animals, and man. Hence one finds that most natural things come in fours.

- (28) These natural things come in fours by the intention of the Creator and the exigencies of His wisdom. The categories of natural things conform the spiritual things which are above natural things and are not

corporeal; for things which are above the natural are (also) set in four ranks. The first of them is the Creator; then under Him, Active Universal Reason⁷; then under it, the Universal Soul; and under it, Primary Matter⁸; and all these are not corporeal.

The relation of the Creator to the universe is like the relationship of the number one (to the other numbers)⁹; and the relation of Universal Reason to the universe is like the relation of the number two; and the relation of the Soul to the universe is like the relation of the number three; and the relation of Primary Matter to the universe is like the relation of the number four.

Every number has its units, its tens, its hundreds, and its thousands or what exceeds them, *ad infinitum*, and the source of all of them are the numbers from one to four: 1 2 3 4. The rest of the numbers are composed and generated from them, and they are the source of all the numbers. You see this when you add one to four, the total is five; and when you add two to four, the total is six; and when you add three to four, the total is seven; and when you add one and three to four, the total is eight; and when you add two and three to four, the total is nine; and when you add one and two and three to four, the total is ten. This is the rule for the rest of the numbers, the tens, the hundreds, and the thousands and what exceeds them, *ad infinitum*. And similarly the elements of writing are four and the rest of the letters are compounded from them, and words are composed from the letters as we will explain later. Consider it, and you will find what we say true and correct. Let those who wish to know how God invented things in (Universal) Reason and how He brought things into existence in the Soul and how He formed them in the Primary Matter, consider what we have discussed in this chapter.

The first thing which the Creator invented and innovated from the light of His unity was an extensive substance called Active Reason, as He made two arise from one, by repetition. Then He made the Universal Soul arise from the light of (Universal) Reason, as He made three from the adding of one to two. Then He made Primary Matter from the movement of the Soul, as He made four by adding one to three. He then made the rest of the created world from Primary Matter and He arranged it by the intermediary of Reason and Soul, as He made the rest of the numbers from four by adjoining what precedes it as in the examples above.

When you think about what we have said concerning the composition

and the generation of numbers from the number one, you will find it one of the clearest proofs of the uniqueness of the Creator, and the process of His creation and invention of things. For although the existence of numbers and their composition can be conceived from the number one, as we explained above, nothing essential to it is changed, i.e., that the number one is indivisible. Similarly, although God is the one who created all things from the light of His unity, and made their beginning and made them grow, and they have their existence, duration, completeness, and perfection through Him, nothing essential to Him is changed, i.e., His unity before His act of creation, as we will explain in the treatise concerning the Principles of Reason. We already informed you that the relation of the Creator to the universe is analogous to the relation of the number one to the numbers; as one is the origin of the numbers and that which generates them, their beginning and their end, similarly God is the cause of all things and their Creator, their beginning and their end; and as one can not be divided, nor can it be compared to any other number, so God can not be compared or likened to anything in His creation; and as one encompasses and accounts for all the numbers, so God knows all things and their natures. Hence God is exalted over what the unjust say in grandeur and magnificence.

The orders of the numbers are four according to most people, as we mentioned already, but the Pythagoreans put them in sixteen ranks and this is their table:¹⁰

(30)	Ones	1
	Tens	10
	Hundreds	100
	Thousands	1,000
	Ten thousands	10,000
	Hundred thousands	100,000
	Millions	1,000,000
	Ten millions	10,000,000
	Hundred millions	100,000,000
	Billions	1,000,000,000
	Ten billions	10,000,000,000
	Hundred billions	100,000,000,000
	Trillions	1,000,000,000,000
	Ten trillions	10,000,000,000,000

Hundred trillions	100,000,000,000,000
Quadrillions	1,000,000,000,000,000

The fractions have many ranks, because every whole number has one part, two parts, and a number of parts. For example, twelve has a half, a third, a fourth, a sixth, and a twelfth, and similarly twenty-eight, etc. But although the ranks and divisions of fractions are numerous, their scheme is in descending order, each rank is smaller than the previous one. All of them are included in ten words, one word which is general and ambiguous and nine words which are special and fixed. Among the nine words is one word without (etymological) derivation (from its whole number), and that is a *half*, and eight words which are derived: a third (from three), a fourth (from four), a fifth (from five), a sixth (from six), a seventh (from seven), an eighth (from eight), a ninth (from nine), and a tenth (from ten). The word which is general and ambiguous is a “part” because one in eleven is called a “part” in eleven, and similarly for thirteen, seventeen, etc. The rest of the expressions for fractions are formed by combining these ten words. For example, one in twelve is called a half of a sixth, and one in fifteen is called a fifth of a third, and one in twenty is called a half of a tenth. The rest of the significations of the fractions are similarly understood as the adjoining of one of them to another. (31)

These two kinds of numbers continue in quantity ad infinitum. Whole numbers start with the smallest quantity, two, and continue to increase without limit. Fractions begin with the largest quantity, a half, and diminish without limit. So both of them begin at a fixed point, but have no end point to limit them.

Chapter Concerning the Special Properties of Numbers

Every number has one or more special properties meaning the particular qualities of the described object which nothing shares with it. The special property of one is that it is the source of all the numbers as we explained above, and it generates¹¹ all the numbers both odd and even. A special property of two is that it is the first whole number and it generates half of the numbers, the even numbers as opposed to the odd numbers. A special property of three is that it is the first odd number and it generates a third

ever derives from it eternally, whatever it may reach? And this is their table:

5-25-625-390,625.

As for six, it is similar to five in this sense, but it is not self-continuing as five is. Its prolongation is 6 36 1296. Six times six is thirty-six; six returns to itself, and thirty appears. When thirty-six is multiplied by itself, the product is 1296, six again appears, but not thirty. So it is evident that six conserves itself but not what is derived from it. But five conserves itself and what derives from it eternally and forever.

It was said that a special property of the number six is that it is the first perfect number, i.e., if the divisors of a number add up to itself, it is called a perfect number, and six is the first of them. Six has a half which (33) is three, and a third which is two, and a sixth which is one, and if these divisors are added up, the sum is equal to six. No number before six has this property, but after it twenty-eight, four hundred and ninety-six, and eight thousand one hundred and twenty-eight are all perfect numbers. And this is their table:

6 28 496 8,128.

