

Pythagorean Geometry and Fundamental Constants

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Abstract

The Cosmological Circle from ancient geometry, with its right triangles, and the ratios of the Pythagorean Table are found to be harmonically related to the fundamental physical constants. After a brief history of harmonic mathematics, harmonic values are calculated for the speed of light constant, gravitational constant, Planck's constant, and the inverse fine-structure constant. We then calculate the harmonic of electron mass and proton mass, showing the related Pythagorean/Cosmological Circle harmonics; and speculate on geometry and symmetry.

Mathematical Subject Classifications, MSC: 51P05, 01A05, 85-03

1 Introduction

Why do the fundamental physical constants have their particular values [1] - [6]? This is one of the classic questions of fundamental importance for foundational physics. What can we learn from ancient geometry that might help answer the question [7]? According to John Michell 6, 12, 37, and their multiples are prominent in the canon of ancient geometry [8]. From Euclid's formula for Pythagorean triples: $a = 12$, $b = 6$; $x = a^2 - b^2$, $y = 2ab$, $z = a^2 + b^2$ [9, 10].

$$x = 144 - 36 = 108, y = 2 \times 12 \times 6 = 144, z = 144 + 36 = 180; \text{triangle } 108, 144, 180. \quad (1)$$

The double of this triangle is 216, 288, 360; and a related triangle we will see is 162, 216, 270. The sides of a 3, 4, 5 right triangle multiplied by 36 gives a 108, 144, 180 right triangle. The sides of a 3, 4, 5 right triangle multiplied by 72 ($6 \times 12 = 72$), gives a 216, 288, 360 right triangle. The arithmetic mean $((A + B)/2)$ between 36 and 72 is $(36 + 72)/2 = 54$. $2 \sin 54^\circ = \phi$, the golden ratio. The sides of a 3, 4, 5 right triangle multiplied by 54 gives a 162, 216, 270 right triangle. Notice that 36, 72, 108 and 144 are also the angular measures found in the pentagon and decagon construction. The importance of the decagon is emphasized by William Eisen [11, 12]. $162 + 216 + 270 = 648$, and $216 + 288 + 360 = 864 = 12^3/2$. 864 is a harmonic of the “solar diameter” [13, 14]. $2160 = 360 \times 6$, a harmonic of the minutes of angular measure in a full circle of 360° [15], and sum of the angles in a cube.

In the Cosmological Circle a central “squared circle” plan is drawn from the sizes and relationships among the Earth, Sun and Moon. Drawings of this circle can be found at [14], [16] - [19]. A 12 around 1 theme, groups of 3 circles on each side, give a general form of equations as $3x^{\pm 1/12}$ [19, 20]. The 3, 4, 5 right triangle is found in the constructions of the Brunes Star [7, 21], Squaring the Circle [13, 18], and Platonic solids [13, 22]. The 3, 4, 5 triangle is also in accounts attributed to the oral tradition [23]. The “Moon diameter” of 2160 miles is placed on an “Earth square” that forms a triangle, 2160, 2880, 3600. John Michell again, “A tradition which has been credited by many learned men over the centuries is that the ancients encoded their knowledge in the dimensions of their sacred monuments” [13]. The “New Jerusalem diagram” squared circle is found in many ancient temples worldwide. In this diagram Earth and Moon form a “squared circle.” The basic Earth square has a perimeter of $4 \times 11 = 44$, and the circle with an inscribed regular heptagon, or star heptagram [24], has a radius of $r = 7$, $C = 2\pi r = 2\pi \times 7 \simeq 44$. $37 + 7 = 44$ (for 37 see notes to Eqs. (2), (8-10), & (13-15)). Triangles 3, 4, 5; 6, 8, 10; 12, 16, 20: are basic triangles from the Cosmological Circle. ϕ is the golden ratio [25] - [27]. $\phi = (1 + \sqrt{5})/2$, $\phi^5 - \phi^{-5} = 11$, $\phi^4 + \phi^{-4} = 7$.

