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The problem of the volume of the sphere was already an old problem in Archimedes' day. In Greece, as in China, in pre-Archimedean days it was associated with the box-lid. Probably it goes back to the common source of Pythagorean and Chinese-like mathematics. [14793]

On the Volume of a Sphere

A. SEIDENBERG *

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1. Archimedes on the volume of a sphere

It is well known that ARCHIMEDES found the volume of a sphere. If S is the volume of a sphere of radius a , then $S = \frac{4}{3} \pi a^3$, where $\pi/4$ is a constant giving the ratio of the area of a circle to the area of its circumscribed square. Hence, too, the volume ($= \pi a^2 \cdot 2a$) of the right circular cylinder of height $2a$ circumscribing the sphere is $\frac{3}{2} S$. ARCHIMEDES expressed the wish that the figure of a sphere and a circumscribing cylinder be engraved on his tombstone (cf. BALL 1901, p. 69).

For us, finding the volume is an easy exercise. If the center of the sphere is taken at the origin of a rectangular coordinate system, the equation of the sphere is $x^2 + y^2 + z^2 = a^2$. The section of this by a plane at level z is a circle of radius $\sqrt{a^2 - z^2}$. Hence the volume, as taught in our first year calculus courses, is $2 \int_0^a \pi(a^2 - z^2) dz$. In accordance with some simple rules also taught in the first year, this integral is evaluated as $\frac{4}{3} \pi a^3$.

Of course, the ancients of ARCHIMEDES' time (c. 250 B.C.) did not have calculus, but they had CAVALIERI-like methods into which the above argument can be easily

* ABRAHAM SEIDENBERG died on May 3, 1988. Thus he could not have seen the proofs of this article, which Professor JAMES CASEY has corrected

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translated. In this method one has a solid or planar region and seeks, say, the volume or area; let us speak of the above-mentioned sphere, s , of volume S . One then cuts s (or the solid) with parallel planes, in our case with the planes of level z . The sections are circles, conceived of, in a not altogether clear way, as cylinders (or discs) of infinitely small thickness; and s as the union of these slices. A corollary of this way of looking at things is that if s' is another solid and the sections of s' at level z are equal in area, respectively, to the sections of s at level z , then the volume of $s' = \text{volume of } s$, and if the area of the section of $s' = c$ times the area of the corresponding section of s for some constant c , then volume of $s' = c$ times volume of s .

In particular, let us circumscribe a square to our circular section of the sphere, taking the sides parallel to the x - and y -axes. This square, or its boundary rather, sweeps out a surface b , called by LIU HUI a *box-lid*; let B be the volume enclosed by b . Then since the section (at level z) of $s = \pi/4$ times section of b , we also have $S = \pi/4 B$.

For convenience, at least in drawing figures, let us confine ourselves to the situation in the first octant. Then the section of B (at level z , in the first octant) is a square of area $a^2 - z^2$ (see Figure 1). Consider the cube of vertices $(\pm a, \pm a, \pm a)$; the section of it in the first octant is a square of area a^2 . Thus the one square fails to equal the other by z^2 . Now consider the pyramid p' having the origin as vertex and the square of vertices $(\pm a, \pm a, a)$ as base; and the part of this in the first octant, p , of volume P . The section of p (at level z) is z^2 . Thus $P + B/8 = a^3$. It was well known to ARCHIMEDES from his predecessors (and besides it is obvious) that $P = a^3/3$, so $B/8 = \frac{2}{3} a^3$ and $S = \frac{4}{3} \pi a^3$.

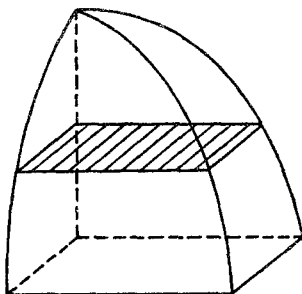


Fig. 1. Plane section of an octant of the box-lid.

This is so simple one might wonder what all the fuss, or acclaim, is about. Yet it is clear that neither ARCHIMEDES, nor anyone after him for about two thousand years, nor indeed anyone even yet regards ARCHIMEDES' achievement as a slight thing.

Let us look at the way ARCHIMEDES found the answer. He gives two treatments, one informal, the other formal. The informal one is heuristic; it leads by a plausible line of reasoning to the view that the volume is $4\pi a^3/3$. ARCHIMEDES explicitly denies that this line of reasoning is a proof, but, as he remarks, it is well to have some notion of what the answer might be before trying to establish the answer by a (rigorous) proof.

ARCHIMEDES explains his heuristic reasoning in *The Method*. The argument is CAVALIERI-like in that it employs slices viewed as infinitely thin solids, but they are *weighed* against each other. Thus not only geometrical notions, but also mechanical notions, come in.