It was said that seven is the first complete number because seven combines in itself the meanings of all the (preceding) numbers. For all the numbers are even or odd, two is the first even number, and four is the second; three is the first odd number and five is the second. If the first odd number is added to the second even number, or the first even number is added to the second odd number, the sum is seven. So, if you add two, the first even number, to five, the second odd number, the sum is seven; similarly, if you add three which is the first odd number to four which is the second even number, the sum is seven. And if one, which is the source of all numbers, is taken with six which is a perfect number, the sum is seven which is a complete number. This is their table: 1 2 3 4 5 6 7. This is a special property of seven which no other number before seven possesses, and it has other special properties which we will discuss when we discuss the fact that the universe is constructed in accordance with the nature of numbers.

It was said that eight is the first perfect cube because of the following argument. If any number is multiplied by itself, it is called a (square) root and the product of two of them is a perfect square as we explained before. But if the perfect square is multiplied by its (square) root, the product is

called a perfect cube. Two is the first number, and if it is multiplied by itself the product is four, which is the first perfect square, then the perfect square is multiplied by its (square) root which is two and the product is eight. Hence eight is the first perfect cube.

- (34) Eight is the first *solid* number because there can not be a solid body without interlocked surfaces and there can not be a surface without mutually adjoining lines and there can not be a line without ordered points as we will explain in the treatise on *Geometry*. The shortest line consists of two points and the narrowest surface consists of two lines, and the smallest solid body consists of two surfaces, so the conclusion from these premises is that the smallest solid body has eight parts. One of them is a line which has two parts. If a line is multiplied by itself, they form a surface which has four parts, and if the surface is multiplied by one of its lengths, it will have depth from it, so then there will be eight parts in all, two of length, two of width, and two of depth.

It was said that nine is the first odd perfect square because three times three is nine and neither seven nor five nor three is a perfect square.

Ten is clearly the first number of the tens' rank as one is the first number of the units' rank, and this is clear without the necessity of commentary. It has another special property similar to a property of the number one, namely, that it only has one number adjacent to it, twenty, and ten is half of it as we explained in the case of one which is half of two.

It was said that eleven is the first *deaf* number because it has no fractional part with a name of its own, but a part is called one part in eleven or two parts in eleven. All of the following numbers are called deaf:

11 13 17 23 29 31 37 41 43 47 53 59 61 67 71 73 79 83 89 91.

It was said that twelve is the first *excessive* number, because if the sum of all the divisors of a number are added and are greater than it, it is called an excessive number, and twelve is the first such number. It has a half which is six, and a third which is four, and a fourth which is three, and a sixth which is two, and a twelfth which is one. If these divisors are added up, the total is sixteen which exceeds twelve by four.

So, in general, every whole number has a special property peculiar to itself but we omit their mention as both easy and superfluous.

- (35) Numbers are divided into two divisions, whole numbers and fractions as we explained above, and whole numbers are divided into two sub-

divisions, even numbers, and odd numbers. An even number is any number which can be divided into two halves which are whole numbers, while an odd number is any number which exceeds an even number by one or which falls short of an even number by one. The generation of even numbers begins from the number two, continuing by repetition without end as is seen:

2 4 6 8 10 12 14 16 18 20.

The generation of odd numbers begins from the number one, to which two is adjoined continually, ad infinitum:

3 5 7 9 11 13 15 17 19.

Even numbers are divided into three kinds: powers of two, pairs of odd numbers, and pairs of pairs of odd numbers¹⁵. Powers of two are all numbers which may be divided into two equal halves of whole numbers which in turn may be so divided, continuing until the process of dividing reaches one. For example, sixty-four: half of it is thirty-two, and half of that is sixteen, and half of that is eight, and half of that is four, and half of that is two, and half of that is one¹⁶. And the generation of these numbers begins with two, which is multiplied by two, and the product is multiplied by two, etc., continuing ad infinitum.

Whoever wishes to understand this thoroughly, ought to double the squares of the chess-board, because he will always remain within the powers of two, and these numbers have other special properties which Nicomachus explained in his book at length, and we will quote a part of it. He says:¹⁷

Let these numbers be set in their natural order, which is one, two, four, eight, sixteen, thirty-two, sixty-four, and so on, ad infinitum. One of their special properties is that if one multiplies the two extreme terms, the product will be equal to the mean term multiplied by itself, if there is only one mean term; or if there are two mean terms, the product of the extreme terms is equal to the product of the two mean terms. For example, let 64 be the last term of the series and one the first. This series has one mean term which is eight, so I say: if one is multiplied by sixty-four, or two times thirty-two, or four times sixteen, the product is equal to eight times eight, and this their table: (36)

1 2 4 8 16 32 64.

And if one adds to it another rank so that there will be two mean terms,

then I say: if one multiplies the two extreme terms, it will be equal to the product of the two mean terms¹⁸. For example: if 128 is multiplied by one, or sixty-four by two, or thirty-two by four, the product will be equal to the product of sixteen times eight. And this is their table:

1 2 4 8 16 32 64 128.

These numbers have another special property. If one adds the numbers of the series starting with one and ending arbitrarily, the sum will be one less than the next number of the series. For example, take one, two, and four, the sum is smaller than eight by one. And if eight is added to it, the sum is smaller than sixteen by one. And if sixteen is added to it, the sum is smaller than thirty-two by one. Similarly, you discover the ranks of these numbers, however great, and this is their table:

1 2 4 8 16 32 64 128 256.

Pairs of odd numbers are all numbers which can be divided in half once, but do not lead to one by division, such as six, ten, fourteen, twenty-two, twenty-six¹⁹. All of these examples are numbers which can be divided once, but do not lead to one. These numbers are obtained by multiplying every odd number by two, and this is their table:

6 10 14 18 22 26 30 34 38 42 46.

(37) Pairs of pairs of odd numbers²⁰ include all numbers which may be divided in half more than once, but do not lead to one by division, such as twelve, twenty, twenty-four, twenty-eight, and similar numbers, and this is their table:²¹

12 20 24 28 36 44 52 60 68.

These numbers are generated by multiplying a pair of odd numbers by two, once or many times. And these numbers have other special properties whose mention we will omit fearing to be redundant.