The harmonic system of William Conner [28, 29], partly derived from the Cosmological Circle and incorporating a modified version of the Pythagorean Table [30, 31], has a natural harmonic scale of 12 tones from 144, ..., 162, ..., 180, ..., 216, 233, ..., 270 in the base octave (showing only those used here). The “harmonic” language has both a quantitative and qualitative meaning where concern for units is not immediate or exclusive to the meaning, and suggesting that some units are not as arbitrary as they are currently thought to be [11] - [13], [16, 28, 29]. The Pythagorean [32] - [39] triangle harmonics 108, 144, 180, are a convergence of decagon angular measure, semicircle, with 144 as the grid speed of light harmonic, Fibonacci number, and fundamental tone-number. The most frequent harmonics seen below are: 108, 144, 162, 180, 216, 233, 288, 360, & 864. Albert von Thimus, a contemporary of Faraday and Maxwell, constructed the Pythagorean Table. Conner’s version, reconstructed from Levy and Levarie’s version, substitutes frequency for string-length on the vertical side of the Table; with 144 as the fundamental tone [29]. Hans Kayser rediscovered Thimus, and regarded the “algebraic and tonal laws” of the Pythagorean Table as a “group-theoretical continuation of the overtone series” [40]. Jocelyn Godwin considers Kayser [41, 42] the founder of modern harmonics [43, 44]. Referring to the Pythagorean Table, Thimus and Kayser noted that

Plato hinted it was the harmonic science of antiquity, and highly valued secret of the oral tradition [43]. In his youth Werner Heisenberg questioned Plato's view of world construction from small right triangles [45, 46], and in his later years believed Platonism provided clues for modern physics [47].

2 Speed of Light Constant c

The constant c is defined as the speed of light in vacuum.

$$c_{harmonic} \cong 3(1008 + 75.6^{-1/4})^{-1/12} 10^{33/4} \cong 299\,792\,458 \quad (2)$$

$$\text{NIST 2006 [48, 49]} \quad c = 299\,792\,458 \text{ ms}^{-1}$$

$7 \times 144 = 1008$, $7 \times 10.8 = 75.6$, $2 \ln(360/\phi) \simeq 10.8$, $864 - 108 = 648 + 108 = 756$. From the base of the Earth square to the top of the Moon square is 10,080. The radius of the Moon circle is 1080. 7 is the radius of the circle with the inscribed heptagon. 144 is the grid speed of light harmonic. The 144 harmonic as angular measure, is associated with the harmonics of the circle, $14.4^\circ \times 60'/1^\circ = 864' = 4 \times 216' = 8 \times 108' = (144 \times 360)''$. $60 = 27 + 33$, the time harmonics [29]. Additional references for 108 [50] and the harmonics of light [51].

3 Planck's Constant h

Planck's constant is the quantum of action, where energy is quantized.

Notice the Pythagorean relation: $662^2 + 606^2 \simeq 896^2 \rightarrow 6.626\,068\,96 = h_{harmonic}$.

$$h_{harmonic} \cong 3(108^2 - (7.39 + (7\pi)^{-2})^{1/6} 10^{-1/3}) \cong 6.626\,068\,96 \quad (3)$$

A reduction from the form $3x^{1/12}$. Again, 7 is the radius of the circle with the inscribed heptagon. $22 \simeq 7\pi$, $2 \times 11 = 22$, $e^2 \simeq 7.39$, $162^2 - 144^2 \simeq 73.9^2$, $739 + 233 = 972 = 9 \times 108$. 233 is the 9th tone in the base octave. 972 is resonant with several harmonics [28]. $972 = 27 \times 36 =$ harmonic of the seconds in a grid day of 27 hours [28, 29]. 22 is half the perimeter of the basic Earth square. $22/7 \simeq \pi$ [52]. $10^{1/3} \simeq$ harmonic of 216.

$$h_{harmonic} \cong 3((23.3 + (864 + 1.44)^{1/3} 10^{-3})/10^5 \pi)^{-1/12} \cong 6.626\,068\,96 \quad (4)$$

Again, the 9th tone is 233, harmonic of 23.3. $7^\phi \simeq 23.3$, $144^2 + 180^2 \simeq 233^2$, $233^2/2\pi \simeq 8640$, and curiously; $\sqrt{\phi_f \phi} \simeq 2.33$. ϕ_f is the reciprocal Fibonacci constant [53]. $972 - 739 = 270 - 37 = 233$, $864 = 216 + 288 + 360 = 8 \times 108 = 4 \times 216 =$ harmonic of the perimeter of the Moon square, 8640. $8.64 \times 10^5 \simeq$ "solar diameter" in miles [29]. $\sqrt{\phi_f/\phi} \simeq 1.44$.

$$h_{harmonic} \cong 3((55 + (8^3 + 13/21)10^{-4})^{1/2} 10^{-5})^{-1/12} \cong 6.626\,068\,96 \quad (5)$$

$$\text{NIST 2006 [48, 49]} \quad h = 6.626\,068\,96\,(33) \times 10^{-34} \text{ Js}$$

From the Fibonacci sequence, ..., 3, 5, 8, 13, 21, 34, 55, 89, 144, 233.... $288 - 233 = 55$. And $33^2 + 44^2 = 55^2$. Other references for Planck's constant [54, 55].