The Method of ARCHIMEDES, though known to have existed, was long believed to have been irretrievably lost, and the story of its recovery by HEIBERG in 1906 is almost as much of a wonder as ARCHIMEDES' work itself. But let us not tarry over this by now well known episode, however engrossing, and proceed to ARCHIMEDES' explanation.

In the following figure (Figure 2) the circle $AB\Gamma\Delta$ is the section of a sphere of radius a by a plane through its center K ; the square $\phi\psi\Omega X$ has its sides tangent to the circle at A, B, Γ, Δ ; the line AA' meets $\Gamma\Omega$ in Z ; and $HZE\Delta$ is a $2a$ by $4a$ rectangle. The other circles that enter the discussion will all be in planes perpendicular to AF . In particular there will be the circle on ZE as diameter, which will be the case of a right circular cylinder of height AF , as well as the base of a right

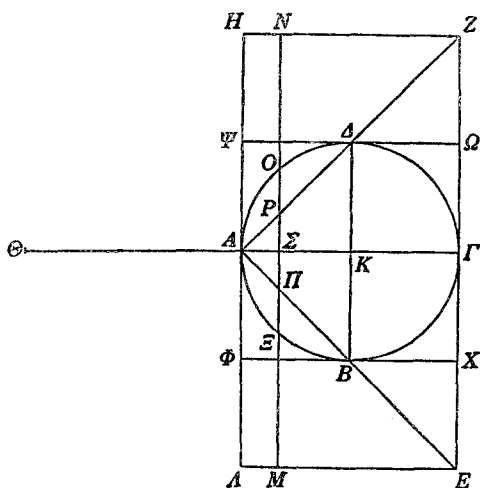


Fig. 2. *Weighing the sphere.*

circular cone of height AT ; and there will be the circles on NM , OE , and PI , which are respectively the sections by a plane of the cylinder, the sphere, and the cone. One now thinks of θAT as a lever with fulcrum at A , and with $\theta A = AT$. Since

$$\begin{aligned}(\Sigma P^2 + \Sigma O^2): \Sigma N^2 &= (\Sigma A^2 + \Sigma O^2): EN^2 = AO^2: \Sigma N^2 = (A\Sigma \cdot A\Gamma): EN^2 \\ &= (A\Sigma \cdot \Sigma N): \Sigma N^2 = A\Sigma: \Sigma N = A\Sigma: A\theta,\end{aligned}$$

the moment of the section of the cylinder where it is (*i.e.*, placed on the lever at Σ) is equal to the moment of the sections of the sphere and cone placed at θ .

Hence the cylinder, which is made up of the sections at Σ as Σ varies from A to I , will balance where it is, *i.e.*, with its center of gravity at K , with the sphere and cone placed with their centers of gravity at θ . Hence

$$(\pi(2a)^2 \cdot 2a) \cdot a = (S + \frac{1}{3} \pi(2a)^2 \cdot 2a) \cdot 2a,$$

whence S , the volume of the sphere, equals $\frac{4}{3} \pi a^3$.

This is certainly ingenious and intriguing, but if all that was wanted was the volume of a sphere, somewhat trickier than it had to be.

From $S = \frac{4}{3} \pi a^3$ ARCHIMEDES conjectured that the surface of a sphere should have area equal to 4 times the area of one of its great circles. Just as a circle may be viewed as made up of infinitely many isosceles triangles with vertex at the center and with infinitely small segments for bases, from which it follows that area = $\frac{1}{2}$ circumference times radius, so a sphere may be viewed as made up of infinitely many tetrahedra of infinitely small triangular bases and altitude a , from which $S = \frac{1}{3}$ surface area times a ; and surface area = $4\pi a^2$.

In his formal treatment, in the work *On the sphere and cylinder I*, ARCHIMEDES first takes up the surface of the sphere and then its volume. For someone interested only in the volume (and not the surface area), this makes the going tough (and furthermore the treatment somewhat impure, as it brings in unnecessary hypotheses on surface area), but since ARCHIMEDES also wanted the surface area, for organizational reasons he may have preferred to proceed as he did.

2. The Chinese on the volume of a sphere

The ancient Chinese also occupied themselves with the volume of a sphere. The *Chiu Chang Suan Shu* (*Nine Books on Arithmetic Technique*) has a problem (IV, 23–24) in which the volume V (or S) of a sphere is given and one asks for its diameter d . The rule for getting d is to multiply V by 16, divide by 9, and extract the cube root. This is just a problem on cube root, but still one learns that $V = \frac{9}{16} d^3$, or so at least it was taught.

LIU HUI, who lived in the third century A.D., realized that the 9/16 is wrong. He wrote:

The creator of the method used the proportions: circumference 3, diameter 1. If it is supposed that the area of a circle occupies $\frac{3}{4}$ of the area of a (circumscribed) square, then a cylinder also occupies $\frac{3}{4}$ of a (circumscribed) cube.