Odd numbers are divided into subdivisions: prime numbers and composite numbers. Composite numbers are of two kinds, those which are associated with one another, and those which are relatively prime²². The distinction is this: the prime numbers include all numbers, together with one, which are not generated by another number, such as: three, five, seven, eleven, thirteen, seventeen, nineteen, twenty-three, etc. The special property of these numbers is that they have no fractional part other than the one named from them²³. So, three has no fractional part

except a third; five has no fractional part except a fifth, and similarly seven has no fractional part except a seventh; and so on for eleven, thirteen, and seventeen. In general, all the deaf numbers can not be generated except by one, and the name of their fractional parts is derived from them.

The composite numbers include all the numbers which are generated by another number, excluding one, such as nine, twenty-five, forty-nine, eighty-one, etc. And this is their table:

9 25 49 81 121 169.

Two numbers are *associated* with one another if both of them are generated by the same number, excluding one. For example, nine, fifteen, and twenty-one are associated because three generates all of them. Similarly, fifteen, twenty-five, and thirty are all generated by five. These numbers and those like them are said to be *associated* by the number which generates them. And this is their table:

9 15 21 25 35.

Two numbers are *relatively prime* if two different numbers other than one generate them, but what generates one of them does not generate the other, such as nine and twenty-five. Three generates nine but does not generate twenty-five, whereas five generates twenty-five, but does not generate nine. So these numbers and others like them are called *relatively prime*.

Chapter concerning Perfect, Defective, and Excessive Numbers

Every odd number has the special property that if it is divided into two parts, in any way, one of the parts will be even and the other will be odd; and every even number has the special property that if it is divided in any way, the parts will be either both odd or both even, and this is their table:²⁴

<i>Odd</i>	<i>Even</i>	<i>Even</i> ²⁵	(38)
10 11 1	4 10 4	9 10 1	
9 11 2	7 10 7	8 10 2	
8 11 3	2 10 2	7 10 3	
7 11 4	1 10 1	6 10 4	
6 11 5	5 10 5	5 10 5	

Numbers may be divided into three kinds by considering them from another point of view: perfect, excessive, and defective²⁶. A *perfect* number is any number whose divisors add up to itself, such as six, twenty-eight, four hundred and ninety-six, and eight thousand one hundred and twenty-eight. If the divisors of each of these numbers are added up, the sum will be equal to itself. There is only one perfect number in each rank of the numbers: six in the units, twenty-eight in the tens, four hundred and ninety-six in the hundreds, and eight thousand one hundred and twenty-eight in the thousands. This is their table:

6 28 496 8128.

An *excessive* number is any number whose divisors add up to more than itself, such as twelve, twenty, etc. Half of twelve is six, and a third of it is four, and a fourth of it is three, and a sixth of it is two, and a twelfth of it is one: all these divisors add up to sixteen which is more than twelve. A *defective* number is any number whose divisors add up to less than itself, such as four, eight, ten, etc. Half of eight is four, and a fourth of it is two, and an eighth of it is one: the sum of them equals seven which is less than eight. The rest of the defective numbers are of the same kind.

Chapter concerning Friendly Numbers

From another point of view the numbers may be divided into two subdivisions, one of them called *friendly* numbers. This means any two numbers, one excessive, and one defective such that the sum of the divisors of the excessive number is equal to the defective number and the sum of the divisors of the defective number is equal to the excessive number. For example consider two hundred and twenty which is an excessive number, and two hundred and eighty-four which is a defective number. The sum of the divisors of two hundred and twenty is equal to two hundred and eighty-four, and the sum of the divisors of the latter number is equal to two hundred and twenty. So these numbers and others like them are called *friendly* and there are (only) a few of them. This is their table:

excessive number	220
half of it	110
fourth of it	55

fifth of it	44
tenth of it	22
twentieth of it	11
eleventh of it	20
twenty-second of it	10
forty-fourth of it	5
fifty-fifth of it	4
hundred tenth of it	2
two hundred twentieth of it	1
Total	284
defective number	284
half of it	142
fourth of it	71
seventy-first of it	4
hundred forty-second of it	2
two hundred eighty-fourth of it	1
Total	220

Multiplication of Numbers

One of the special properties of numbers is that numbers increase by multiplication and addition without limit. That happens in five ways. Firstly in the natural order: 1 2 3 4 5 6 7 8 9 10 11 12, and so on, ad infinitum. Secondly, in the order of even numbers: 2 4 6 8 10 12 14 and so on, ad infinitum. Thirdly, in the order of odd numbers: 1 3 5 7 9 11 13 15 17, and so on, ad infinitum. Fourthly, by subtraction by any of the preceding methods. And fifthly, by multiplication which we explain later.

Chapter concerning the Special Properties of the Subdivisions

Each subdivision of the numbers has many properties which have been recorded in the Book of Arithmetic in detail, but we will restate part of it in this chapter.

One of the special properties of the natural order of numbers is that the sum from one to any arbitrary number is equal to the product of one more than the last number multiplied by half of the last number

[i.e., $S = \frac{n}{2} (n + 1)$]. For example, when we say: what is the sum of the numbers from one to ten; we add one to ten and multiply it by half of ten and we get fifty-five; or multiply five by itself which is twenty-five, then multiply five by the other "half" which is six [i.e., $11 - 5 = 6$] and we get thirty; the sum is fifty-five, and this is its solution and the pattern which was sought.

The order of even numbers is one, two, four, six, eight, ten, twelve, etc. ad infinitum. One of the properties of this order is that the sum is always odd. Moreover, the sum of one to any arbitrary number is equal to the product of [half of]²⁷ this times one more than the other half of this number, adding one to the total. For example, when we say to you: what is the sum of the numbers from one to ten according to the even order; you take half of ten and add one to it, then you multiply it by the other "half" and add one to the total and that is thirty-one, and similarly for the rest of the numbers.

The order of odd numbers is one, three, five, seven, nine, eleven, etc., ad infinitum. One of its properties is that when these numbers are added according to their natural order, there are two (kinds of) sums, one even, and the other odd, one following after the other, continuing ad infinitum²⁸, and all of the sums will be perfect squares. Moreover, when they are added according to their natural order from one to any arbitrary number, the sum is equal to half of the last number rounded off to the next whole number and then squared. For example, when we say: what is the sum from one to eleven; its solution is that you take half of the number which is five and a half and round it off to six, then multiply it by itself which equals thirty-six, and that is its solution, so take it as a pattern.