4 Gravitational Constant G

The gravitational constant G is found in Newton's law of gravitation: $F = GMm/r^2$.

$$G_{\text{harmonic}} \cong 3(11 + 86.4^{-1})^{1/3} \cong 6.674\,28 \quad (6)$$

Again, 86.4 harmonic from Eq. (2), and $11 =$ side of the basic Earth square. And $3x^{4/12} = 3x^{1/3}$.

$$G_{\text{harmonic}} \cong 3((3.6/\phi) - (\phi/10^4)) \cong 6.674\,28 \quad (7)$$

$$\text{NIST 2006 [48, 49]} \quad G = 6.674\,28\, (67) \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}$$

$360 - 360/\phi \simeq 137.5$, harmonic of the “golden angle” of plant phyllotaxis [56], and $2 \cos 36^\circ = \phi$. $144 - 108 = 36$. 10^4 is notable power of ten [29], $8.64 \times 10^4 = 24 \times 60 \times 60 =$ harmonic of the seconds in a 24 hour day, and $2.16 \times 10^4 =$ harmonic of the minutes of arc in a circle, 60×360 . Another interesting approximation for the G harmonic $\cong (10.8/\phi) - (10\sqrt{\phi})^{-3} \cong 6.67428$, see notes to Eq. (2). Other references for the gravitational constant [57] - [59].

5 Inverse Fine-Structure Constant, 1/Alpha α^{-1}

The Fine-Structure Constant determines the strength of the electromagnetic interaction.

$\alpha^{-1} = \hbar c/e^2$ in cgs units. $\hbar = h/2\pi = \hbar\text{-bar}$, the reduced Planck's constant, $c =$ speed of light constant, $e =$ elementary charge.

$$\alpha^{-1} \cong 60H_5 + (360 - 120^{-1})10^{-4} \cong 137.035\,999\,16 \quad (8)$$

H_5 is the sum of the first five terms of the harmonic series, $1 + 1/2 + 1/3 + 1/4 + 1/5 = 137/60$ [60]. $60 + (7 \times 11) = 370 - 233 = 137$, $\sqrt{3600} = 60$, $120 \cong 2 \times 37 \times \phi$. 144 is the harmonic mean ($2AB/(A + B)$) between 120 and 180. The 120 harmonic is also a significant angular measure [29].

With the main Pythagorean triangle of the Cosmological Circle: $2160 + 2880 + 3600 = 8640 \rightarrow 2.160\,2880\,3600 - 8.64/10^6 = x$.

$$\alpha^{-1} \cong 20\phi^4 - 10^{-1/3}x^{-3} \cong 137.035\,999\,16 \quad (9)$$

$2880/144 = 20$, $\phi^4 = (7 + 3\sqrt{5})/2$, $\phi^4 + \phi^{-4} = 7$, $108\phi^4/2 \cong 370$. For 864 see Eqs. (4), (5), and (6). $2.16 + 739/(10^6\sqrt{7}) \cong x$. See Eq. (3) for 739, and $10^{-1/3}$.

$$\alpha^{-1} \cong 1/3(2330 + \phi^{-1} + 288^{-2})^{1/12}10^{7/3} \cong 137.035\,999\,16 \quad (10)$$

$$\text{NIST 2002[61]} \quad \alpha^{-1} = 137.035\,999\,11\, (46)$$

233, the harmonic of 2330, Eq. (4). $233 + 55 = 288$, $12 \times 10^{-6} \cong 288^{-2}$, $233^2/2\pi \cong 8640$, perimeter of the Moon square and harmonic of the solar diameter. $370 - 137 = 233$. From the Fibonacci sequence, $89 + 144 = 233$, and $144\phi \cong 233$. $89^2 = 7921 \cong$ “Earth diameter” harmonic. 89 is also the product of the four Einstein-Conner harmonic energy equations [28]. $(89 \times 233)/144 = 144 + 1/144$ [29]; and of harmonic interest to compare this form with: $i + 1/i = 0$ [62, 63], $\phi i + 1/\phi i = i$ [21], $\phi + 1/\phi = \sqrt{5}$. Also for Eq. (10) see [64]. The latest experimental and QED calculation of the Gabrielse Research Group 2007 [65], the value of $\alpha^{-1} = 137.035999068$ (96), is a correction from 2006 [48, 49]; which will probably change the next determination by NIST [49]. Since Gabrielse 2006 was the key datum in NIST 2006 [49], the NIST 2002 value is used for comparison.