Right! Next LIU HUI conjectured that the creator of the method supposed that the sphere occupies $\frac{3}{4}$ of the circumscribed cylinder, whence the $\frac{9}{16}$ (or $\pi^2/16$ with $\pi = 3$). LIU HUI realized this was wrong and tried to set things right. He proved by a CAVALIERI-like argument that $S = \frac{\pi}{4} B$, but could not find B . Or perhaps all this was done by TSU CH'UNG-CHIH, a later commentator, who lived in the fifth century. Anyway, TSU KENG-CHIH, TSU CH'UNG-CHIH's son, finally showed by a CAVALIERI-like argument that $B = \frac{2}{3} d^3$. TSU KENG-CHIH expressed his satis-

faction in a poem:

The proportions are extremely precise
And my heart shines ...

The first notice of the *Nine Books* dates from 179 A.D., but the oldest manuscript is an edition from the middle of the third century, with a commentary by LIU HUI. Or perhaps, in accordance with some doubts of D. WAGNER [1978a] (*cf.* B. L. VAN DER WAERDEN, *Geometry and Algebra in Ancient Civilizations*, pp. 195, 206), parts of the commentary may be due to TSU CH'UNG-CHIH. LIU HUI, or the commentator, [who] says that the work was put together from older materials by CHANG TSANG (fl. 165–142) in early years of the Han dynasty (202 B.C. to 9 A.D.). The *Nine Books* was based on an earlier collection, all copies of which were ordered burned by the emperor CH'IN SHIH HUANG (221–206 B.C.).

It is clear that from such notices we do not get the date of the contents of the *Nine Books*. At best we could put the *Nine Books* back to 221 B.C., yet it is obvious that the contents might be vastly older. Indeed, the thesis that they *are* vastly older was advanced in my paper "On the Area of a Semi-Circle" [1972].

3. The relation between Pythagorean and Old-Babylonian mathematics

Let me recall the argument briefly. First, though, I must recall my position on the relation of Pythagorean to Old-Babylonian mathematics. Up till about 1930 the generally held view on the origin of mathematics (or any mathematics worthy of the name of Science) was that it all started in Greece with THALES at about 600 B.C. The theorem of PYTHAGORAS was considered to be "of PYTHAGORAS". By 1930 NEUGEBAUER was busy deciphering the Old-Babylonian cuneiform texts; already by 1928 he published evidence showing that the Old-Babylonians, of about 1700 B.C. (or better: 1800–1600 B.C.), knew the Theorem of PYTHAGORAS. In 1937 he conjectured that "what is called Pythagorean in the Greek tradition had better be called Babylonian"; and in 1943, after a cuneiform text concerning "Pythagorean number triples" was discovered, he considered that his conjecture had been validated. The notion that Greek geometry stood at the beginning collapsed. The view that it was the Old-Babylonian mathematics that stood at the beginning became dominant. That this is, or should be, the proper view is more or less spelled out by NEUGEBAUER in *Exact Sciences in Antiquity*, p. 29f. (1962).

In 1930, and even much earlier, the existence of ancient mathematical works outside of Greece was known. For example, there were the *Sulvasūtras* (roughly, "Rules of the Cord"), a Vedic manual on altar construction. In 1875 G. THIBAUT had translated a large part of the *Sulvasūtras*, and these showed that the Indian priests possessed no little mathematical knowledge. In particular they knew the Theorem of PYTHAGORAS. A polemic ensued: Did the Indians get the Theorem from the Greeks? Or did the Greeks get it from the Indians? Or was it independently invented by the Greeks and the Indians? The Greek scholars could hardly contain themselves upon hearing the suggestion that perhaps the Greeks had learned

it from the Indians. A great number of objections to the view that the Indians might have priority was put forward, and the *Sulvasūtras* were generally dismissed as a product of Alexandrian, and even Heronic, knowledge. Then in 1928 NEUGEBAUER, in connection with his disclosure that the Theorem of PYTHAGORAS existed well over a thousand years before PYTHAGORAS, mentions the *Sulvasūtras* and writes that "the difficulties [involved in the view] of a direct borrowing by the Greeks from India fall away on the assumption of a common origin in Babylonia." This is really an excellent suggestion and the only trouble with it is that many of the common elements of Greek and Indian mathematics, and especially the geometrical constructions at issue, are not found in Old-Babylonia.

There are two great traditions in the history of mathematics, the geometric or constructive and the algebraic or computational, Traditions I and II. Correspondingly there are two aspects under which the Theorem of PYTHAGORAS can be regarded: in Aspect I one views the hypotenuse of a right triangle (or diagonal of a rectangle) as generating a square whose area is the sum of the areas of the squares on the sides; in Aspect II, the diagonal is said to be the square root of the sum of the squares of the sides. In Aspect I one views the situation with a construction in mind; in Aspect II there is a computation in view. Aspect I is geometrical, Aspect II is computational.