The meaning of multiplication is the duplication of one of the numbers by the number of ones in the other number, as for example, when we say: how much is three times four; its meaning is how much is the sum of three taken four times.

- Numbers are of two kinds, whole numbers and fractions as we explained before, and moreover, the multiplication of numbers is of two kinds, simple and compound. Simple multiplication is of three kinds, a whole number by a whole number, like two times three, or three times (41) four, etc.; a fraction by a fraction, like a half times a third or a third times a fourth, etc.; and a whole number times a fraction, like two

times a third, or a third times four, etc. Compound multiplication is also of three kinds: a fraction and a whole number times a whole number like two and a third times five, etc.; a whole number and a fraction times a whole number and a fraction like two and a third times three and a fourth, etc.; and a whole number and a fraction times a fraction like two and a third times a seventh.

Chapter concerning Whole Numbers

The multiplication of whole numbers is of four kinds and there are ten categories for the multiplication of all of them. The four ranks of numbers are units, tens, hundreds, and thousands. The ten categories are: units times units, one of them is one and ten of them are ten; units times tens, one of them is ten and ten of them are a hundred; units times hundreds, one of them is a hundred and ten of them are a thousand; units times thousands, one of them is a thousand and ten of them are ten thousand; and these are four categories. As for tens times tens, one of them is a hundred and ten of them are a thousand; and tens by hundreds, one of them is a thousand and ten of them are ten thousand; and tens by thousands, one of them is ten thousand and ten of them are a hundred thousand; and these are three categories. And as for hundreds times hundreds, one of them is ten thousand and ten of them are a hundred thousand; and hundreds times thousands, one of them is a hundred thousand and ten of them are a million; and these are two categories. As for thousands times thousands, one of them is a million and ten of them are ten million, and this is one category, so there are ten categories in all and this is their table:

units times units; units times tens; units times hundreds;
units times thousands; tens times tens; tens times hundreds;
tens times thousands; hundreds times hundreds; hundreds times
thousands; thousands times thousands.

Chapter concerning Multiplication, Square Roots, and Perfect Cubes (42)

The words which algebraists and geometers employ and their meanings. So we say: for any two numbers whatever, if one of them is multiplied by the other, the product is called a rectangular number. But if the two

numbers are equal the product is called a perfect square and the two numbers are called square roots of this number. For example, if two is multiplied by two, the product is four, or three times three is nine, or four times four is sixteen. Four, nine and sixteen, and similar numbers are all called perfect squares; while two, three, and four are called square roots, so two is the square root of four, three is the square root of nine, and four is the square root of sixteen, and one considers the rest of the perfect squares according to this pattern. The square roots are as follows:

2	3	4	5	6	7	8	9
4	9	16	25	36	49	64	81.

If one multiplies any number by any other number, then the product of them is called a rectangular number which is not a perfect square, and the two different numbers are called its factors and they are called sides of this rectangle, which is the geometric term. For example, two times three, or three times four, or four times five, etc. The product of these numbers which are multiplied is called a rectangle which is not a square.

Chapter concerning Rectangular Numbers²⁹

- (43) When any rectangular number, whether a perfect square or not, is multiplied by any number whatever, the product is called a *solid* number, but if the number was a perfect square and it was multiplied by its square root, then the product is called a perfect cube, as for example, if four which is a perfect square is multiplied by two which is its square root, the product is eight; and similarly if nine which is also a perfect square is multiplied by three which is its square root, the product is twenty-seven. And similarly if sixteen which is a perfect square is multiplied by four which is its square root, the product is sixty-four. Hence, eight, twenty-seven, and sixty-four, and similar numbers are called perfect cubes. A *perfect cube* is a solid such that its length, its width, and its depth are equal, and it has six rectangular faces whose sides are equal and perpendicular (to each other); and it has twelve edges, eight solid angles, and twenty-four plane angles.

If a perfect square is multiplied by a number less than its square root, the product is called a *diminished solid number*³⁰ which is a solid whose length and width are equal, but whose height is less than they are. It has

six rectangular faces whose sides are perpendicular; but it has only one pair of opposite faces which are rectangular whose sides are equal and perpendicular and four faces which are elongated [i.e., whose sides are unequal]; twelve edges, every pair of which are parallel; eight solid angles; and twenty-four plane angles. If a perfect square is multiplied by a number greater than its root, the product is called an *augmented solid number*, such as if four which is a perfect square is multiplied by three which is greater than its square root, the product is twelve; similarly, if nine is multiplied by four which is greater than its square root, the product is thirty-six. Hence, twelve, thirty-six, and similar numbers are called augmented solid numbers and an augmented solid is one whose height is greater than its length and width. It has six rectangular faces, one pair of opposite faces are rectangles whose sides are equal and perpendicular, and four oblong faces whose sides are parallel and perpendicular. It has twelve edges, every pair of which are equal and parallel; eight solid angles; and twenty-four plane angles.

If any rectangular number which is not a perfect square is multiplied by its shorter side, the product is called a diminished solid; and if it is multiplied by its longer side, the product is called an augmented solid; and if it is multiplied by a number smaller than both of them or greater than both of them, the product is called a *free solid*, as for example if twelve which is a rectangular number not a perfect square, one of its sides being three and the other four, is multiplied by three, the product is thirty-six, which is a diminished solid number; and if it is multiplied by four, the product is forty-eight which is an augmented solid number; and if it is multiplied by a number less than three or more than four it is called a free solid. A free solid is one whose length is greater than its width, and its width is greater than its height. It has six faces, every pair of which are equal and parallel; twelve edges, every pair of which are parallel; eight solid angles and twenty-four plane angles. (44)

Chapter concerning the Properties of Perfect Squares

We say: if one more than twice its square root is added to a perfect square, the sum is a perfect square [i.e.,

$$x^2 + 2x + 1 = (x + 1)^2].$$

If a perfect square is diminished by one more than twice its square root, the remainder is a square [i.e.,

$$x^2 - 2x + 1 = (x - 1)^2].$$

For every two perfect squares which follow each other: if the square root of one of them is multiplied by the square root of the other and a fourth is added to it the total will be a perfect square. For example: if the square root of four which is two is multiplied by the square root of nine which is three, the product is six, to which is added a fourth, totaling six and a fourth, and its square root two and a half. The product of two and a half times itself is six and a fourth whose square root is two and a half. For every two perfect squares which follow each other: if the square root of one of them is multiplied by the square root of the other, the product is the geometric mean between them and the three numbers are in one proportion. For example, four and nine are perfect squares whose roots are two and three: two times three is six, and four is to six as six is to nine. The other cases follow the same pattern.