Like Isaac Newton [66], Johann Balmer was a student of the Pythagoreans and temple geometry [67]. Balmer’s formula for the spectral lines from hydrogen was a Pythagorean relation that found explanation in Bohr’s atomic model [68] - [70]. The fine-structure constant [71] - [80], which gives the electromagnetic interaction strength, was introduced by Arnold Sommerfeld [81]; and later gained more attention when it was derived from the Dirac equation [82, 83]. From Max Born’s research [84], the fine-structure constant presents a challenge to both determine its numerical value and discover the deeper relationship that it suggests between quantum theory and electrodynamics [85, 86]. Wolfgang Pauli had one of the deepest understandings and curiosity about the importance of the fine-structure constant and spectra, as the Pauli/Jung letters reveal [45, 87].

Continuing notes for Eq. (10), the largest of sunflowers can have a 233/144 ratio of clockwise and counterclockwise spiral patterns [27, 88, 89]. 89, 144, and 233 are the basis for a demonstration of a harmonic version of Kepler’s [90] third law [29]. According to Conner, 233 and 288 respectively represent the two lowest harmonic values of energy in one of the spirals of his “Quadrspirals,” and possibly helping to explain the low energy limit of alpha as a “running coupling constant.” 120, from Eq. (8), is the lowest harmonic value of energy in the spiral counter to the 233, 288 spiral. The Quadrspirals model is derived from π , ϕ , the Fibonacci sequence, and four generator tones from the base octave of the Pythagorean Table, (the most important being 216, here it is called the “center of the vortex” harmonic). The Quadrspirals is an analog harmonic mapping of double helical vortices. The conical Fibonacci spiral is conjectured to be a least energy pattern [91]. Speculation concerning the underlying dynamics of the Quadrspirals raises questions about the nature of space, time, and energy addressed by others in various ways; especially with reference to the continuous field concept and linear “approximations” for discrete quanta [20, 29], [92] - [98].

6 Electron Mass m_e

From the Pythagorean triangle, $144^2 \cong 108^2 + 96.4^2 \rightarrow 1.44 \ 108 \ 964$.

$$m_e \text{ harmonic} \cong (1.44108964)^{-1/9} 90^{1/2} \cong 9.109 \ 382 \ 15 \quad (11)$$

$$\text{NIST 2006[48, 49]} \ m_e = 9.109 \ 382 \ 15 \ (45) \times 10^{-31} \text{kg}$$

Eq. (11) is reduced from $3((1.44\dots)^{4/3}10^{-6})^{-1/12}$, similar in form to the proton mass harmonic equation, Eq. (12). $1.44 \ 108 + 20/1440^2 \cong 1.44 \ 108 \ 964$. $180/2 = 90$, $90^{1/2} = 3\sqrt{10}$. An approximation, $10/(1 + \sqrt{10})^2 \cong \gamma$, Euler's constant; and for its relation to the harmonic series see [60]. $\sqrt{10}/3 \simeq \hbar$ harmonic. For $\sqrt{10}$, a discussion of powers of ten, and units; see White [98]. And for fundamental constants, units, and the origin of mass see Wilczek; who notes the “profound geometric meaning” of the fundamental constants [99, 100].

7 Proton Mass m_p

From another Pythagorean triangle composite, $1080^2 + 1096^2 \simeq 1538^2$, $2 \times 1096 = 2192 \rightarrow 1080.219 \ 215 \cong 1080 + 162/739$.

$$m_p \text{ harmonic} \cong 3((1080 + 162/739)^{4/3}10^{-1})^{-1/12} \cong 1.672 \ 621 \ 637 \quad (12)$$

$$\text{NIST 2006 [48, 49] } m_p = 1.672 \ 621 \ 637 \ (83) \times 10^{-27} \text{ kg}$$

For 1080 see Eq. (2), $(12 \times 10^5)^{1/2} \simeq 1096$. 1538 is a harmonic of the inner radius of the decagon with side = 1, $1.538^2 + 0.5^2 \cong \phi^2$. For 162 see notes to Eq. (1), and (3). 739 again, from Eq. (3) for Planck's constant, and see notes for Eq. (9).