Consider the problem: Given the sides a and b of two squares A and B . To find the side c of a square C whose area is that of A and of B together. In Greece (in *The Elements*) and in India (in *The Sulvasūtras*) this problem would be solved by constructing a right triangle ACB with sides CB , CA congruent to a , b respectively. By the Theorem of PYTHAGORAS the square on AB has the area of C ; and AB is declared to be the desired side c . No arithmetic whatsoever comes in. In Old-Babylonia a and b are regarded as numbers, a is multiplied by itself and b is multiplied by itself, the products are added, and the square root of the sum taken. This square root, $\sqrt{a^2 + b^2}$, is then declared to be the desired side c . One has to know what a square is and how to compute its area from a side, but otherwise no geometry comes in, and in particular the Theorem of PYTHAGORAS is not employed.

Greece and India know both aspects—for example, they know that a right triangle with sides 3 and 4 has hypotenuse 5. On the other hand, Old-Babylonia, which knows Aspect II, has no use for Aspect I. From these facts I argue, first, that the common source of the Theorem in Greece and in India could not be the Old-Babylonia pictured by NEUGEBAUER; and second, that the occurrences of the Theorem in Greece, India, and Old-Babylonia have a common source and that something was lost in the transmission to Babylonia. Of course, this source would be pre-Babylonian. Hence there must have been a source for Babylonian mathematics that was more like the Greek and Vedic mathematics than like the Babylonian. It had a clearer geometric component.

I did not let my notion of the relation of Pythagorean to Old-Babylonian mathematics rest on these observations alone. Rather I scrutinized the Old-Babylonian, the classical Greek, the Vedic Indian (and the Egyptian) mathematics under the hypothesis that they all derive from a *Sulvasūtra*-like source, to see what light this hypothesis might throw on the evidence; and having become convinced that it threw plenty, I considered the hypothesis validated. (Please note that when I

say that the Old-Babylonian mathematics is derivative of a *Sulvasūtra*-like mathematics, I am not saying that the Babylonian mathematics, which dates from 1700, or perhaps 1800, B.C., derives from the *Sulvasūtras*, which have never been given a date before 500 B.C.).

For example, I considered the so-called geometric algebra of the Greeks. The fundamentals of geometric algebra are given in Book II of *The Elements*, except for the Theorem of PYTHAGORAS and its converse from Book I and Theorem I, 43 on the gnomon. The *Sulvasūtras* also have geometric algebra. My attention was especially struck by the geometric algebra of the *Sulvasūtras*, because there it is associated with sacrifice. RAGLAN [1939] had put forward a theory that civilization itself had a ritual origin, and I would have liked to add to this theory inductively. Now NEUGEBAUER, and after him VAN DER WAERDEN in his *Science Awakening* (1954), explained the rise of geometric algebra in classical Greece as follows: The Greeks inherited the algebra of the Babylonians, but they discovered that $\sqrt{2}$ is not rational, or, as they put it, that the diagonal of a square and its side have no common measure. For the Greeks, number was, by definition, what we would call positive whole number, and they adhered to the definition, though they also spoke of the ratio of numbers. Thus they could not solve $x^2 = 2$ in the domain of number, even if they had allowed rational numbers or some logical equivalent. But they could solve it in the domain of geometry. This led them to translate the purely algebraic treatment of quadratic equations by the Babylonians into geometric language and thus they invented geometric algebra. All this was said without the slightest reference to the *Sulvasūtras*. And now the question arises as to how the *Sulvasūtras* got their geometric algebra. Of course, if they got it from the Greeks, fine; but what if the Vedic geometry was older than 600 B.C. in India? Here, of course, we come upon a chronological question, and such questions are notoriously difficult and delicate. Still, I concluded, at least to my own satisfaction, that the Vedic geometric algebra dates back to before 600 B.C.; and even some Greek scholars conceded that the Theorem of PYTHAGORAS occurred in India before 600 B.C. (see, for example, BECKER 1951). The fact that the geometry of the *Sulvasūtras* was closely associated with sacrifice, whereas in Greece the geometry was secular except for some philosophic overtones, also weighed in my judgment; for the contents of many ritual activities have become secular—the ritual falls away, leaving the contents to go on apart from ritual—but I could not imagine how the secular and presumably highly technical geometry of the Greeks might be taken up by the Vedic priests and made part of their most sacred activities (nor has anyone else even tried to imagine how it could have happened). Thus NEUGEBAUER's theory simply could not be maintained in the form presented.

Or consider the view expressed by NEUGEBAUER in 1962. NEUGEBAUER writes (*Exact Sciences*, p. 42): "It is easy to show that geometrical concepts play a very secondary part in Babylonian algebra, however extensively a geometrical terminology may be used," and on p. 45, "The mathematical importance of a problem lies in its arithmetical solution; 'geometry' is only one among many subjects of daily life to which the mathematical procedures may be applied." This view was shared by VAN DER WAERDEN in the first edition of his *Science Awakening I*, although already by the time of the second edition there was some change (following remarks by P. HUBER on the Old-Babylonian text VAT 8512).