*Chapter concerning Problems from the Second Book of
Euclid's Elements*

Given any two numbers, if one of them is divided into any number of parts, then the product of the two numbers is equal to the product of the one which was not divided, times all the parts of the number which was divided, one part after the other³¹. For example, given ten and fifteen, and let fifteen be divided into three parts: seven, three, and five, then we say:

- (45) 1. The product of ten times fifteen is equal to the product ten times seven plus ten times three plus ten times five [i.e., the law of distributivity (Euclid, II, 1):

$$a(b + c + d) = ab + ac + ad].$$

2. Let any number be divided in parts arbitrarily, then the product of this number by itself is equal to the product of this number times all its parts (Euclid, II, 2). For example, let ten be divided into two parts: seven and three, then I say: the product of ten times itself is equal to the product of ten times seven plus ten times three [i.e.,

$$(a + b)(a + b) = (a + b)a + (a + b)b;$$

$$\text{or } (3 + 7)(3 + 7) = (3 + 7)3 + (3 + 7)7].$$

3. Let any number be divided into two parts, then we say: the product of this number times one of its parts is equal to the product of this part times itself plus the product of the two parts (Euclid II, 3). For example, let ten be divided into two parts; three and seven, then we say: the product of ten times seven is equal to the product of seven times itself plus three times seven [i.e.,

$$(a + b)b = ab + b^2;$$

$$\text{or } (3 + 7)7 = 3 \cdot 7 + 7^2].$$

4. Let any number be divided into two parts, then we say: the product of this number times itself is equal to the product of each part times itself plus twice the product of the two parts (Euclid II, 4). For example, let ten be divided into two parts: seven and three, then we say: the product of ten times itself is equal to the product of seven times itself plus three times itself plus twice seven times three [i.e.,

$$(a + b)^2 = a^2 + b^2 + 2ab;$$

$$\text{or } (7 + 3)^2 = 7^2 + 3^2 + 2 \cdot 3 \cdot 7].$$

5. Let any number be divided in two halves, then in two different parts; the product of one of the different parts times the other, plus half the difference between them multiplied by itself is equal to the product of half of the number times itself (Euclid II, 5). For example, let ten be divided in two halves, then into two unequal parts: three and seven. Now we say: the product of seven times three plus half the difference between them, which is two, times itself is equal to the product of five times itself [i.e.,

$$[(a + b)/2]^2 = ab + [(a - b)/2]^2;$$

$$\text{or } [(3 + 7)/2]^2 = 3 \cdot 7 + [(7 - 3)/2]^2;$$

$$\text{or } 25 = 21 + 4].$$

6. Let any number be divided into two halves, then add something to it. We say: the product of this number together with its increment times that increment plus half of the number times itself is equal to the product of half of that number together with the increment times itself (Euclid II, 6). For example, let ten be divided in two halves, then add two to it.

We say: the product of twelve times two plus five times itself is equal to the product of two plus five together times itself [i.e.,

$$\begin{aligned}(x + a)a + (x/2)^2 &= [(x/2) + a]^2; \\ \text{or } (10 + 2)2 + 5^2 &= (5 + 2)^2; \\ \text{or } 24 + 25 &= 49].\end{aligned}$$

7. Let any number be divided into two parts, then we say: the product of that number times itself plus the product of one of its parts times itself is equal to twice the product of that number times that part plus the product of the other number times itself (Euclid II, 7). For example, let ten be divided into two parts: seven and three. Then we say: the product of ten times itself plus seven times itself is equal to the product of twice ten times seven plus three times itself [i.e.,

$$\begin{aligned}(a + b)^2 + b^2 &= 2(a + b)b + a^2; \\ \text{or } (3 + 7)^2 + 7^2 &= 2(3 + 7)7 + 3^2; \\ \text{or } 100 + 49 &= 140 + 9].\end{aligned}$$

- (46) 8. Let any number be divided into two parts, then add one of the parts to the original number. We say that the product of all that (the number plus the part) times itself is equal to four times the product of that number times the part plus the other part times itself (Euclid II, 8). For example, let ten be divided into two parts: seven and three, then add three to it. Now we say: the product of thirteen times itself is equal to the product of ten times three taken four times plus the product of seven times itself [i.e.,

$$\begin{aligned}(2a + b)^2 &= 4a(a + b) + b^2; \\ \text{or } (10 + 3)^2 &= 4 \cdot 3(10) + 7^2; \\ \text{or } 169 &= 120 + 49].\end{aligned}$$

9. Let any number be divided into two unequal parts, then the sum of the product of each of them times itself is double the product of half of that number times itself plus the product of half the difference of what is between the two numbers times itself (Euclid II, 9). For example, let ten be divided into two halves, then into two unequal parts: three and seven. We say that the product of seven times itself plus three times itself is twice the product of five times itself together with the product of two (which is half the difference between the two parts) times itself [i.e.,

$$\begin{aligned}a^2 + b^2 &= 2[(a + b)/2]^2 + 2[(a - b)/2]^2 \\7^2 + 3^2 &= 2[(7 + 3)/2]^2 + 2[(7 - 3)/2]^2 \\49 + 9 &= 2 \cdot 25 + 2 \cdot 4 \\49 + 9 &= 58].\end{aligned}$$

10. Let any number be divided into two halves, then add some increment to it. Now the product of that number with its increment times itself plus the product of the increment times itself is twice the product of half the number with the increment times itself together with the product of half the number times itself (Euclid II, 10). For example, let ten be divided in half, then add two to it. We say: the product of twelve times itself plus the product of two times itself is twice the product of seven times itself together with the product of five times itself [i.e.,

$$\begin{aligned}(a + x)^2 + x^2 &= 2[(a/2 + x)^2 + (a/2)^2] \\(10 + 2)^2 + 2^2 &= 2[(5 + 2)^2 + 5^2] \\144 + 4 &= 2(49 + 25) \\148 &= 2 \cdot 74].\end{aligned}$$

Chapter concerning the Science of Numbers and its Nature

The philosophers have put the study of the science of numbers before the study of the rest of the abstract sciences, because this science is potentially embedded in everyone and a man ought to reflect (on it) with his reasoning power alone without taking examples from another science, but from it one takes examples for everything else that can be known.