8 Geometric Model

Volume of a torus with a vanishingly small hole, or the surface area of an “archetypal” Einstein-Eddington hypersphere, $V = 2\pi^2 R^3$ [101, 102]; with an archetypal $R = 360/2\pi$ [29].

$$\text{Volume of the torus : } V = 2\pi^2 R^3 = 108^2/10^3\pi \cong \text{harmonic of } 60/\phi \cong 37 \quad (13)$$

$108^2 = 1.1664 \times 10^4$, see Eq. (7) notes. Also of interest in relation to this volume are two harmonic approximations for the speed of light.

$$c \text{ harmonic} \cong (2\pi \log 3 + (12 \times 10^3)^{-1})10^8 - 1 \cong 299 \ 792 \ 458 \quad (14)$$

$2\pi \log 3$ is from Shinichi Seike's discussion of hypersurfaces [103]. $108/9 = 12$.

$$c \text{ harmonic} \cong 3((1.008 + 1.08\pi/10^4)^{-1/12})10^8 + 4 \cong 299 \ 792 \ 458 \quad (15)$$

Here again from Eq. (2) we have the harmonic $7 \times 144 = 1008$, and a factor of 108. $3e/5\phi \cong 1.008$. And for 10^{-4} or 10^4 , see Eqs. (5), (7) notes and Eq. (8). The harmonic of 37 [104, 105], is also the approximate geometric mean, $\sqrt{AB}$, between 7 and 14^2 . $1/27$ is a harmonic of 37, $3 \times 27 \times 37 = 2997$, and $\sqrt{2} \times \phi^2 \cong 3.7$. 37 is also a harmonic of the covalent bond radius of the hydrogen atom, H-H, 37 pm [16, 106]. The perimeter of the rectangle enclosing a Vesica Piscis with a radius of 37 is 370; and $233^2 + 288^2 \cong 370^2$.

For additional references to 37 see [12, 16, 107, 108]. 12, 35, 37 right triangle, $\sin 37^\circ \simeq 3/5$. For the golden ratio and general structure of the hydrogen atom, see Heyrovska and Narayan [109]. π is curiously found in both of these speed of light approximations and Planck's constant, Eqs. (3), & (4). The role of π in the hypersphere and Planck's constant is a key to understanding the relationship between relativity and Heisenberg's uncertainty in quantum theory, according to Arthur Young; "Eddington recognizes that the curvature of relativity is the same thing as the uncertainty of quantum theory!" [101]. Ian Durham has a related perspective on Eddington and uncertainty [110]. And the importance of toroidal topology was demonstrated by John Wheeler [101].

9 Conclusion

From these calculations we have shown how the Pythagorean Table/Cosmological Circle encodes the harmonic compositions of four fundamental physical constants, the electron mass, and proton mass. A careful analysis of the calculations show significant and interrelated harmonic connections to the Cosmological Circle. For further connections, the references should provide more detail. These results support the ancient view of this "mandala" as an archetypal scale model of the universe, from the atom to the solar system, and beyond; part of the "cosmic canon of order, harmony, and beauty" [29]. A mandala is often defined as a "map of the cosmos" [111]. The mandala of Franklin Merrell-Wolff gave us an understanding of its dynamics as an essential aspect of the Cosmological Circle, a "squaring of the circle" [112] - [115]. M.-L. von Franz writes about a vision of a mandala of regular polygons thought to be related to the origin of the Pythagorean system, and found by Rudolf Haase to be a significant historical pattern [116]. From another viewpoint, David Fideler shows that "a logarithm representation of the octave on a circular graph or "tone mandala" allows one to visually discern any harmonic symmetries" [117]. Robert Lawlor shows how the musical ratios can be derived from a 3, 4, 5 right triangle [22], and Anne Macaulay reveals another derivation from the ancient diagram of the Hyperborean Apollo's Circle, with connections to megalithic Britain [32, 118]. Although the heptagon and hexagon (usually shown as a star heptagram and a double of the hexagon, a star dodecagram) are the main regular polygons in the Cosmological Circle, the 3, 4, 5 right triangle gives phi and the pentagon [119]. These structures are the basis for constructing the Platonic solids that have important symmetry groups associated with them [120] - [123]. For Murray Gell-Mann, "Disciplined judgment about what is neat and symmetrical and elegant has time and time again proved an excellent guide to how nature works." [124]. Considering the recommendation by Pauli to be more "fundamental" in our methods [83], we have also tried to keep in mind the reasoning behind the quote from Albert Einstein, "Our experience hitherto justifies us in believing that nature is the realization of the simplest conceivable mathematical ideas" [125, 126].

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