To me it seemed that such an evaluation could not be accurate: if the Old-Babylonians knew the Theorem of PYTHAGORAS, then someone, perhaps a predecessor, could prove the Theorem, and this required a greater insight into geometry than any that comes from the activities of daily life. The same could be said for their knowledge of a formula for the area of the trapezoid and of their correct understanding of the relations between the area, diameter, and circumference of a circle, and of other geometrical phenomena. I do think that there is some truth in NEUGEBAUER's statements, but that they miss a key point, namely, that Old-Babylonian mathematics is derivative of a mathematics having a much clearer geometric component. The Old-Babylonians inherited this older mathematics, but by the Old-Babylonian period, the interest in exact geometry was lost and at best we remain with computational problems having a geometrical background.

I should add that other writers, too, objected to getting the Pythagorean geometric algebra out of the Babylonian algebra (*cf.* MAHONEY 1971 and UNGURU 1975), but at the same time they denied the genetic relatedness of Pythagorean and Babylonian mathematics. There was the tacit assumption that the Pythagorean (or Pythagorean-like) developments were subsequent to the Old-Babylonian—at least the *a priori* possibility of their precedence was never considered. And there was (I presume) an appeal to the notion of independent invention.

4. Comparison of Chinese and Old-Babylonian mathematics

Let us return to the dating of the contents of the *Chiu Chang Suan Shu*. We had left the *Nine Books* at 221 B.C.

Let A and C be the area and the circumference of a circle of radius r or diameter d . Then $A = \frac{\pi_1}{4} d^2$ and $C = \pi_2 d$, where $\pi_1/4$ is the ratio of the area of a circle to that of its circumscribed square and π_2 is the ratio of the circumference of a circle to its diameter. Now we know that $\pi_1 = \pi_2$, but the question is whether the ancients knew it. Frequently these two constants (which are equal) are denoted simply by π , thus slurring the question of their equality and debasing the discussion where area, diameter, and circumference come in.

In 1962 I wanted to show that the ancients knew that $\pi_1 = \pi_2$. Of course, saying $\pi_1 = \pi_2$ here is just a shorthand way of saying that the ancients had a correct understanding of the relations of the area, circumference, and diameter of a circle, and not literally that they considered π_1 and π_2 , much less their equality. Or, still more explicitly, that they viewed the circle as made up of a large number of small isosceles triangles having the center of the circle as vertex and two radii as sides. I argued this way because I could not see then, any more than I can now, how else they might come to the supposition that $\pi_1 = \pi_2$. I am presuming that the Weierstrassian apparatus was still not known to the ancients by 1700 B.C.

I wanted to show that the ancients realized that $\pi_1 = \pi_2$ in order to counter the notion held by VAN DER WAERDEN at that time that Egyptian geometry (and, tacitly, the Babylonian too) was “merely applied arithmetic”, for no-one would claim that $\pi_1 = \pi_2$ would result from an application of mere arithmetic.

In arguing against VAN DER WAERDEN, I nevertheless appealed to some of his remarks on Problem 10 of the Moscow Mathematical Papyrus, which dates from the Middle Kingdom. In 1930 STRUVE published a translation of this papyrus and astounded the world by declaring that in MMP 10 the Egyptians had computed the area of a hemisphere as twice the area of its opening. Eventually T. E. PEET [1931] and NEUGEBAUER expressed skepticism. VAN DER WAERDEN gives an explanation of PEET and also one of NEUGEBAUER. According to PEET's explanation, what is being calculated is the area of a semi-cylindrical surface, whereas NEUGEBAUER thinks that an approximation to the area of a dome-like Egyptian basket is being computed. VAN DER WAERDEN prefers PEET's explanation. Now this explanation, simple though it be, involves the assumption $\pi_1 = \pi_2$ (as does NEUGEBAUER's). Thus I concluded that Egyptian geometry could not be "merely applied arithmetic."

As a *reductio ad absurdum* I can find no fault with this argument, but by 1972 I began to doubt the explanations of PEET and NEUGEBAUER mentioned. Instead I began to think that the area being calculated in MMP 10 was that of a semi-circle. Indeed, PEET had already suggested this, though he lost the way at a couple of points. Curiously, this explanation, though also occurring in the same paper of PEET to which VAN DER WAERDEN refers, has fallen out of his book *Science Awakening*. In my paper "On the Area of a Semi-Circle," I show that the area being computed is that of a semi-circle, and moreover that the scribe is following the prescription: area = semi-circumference times radius with $\pi_2 = 4 \cdot (8/9)^2$, and I also had $\pi_1 = 4 \cdot (8/9)^2$ from the Rhind mathematical papyrus, from which I argued that the Egyptians (of the Middle Kingdom) realized that $\pi_1 = \pi_2$.