The examples which we expressed in figures in this treatise are for the beginner students whose mental powers are weak, but for those who are sharp-witted, these examples are not necessary.

One of our goals (in writing) this treatise is what we explained in the beginning, and the other goal is to bring attention to the *Science of the Soul* and incitement to the knowledge of its essence. For when the understanding intelligent man studies the science of numbers and reflects upon the quantity of its species, the divisions of its several branches, and the special properties of these several branches, he knows that all of them are accidental and have their being and existence in the soul. So the soul is an essence, because accidents do not have existence other than in essence, and can not exist except through it. (47)

The Goal of the Sciences

The goal of philosophers is the study of the abstract sciences and the training of their students in it. Indeed, it is the path to the natural sciences; the goal of studying the natural sciences is the ascent from it to the theological sciences which are the highest goal of the philosopher and the aim to which they are ascending with true knowledge. The first step of the study of the theological sciences is the knowledge of the essence of the soul, and the search for its source, where it was before its fastening to the body; and inquiry into its life to come, where it will be after its separation from the body, which is called death; and inquiry into the manner of reward for the good people, and how it will be in the world of spirits; and inquiry into the lot of evil doers and how it will be in the other place. Moreover, another quality which men are recommended to acquire is the knowledge of their Lord, and there can be no means of knowing Him except after knowing oneself, as God, the Exalted, has said: "Who forsakes the religion of Abraham but he who is ignorant of himself" (Sura 2: 124) meaning he is ignorant of the soul. And as it is said, if one knows himself, he knows his Lord. And it has been said, if he informs you of himself, he informs you of his Lord. It is binding on every scholar to study the science of the soul and the knowledge of its essence and its arrangement. God has said: "And by the soul, and He who fashioned it, and who taught it its sin and its piety, he who keeps it pure will be happy, and he who corrupts it will be disappointed" (Sura 91: 7-10). And God said in the story of the beloved woman in the narrative of Joseph: "The soul is inclined to evil unless my Lord has had mercy" (Sura 12: 53). And God has said: "As for him who fears to stand in the presence of his Lord and forbids his soul from low desires, surely Paradise will be (his) abode" (Sura 79: 40-41). And God has said: "On the Day (of Judgment) every soul will plead for itself" (Sura 16: 112).

(48) And God has said: "Oh soul that art at rest: Return to your Lord completely satisfied" (Sura 89: 27-28). And God has said: "God takes the souls at the time of their death and as for those that die not, (he takes them) during their sleep" (Sura 39: 43).

Thus there are many verses and proofs in the Quran on the existence of the soul and on its changeable conditions and they are decisive against anyone who denies the existence of the soul.

When those philosophers, who used to discuss the science of the soul

before the descent of the Quran, the New Testament, and the Torah, inquired into the science of the soul with the natural talents of their minds, they deduced the knowledge of its essence by the conclusions of their reasoning. This induced them to compose philosophical books which were mentioned previously in this first treatise. But because of the extensive discourse in them and their transmission from language to language, one can not understand their meaning or know the goal of their authors. The understanding of the meaning of these books is closed to those who inspect them and the goals of their authors trouble those who examine them. We have taken the core of their meaning and the highest goals of their authors and we have presented them as briefly as possible in fifty-two treatises, of which this is the first. The others follow it and you find them according to the order of the numbers.

The treatise is completed, praise to God, Lord of the universe, and may God bless his apostle, Muḥammad the prophet, and his family who are righteous, and may He surely grant them peace.

NOTES

1. Nicomachus, *The Introduction to Arithmetic*, I iii. English translation in: D'Ooge, Robbins, Karpinski, *Nicomachus of Gerasa*, (London, 1926).
2. Nicomachus I iv.
3. D'Ooge, Robbins, Karpinski, p. 113.
4. There are no ordinals in Arabic higher than a tenth. In order to compensate for this deficiency, an eleventh is called "one part in eleven" and a twelfth is called "a half of a sixth". In general, if a number has no factors, a fraction of that rank must be called one part in it. If the number does have factors, then one uses a compound name for a fraction of that rank.
5. Nicomachus II i.
6. The four cardines are the rising and setting points of the zodiac, and the upper and lower culminating points of the zodiac.
Cf. Al-Bīrūnī, *The Book of Instruction in the Elements of the Art of Astrology*, ed. and transl. R. Ramsay Wright, London, 1934, para. 247.
7. Cf. Plotinus, *Ennead* V vii. "We call Intelligence the image of the One. Let us explain this. It is its image because that which is begotten by the One must possess many of its characteristics and resemble it, as light resembles the sun. But the One is not Intelligence. How then can it produce Intelligence? By its turning towards itself the One has vision. It is this vision which constitutes Intelligence." *Ennead* V vi. "The Soul is the Word and a phase of the activity of Intelligence just as Intelligence is the word and a phase of the activity of the One." According to van den Bergh, "the

- Intellect and World-Soul stand in Plotinus' system in the relation of Aristotle's active and passive intellect (*De Anima* III v)." *Averroes' Tahafut al-Tahafut* (London, 1954), II 13.
8. Primary Matter refers to the Platonic forms and hence is spiritual and not corporeal. Cf. F. Dieterici, *Die Philosophie der Araber* (Leipzig, 1875), I 164.
 9. In the arithmology of Nicomachus, as well as other Greeks, the monad was identified with God. D'Ooge, Robbins, Karpinski, p. 104.
 10. The numbers beginning with 10^5 are given names which I was unable to identify: naw^oât, ghayât, sūrât, ḥalbât, al-baṭṭât, haniyât, da^oūrât, wahuwât, majwât, wamūr, mārū.
 11. A number is said to *generate* all its multiples, and they are *generated* by it.
 12. Nicomachus I viii.
 13. Three generates all its multiples, i.e. 3 6 9 12 15 18, etc. One sees that they alternate between odd and even numbers.
 14. Nicomachus II xvii.
 15. Nicomachus I viii.
 16. *ibid.* This example is used by Nicomachus.
 17. *ibid.*
 18. *ibid.*
 19. Nicomachus I ix.
 20. Read *al-fard* instead of *wal-fard* in the Arabic text.
 21. Nicomachus I x.
 22. Nicomachus I xi.
 23. A prime number is a number which has no divisors other than itself and one.
 24. Nicomachus I viii.
 25. Square brackets are used here to emend the text. The even column in our edition makes no sense. Square brackets will also be used to indicate mathematical equations in modern notation.
 26. Nicomachus I xiv. ff.
 27. Emendation based on the example which follows in the text, "you take *half* of ten and add one to it, then you multiply it by the other *half* ...".
 28. The sums are: 1 ; $1 + 3 = 4$; $1 + 3 + 5 = 9$; $1 + 3 + 5 + 7 = 16$; $1 + 3 + 5 + 7 + 9 = 25$; etc. These sums are all perfect squares and are alternatingly odd and even.
 29. Plane and solid numbers are much more extensively treated in Nicomachus Bk. II.
 30. Literally a brick. Cf. Nicomachus II xvii.

A Treatise on Number Theory from a Tenth Century Arabic Source

by BERNARD R. GOLDSTEIN*

The Ismā'īlī sect which arose in the ninth century was responsible for the composition of a number of encyclopedic works in Arabic. One of the most important was the encyclopedia composed by the Ikhwān al-Ṣafā ca. 960, a sect from Basra closely related or perhaps identical with the Ismā'īliyya¹. Five of the collaborators on this encyclopedia are known: Abū Sulaiman Muḥammad b. Muṣṣir al-Bustī; Abū'l-Ḥasan °Alī b. Hārūn al-Zanjānī; Muḥammad b. Nāhrajūrī; al-°Awfī; Zaid b. Rifā°a. The central theme of the Ismā'īlī doctrine was that God is far away from the world and works in the world only through his attributes. According to the Ikhwān al-Ṣafā the basic attribute of God is His unity, which is compared to the unity of the number one. The relation of God to the universe is compared to the relation of the number one to the rest of the numbers. Creation is abstract and impersonal and proceeds from the light of God's unity, a kind of theory of emanation. The universe is composed of spiritual things and material things on different levels. The spiritual things are God, Active Universal Reason, Universal Soul, and Primary Matter, and material things are similarly arranged in four ranks (see page 27 of Arabic text). The final goal of philosophy is the knowledge of the soul and of God, but to achieve this one must begin with the science of numbers and then study all the other sciences in turn.

The entire encyclopedia is known as the *Rasā'il Ikhwān al-Ṣafā* and contains fifty-one (sometimes given as fifty-two) treatises. They are divided into four major sections: Abstract Sciences; Logical Sciences; Natural Sciences; and Theological Sciences. The treatise on Numbers is both the first of the Abstract Sciences and the first treatise of the entire encyclopedia. The fundamental importance of arithmetic for the study of philosophy is found in Plato's *Republic* as well as in the writings of many Arabic philosophers.

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It is possible for us to trace the theory of numbers from its Greek sources to this Arabic treatise on the subject. Within the text itself there are several references to Nicomachus as having recorded the properties of numbers. The Arithmetic of Nicomachus is the major source for this treatise though there are some important deviations from his ideas. One section of this treatise is entitled "A Commentary to the Second Book of Euclid" and it includes algebraic formulations of the theorems on geometrical algebra found in the first ten propositions of Book II of the *Elements*. The last four propositions of Book II are omitted since they contain solutions of special problems rather than algebraic identities.

Nicomachus of Gerasa was a second century (A.D.) Greek mathematician in Alexandria. Nearly all the Greek mathematicians were connected with Alexandria, e.g. Euclid, Eratosthenes, Apollonius of Perga, and Heron, and so one speaks of the Alexandrine school of mathematics in the ancient world. Nicomachus was considered a prominent Pythagorean and a link in the "golden chain" of the Pythagorean sect². His text, called *The Introduction to Arithmetic* consists of two books, the first containing 23 chapters and the second containing 29 chapters. It may be useful here to give an outline of the contents of the Arithmetic of Nicomachus and compare it to that of our treatise.

Book I:

- Chapter 1. Introduction – discussion of philosophy and its terms: wisdom, science, truth, matter.
- Chapter 2. The distinction between quality and accident.
- Chapter 3. Arithmetic is the study of absolute quantity whereas music is the study of relative quantity. Geometry treats size at rest, and astronomy treats that which moves.
- Chapter 4, 5. The root and "mother" of all of them is Arithmetic.
- Chapter 6. Two species of number are odd and even woven into harmony with each other.
- Chapter 7. Definition of number and its divisions of odd and even.
- Chapter 8. Even-times-even numbers.
- Chapter 9. Even-times-odd numbers.
- Chapter 10. Odd-times-even numbers.
- Chapter 11. Odd numbers and its first species: prime numbers.
- Chapter 12. Composite odd numbers.

- Chapter 13. Relative prime.
- Chapter 14. Excessive numbers.
- Chapter 15. Defective numbers.
- Chapter 16. Perfect numbers.
- Chapter 17–23. Relative quantity.

Book II: Geometrical properties of numbers, and proportions.

In contrast we have the contents of the treatise on Arithmetic of the Ikhwān al-Ṣafā:

1. Introduction. Arithmetic is the first of the philosophical sciences.
2. Definition of the number one and of plurality.
3. The generation of whole numbers and fractions from the number one.
4. The ranks of whole numbers are four conforming to the ranks of natural things.
5. The comparison of creation to the generation of numbers.
6. Ranks of fractions.
7. Special properties of numbers.
8. Even and odd numbers and their subdivisions.
9. Associated and relatively prime numbers.
10. Perfect, defective and excessive numbers.
11. Friendly numbers.
12. Order of numbers.
13. Sums of arithmetic series.
14. Multiplication of numbers.
15. Perfect squares.
16. The law of distributivity.
17. Goals of Arithmetic.