In my first paper in this *Archive*, in 1962, as well as in the subsequent ones, I insisted on the importance of the comparative method for the history of mathematics. In particular I wanted to know whether the Old-Babylonian also knew the relations between the area, circumference, and diameter of a circle. I found this harder to establish than I had thought. In 1962 I was content to write:

The Babylonians used the formulae $A = C^2/12$ and $C = 3d$, from which it would appear that they [also] knew that $\pi_1 = \pi_2$, though here it is a bit more difficult to judge.

Let me explicate this. Let us introduce π'_1 and π'_2 for approximations to π_1 and π_2 . Then from $\pi'_1 = \pi'_2$ we cannot conclude $\pi_1 = \pi_2$. If π'_1, π'_2 are complicated expressions, even fractions, the conclusion $\pi_1 = \pi_2$ is plausible, but if π'_1, π'_2 are integers, the conclusion begins to lack plausibility: one has to face the possibility that π'_1, π'_2 are both but crude integral approximations independently arrived at. This is precisely the situation with $A = \frac{1}{12} C^2$ and $C = 3d$, from which $\pi'_1 = 3$ and $\pi'_2 = 3$. By 1972 I wanted to press this point further.

J. E. M. SMEUR has insisted precisely on this difficulty. In his work "On the Value Equivalent to π in Ancient mathematical Texts, a New Interpretation," which appeared in this *Archive*, vol. 6 (1972), he puts forward the thesis that the relation $\pi_1 = \pi_2$ is Archimedean, and in particular that the equality of π'_1 and π'_2 amongst the Old-Babylonians is mere accident and signifies nothing. In his own words (p. 264):

Thus we can be sure the Babylonians were not familiar with a formula like $A = \pi R^2$. We have to admit that the Babylonians, surely at least in the beginning, were not aware of any relation between the numbers 3, 5', and 15' [*i.e.*, $3, 5/60 = 1/12$ and $15/60 = 1/4$], and certainly not that those numbers were connected by one and the same factor of proportionality, our number π .

While I considered the difficulties posed by SMEUR as *bona fide*, still I was skeptical. What, I wondered, could the $1/12$ in the $1/12 C^2$ be a crude approximation of? Could it have been the result of an experiment? Since I know not even a shred of evidence that geometry was ever an experimental science, I was willing provisionally to put this possibility aside. The only other possibility that I could see was that $A = 1/12 C^2$ was the transform of a relation standing closer to the intuition. We have

$$C = 3d \quad (1)$$

and

$$A = \frac{1}{12} C^2. \quad (2)$$

These are given. Replacing one of the factors C by $3d$, we get

$$A = \frac{C d}{4}, \quad (3)$$

or

$$A = \frac{C}{2} \times \frac{d}{2}; \quad (3')$$

and replacing both factors C by $3d$ we get

$$A = \frac{3}{4} d^2. \quad (4)$$

Conversely, from (1) and (3) or (3'), we can get (2); and likewise from (1) and (4) we can get (2). Both (3) (or (3')) and (4) stand closer to the intuition than (2), though to see (3) or (3') would require considerable sophistication; the $3/4$ of (4) might, conceivably, have resulted from a crude comparison of a circle to its circumscribed square. Thus (2) was presumably the transform of (3) (or (3')) or of (4). I inclined to the implication $(3) \Rightarrow (2)$; still I did not see how to exclude $(4) \Rightarrow (2)$.

Now, however, a further, crucial, piece of information came to my attention. In the Old-Babylonian text BM 85120 (Rs. I, 18) the area of a semi-circle is computed according to the prescription $\text{arc} \times \text{diameter}/4$ (*cf.* my paper 1972, p. 185, n. 26). This is explicitly what I wanted and confirmed my guess that $A = C^2/12$ is the transform of $A = Cr/2$ (and not of $A = \pi_1/4d^2$). Thus I had proved that for the Babylonians, too, $\pi_1 = \pi_2$. But I had gotten through only by the skin of my teeth. The comparison of the Chinese and Old-Babylonian mathematics will support this conclusion.

In 1962, although emphasizing the importance of comparative studies, I did not take up the Chinese mathematics. I explain why in 1972. Anyway, by 1972

I had K. VOGEL's translation of the *Nine Books* available, and I began a comparison of the Chinese and Old-Babylonian mathematics, in the section on *The area of a semi-circle in Babylonian and China* in my semi-circle paper. We had better go into this, as it will be important to have the facts, or a goodly number of them, and the argument in view.