It is clear that the two treatises do not completely overlap in their contents but several chapters, especially on the various subdivisions of numbers, closely parallel each other. Much more attention is given in our treatise to the number one and its significance because this work is the product of a monotheistic sect which was interested in emphasizing the unity of all things. Nicomachus, on the other hand, was a dualist who divided the world into sameness and otherness, represented numerically by the numbers one and two³. For the history of mathematics this difference is very important, for the number one does indeed generate all the other numbers in exactly the way that our treatise describes it. An-

other sophistication introduced in our treatise is the notion of beginning with an undefined term, a *thing*, a principle which no Greek had applied to Arithmetic. Euclid, in books seven, eight, and nine of the *Elements* does discuss Arithmetic, but on the basis of geometry; whereas both Nicomachus and the *Ikhwān al-Ṣafā* consider Arithmetic to be prior to Geometry. Nicomachus writes:

"Which of the four methods (Music, Arithmetic, Geometry, Astronomy) must we first learn? Evidently, the one which naturally exists before them all, is Superior and takes the place of origin and root and, as it were, of mother to the others. And this is Arithmetic, not solely because we said that it existed before all the others in the mind of the creating God like some universal and exemplary plan, relying upon which as a design and archetypal example the creator of the universe sets in order his material creations and makes them attain to their proper ends; but also because it is naturally prior in birth, inasmuch as it abolishes other sciences with itself, but is not abolished together with them." (Book I Ch. 4).

The *Ikhwān al-Ṣafā* write:

"The philosophers have put the study of the science of numbers before the study of the rest of the abstract sciences, because this science is strongly embedded in the essence of all of them and indeed it is necessary for a man to reflect deeply and sufficiently on it, without taking an example from another science, but rather one takes example for other scientific subjects from it." (Below p. 46 of Arabic text).

In contrast to the fourfold division of the abstract sciences of both Nicomachus and *Ikhwān al-Ṣafā*, Theon of Smyrna who lived approximately at the same time as Nicomachus had a different division. His list was Arithmetic (including harmony), Plane Geometry, Solid Geometry, and Astronomy⁴. In practice, however, Nicomachus agrees with this division because he does include harmony (i.e., Music) in his Arithmetic. In spite of this inclusion by Nicomachus, our treatise omits any mention of harmony.

A great deal has been written on the transmission of Alexandrine science to the Arabs and it has a strong bearing to the case before us. In Alexandria the Neo-Platonic school still existed in the beginning of the sixth century though the growing fanaticism of the Byzantine governors was a serious obstacle to scientific and philosophical studies. These temporary difficulties had a long lasting effect on scientific inquiry giving it a decidedly scholastic turn⁵. According to the Arabic and Syriac writers of later centuries, John Philoponos was the last great scholar of

the Alexandrine school. He is of special interest to us because he wrote a scholion to the text of Nicomachus' *Arithmetic*⁶. Though adding no mathematical information to that of Nicomachus⁷, it is important as a link in the transmission of the text to the Arabs. The dates for the birth and death of John Philoponos (also known as John the Grammarian) are disputed, but we shall accept the conclusion of M. Meyerhof who dated his birth ca. 485 and his death ca. 558⁸. He wrote eleven commentaries on books of Aristotle, as well as studies of Greek grammar, optics and mathematics. His religious affiliation has also been questioned but we will follow Meyerhof in assuming that he was first a pagan, then an orthodox Christian, and finally a heretic⁹.

On the authority of several Arabic sources, we learn that the Alexandrine school was transferred to Antioch during the reign of Omar II (717–720). The reason for the transfer is not clearly given but it has been assumed that it was due to the general decline of Alexandria at that time. The school was again transferred, after the collapse of the Umayyad Caliphate, to Harran in the time of the Caliph Mutawakkil (ca. 850). Harran in northern Mesopotamia was then a center of the Nestorian Christians as well as the seat of the pagan Šabians many of whom studied astronomy, astrology, and mathematics. During the next half century the scientific movement spread to Baghdad where it remained for a long period¹⁰. The continuity of the tradition from Alexandria to Baghdad is asserted by the great translator Ḥunain b. Isḥaq (d. 877), who said that in the middle of the ninth century the habits and traditions of the Alexandrine school were still plainly followed by the Christian scholars and medical men of Baghdad¹¹.

Thābit b. Qurra (d. 901) who was born in Harran was responsible for many original scientific works and translations, among them a translation of Nicomachus' *Introduction to Arithmetic*. The Arabic text, a manuscript of which is in the British Museum¹², was written within a century before the composition of our treatise and we may assume that it had a great influence on it. We can thus trace the probable chain of transmitters from Nicomachus to the *Ikhwān al-Šafā*.

In the following pages we present a translation of the first treatise of the division of Abstract Sciences, *On Numbers*, of the *Rasā'il Ikhwān Al-Šafā* based on the text of the Cairo edition, 1347/1928. The numbers in parentheses in the margin of our translation refer to the pages in that edition.

NOTES:

1. P. Kraus, Dschabir ibn Hajjan und die Isma'ilijja, *Forschungs-Institut für Geschichte der Naturwissenschaft*, Dritter Jahresbericht (Berlin, 1930), p. 41.
2. D'Ooge, Karpinski, Robbins, *Nicomachus of Gerasa* (London, 1926), p. 78.
3. Nicomachus, The Introduction to Arithmetic, I xvii (English text in D'Ooge, Karpinski, Robbins).
4. D'Ooge, Karpinski, Robbins, p. 113 note.
5. Max Meyerhof, On the Transmission of Greek and Indian Science to the Arabs, *Islamic Culture* XI (1937), 18.
6. John Philoponos, *Exēgēsis eis to prōton tēs Nikomachū arithmētikēs eisagōgēs*, *Specilegium Romanum* (Rome, 1839) vol. II, and another edition, ed. Hoche (Leipzig, 1864-67).
7. D'Ooge, Karpinski, Robbins, p. 125.
8. Max Meyerhof, Philoponos, *Mitteilungen des Deutschen Instituts für Aegyptische Altertumskunde in Kairo* II (1931), 5.
9. *Ibid.*, p. 4.
10. Max Meyerhof, Transmission, *Islamic Culture* XI (1937), 19.
11. Max Meyerhof, New Light on Hunain Ibn Ishāq and His Period, *Isis* VII (1926), 702.
12. Arabic manuscript in the British Museum 426,15°, published as: Thabit b. Qurra's arabische Uebersetzung der Arithmētikē Eisagōgē des Nikomachus von Gerasa, W. Kutsch (Beyrouth, 1959).