I began by giving briefly the contents of the *Nine Books* seriatim and each time commenting on whether the Old-Babylonians had the corresponding result. I was especially concerned with the geometric parts, though I also commented sometimes on the arithmetic and algebraic parts. For example, I noted that "several problems in Book VI deal with arithmetical progressions", and remark that "the Egyptians and Babylonians have similar problems." Likewise I noted that "Book VIII deals with simultaneous linear equations," and "The Old-Babylonians could also handle simultaneous linear equations." In Book I it is explained how to work with fractions m/n . I made the remark that the Euclidean algorithm was used to reduce a fraction m/n to lowest terms, but added no remark for Old-Babylonia. What I should have said is that Old-Babylonia *does not* have the Euclidean algorithm.

Although I was mainly interested in the geometry, still I had to pay some attention to the arithmetic, as the geometry is always presented, in China as in Babylonia, in computational form.

Book I starts with *the area of a rectangular field*. Problem 1 reads: "Now one has a field; it is 15 steps wide and 16 steps long. The question is: How large is the field?" The answer (= 1 Mou) is given; and a second problem of a similar kind is posed and the answer given. Then the general rule is stated. With a couple of minor exceptions this is the format used throughout the work: one or two problems of some type are posed, the answers given, and the general rule stated. This is different, in the main, from the Babylonian procedure, where the problems are stated and *worked out*, but the general rule is not given (though there are a few instances of general statements). Book I continues with arithmetical problems (addition, subtraction, etc., of fractions m/n), returning to geometry with Problem 25, which asks for *the area of a triangle*. Then comes *the trapezoid*. With Problem 31 we come to *the circle*: "Now one has a round field; the circumference is 30 steps, the diameter 10 steps. The question is: How large is the field? The answer says: 75 Pu." Clearly the ratio of circumference to diameter is taken to be 3, though curiously no problem requires this as prior knowledge, and throughout superfluous information is supplied. The value 3 is typically Babylonian, but the Babylonian scribe needs to know this in working his problems. The rule (in the *Nine Books*) for *the area of a circle* is to multiply one-half the circumference by one-half the diameter; three further rules are given, the third of which says to square the circumference and divide by 12—this is the Babylonian procedure. Then the *sector* is considered. Then come a couple of problems on the *segment*: given the chord s and the "arrow" p (= distance from midpoint of chord to midpoint of arc), the rule is: $\text{area} = (sp + p^2)/2$, and so appears to approximate the segment with a trapezoid of bases s and p and width p . I cannot say that the Old-Babylonians have this problem, but HERON, who is sometimes considered to be continuing the Babylonian tradition, has it. His solution is $\frac{1}{2}(s + p)p + \frac{1}{14}(\frac{1}{2}s)^2$, where the correction term comes from the Archimedean value $22/7$ for π ; he mentions that

the “ancients” took $\frac{1}{2}(s+p)p$ and even conjectured that they did so because they took $\pi = 3$. Book I ends with a problem on the area of a *ring* (= circle minus concentric circle); the Babylonians also have ring problems.

Book IV poses the problem: Given the area of a rectangular field and its width, what is its length? If the area is given via a rectangle, this problem can be solved by *Elements* I, 44, but here, of course, is solved by a simple division. The first 11 problems are on this. Problem 12 asks for the side of a square of 55225 Pu (*Ans.* 235 steps). Thus the square root comes in. The work is in the decimal system, not, as in Babylonia, in the 60-system. In problem 17 one is given the area of a circle, to find its circumference—again a problem in square root. Then comes cube root, and first to find the edge of a cube of given volume. Book IV ends with the problem on the sphere mentioned above, to find the diameter of a sphere of given volume. I commented:

The answer is wrong, of course, but what [I] find surprising is that the problem was set up at all. We have no corresponding problem from Old-Babylonia (or from Egypt of about the same time, *i.e.*, of the Middle Kingdom).

Book V returns to geometry. Here volume is taken up. Problem 9, for example, computes the volume of a *cylinder* ($V = C^2h/12$). Problem 10 gives the rule for a *truncated pyramid of square base*, problem 11 considers a *truncated cone*; problem 12, a *square based pyramid*; problem 13, a *circular cone* ($V = C^2h/36$); problem 14, a *prism*; problem 15, an *oblong based pyramid*; problem 16, a *tetrahedron*; problem 17, a *wedge* having two trapezoidal faces at right angles; problem 18, a special case of the next problem; problem 19, a *truncated pyramid-like body* having rectangular but dissimilar bottom and top.

Thus the *Nine Books* know the basic facts about pyramids. The Egyptians of the Middle Kingdom had a correct formula for the truncated pyramid, and I think the Old-Babylonians did, too, though the evidence is not as clear as one might wish; I'll come back to this. HERON considers a pyramid-like body as described in Book V, Problem 19, but the formula is different (*cf.* T. L. HEATH, *Greek Mathematics*, vol I, p. 332f.).

Book IX is geometric, treating the right triangle, especially problems involving the Theorem of PYTHAGORAS; the Theorem of THALES, that an angle inscribed in a semi-circle is right, also comes in. In the course of this, a familiarity with Pythagorean number triples is disclosed. All this is familiar ground for the Old-Babylonians.

These few remarks already show that the Chinese mathematics of the *Nine Books* and the Old-Babylonian overlap to a considerable extent, and even that the characters of the two mathematics are largely the same. These conclusions are reinforced upon examination in detail of the problems in Book IX.

VAN DER WAERDEN has done this in his book [1983]. To avoid overloading the paper with detail, I skip reporting on this, though doing so would be very useful in comparing the algebra of the two mathematics.

For another discussion of Book IX, see MATHEWS [1985]. MATHEWS' thesis is that the geometry underlying Book IX just about coincides with that of *The Elements*, Book II minus II 11–13 plus I 43 (the theorem on the gnomon) and I

47 (the Theorem of PYTHAGORAS). It might be helpful to call the gnomon of I 43 the *multiplicative gnomon* and the gnomon of II 4 the *additive gnomon* (cf. SEIDENBERG 1962, p. 519, n. 80).

5. Chronology of the Old-Babylonian, Chinese-like and Vedic-like mathematics

In my *Area of a Semi-Circle* (1972) I wrote (p. 186f.):

Our considerations on Babylonia have depended, so far, on Babylonian material only. Now we may compare with China. As has been made evident, the *Nine Books* have a Babylonian look—in saying this we do not intend to say that Chinese mathematics is a derivative of the Babylonian, or vice-versa, but merely that they have a common source. Now in the *Nine Books* we find explicitly what has to be reconstructed for Babylonia, namely, the formula $A = C/2$ times $d/2$... Although the *Nine Books* proceed from the simpler to the more difficult, we cannot be sure that the development was intended to be logical. Still, we note that in the section dealing with the area of a circle, the formula $A = C/2$ times $d/2$ comes first; and one might be tempted, then, to see the others as derivative of this and $C = 3d$. This does not imply that $A = C^2/12$ was not the favorite formula; and there is some reason to think it was, since in the rules for a circular cylinder and circular cone the rule $A = C^2/12$ is followed. Thus if the Chinese had worked out their examples, and their rules had been lost, we would probably see exactly what we see in Babylonia. And what we see in Babylonia we explain in precisely this way: the rules are lost, but the favorite formula remains clear.

In other words I was saying that in some respect the Old-Babylonian mathematics is derivative of a Chinese-like mathematics. This Chinese-like mathematics would, of course, have to be pre-Babylonian. In my *Origin of Mathematics* (1978) I argued, though without reference to the Chinese mathematics, that the Old-Babylonian mathematics had a predecessor having a clearer geometric component.

VAN DER WAERDEN [1983] has extended my comparisons of 1972 and 1978, coming also to the conclusion that the *Nine Books* and the Babylonian texts derive from a common source, and he adds that *the mathematics of the Nine Chapters represents the common source more faithfully than the Old-Babylonian does*. Because of this vivid and forceful statement one may think of Old-Babylonian mathematics as deriving from a Chinese-like mathematics. Note, though, that VAN DER WAERDEN is no more saying that Old-Babylonian mathematics derives from the *Nine Books* than I am saying that it derives from the *Sulvasūtras*.

As I see it, the Old-Babylonian mathematics started from a Chinese-like mathematics, but by the Old-Babylonian period the interest in exact geometry had faded and had shifted to the algebra, to the computation with numbers. The geometry is to a large extent still there, though not in as clear a form. The algebra has become more extensive and richer, and has also changed character. In the Chinese-like mathematics, the numbers are always the measures of geometric

(or related) objects, whereas in Babylonia the connection is not insisted upon. Thus the Old-Babylonian could add lengths to areas and even men to days, and they could multiply areas: algebra is being freed from its geometric roots. Let me illustrate with one example, a problem from the Old-Babylonian text BM 13901: given $x^2 + y^2 = 21,40$, $xy = 10,0$. The text squares the second equation, reducing the system to the type $u + v = S$, $uv = F$ (with $u = x^2$, $v = y^2$), then gets u and v (using the identity $uv = \left(\frac{u+v}{2}\right)^2 - \left(\frac{u-v}{2}\right)^2$). A Chinese-like mathematics would do no such thing. Rather it might compute $(x+y)^2$ and $(x-y)^2$, extract the square roots $x+y$ and $x-y$, and then solve for x, y . If x, y are understood to be measures of length, then the numbers computed in the second solution are also measures of geometric objects. (The second method of solution is actually found in a Demotic text dating from the first century B.C.; cf. VAN DER WAERDEN, *Geometry and Algebra*, p. 169f.)

The view that the Old-Babylonian mathematics derives from a Chinese-like mathematics is quite compatible with the view that it derives from a *Sulvasūtra*-like mathematics. In my paper [1978, p. 329], I argued that the Vedic and Old-Babylonian mathematics had a common source preceding the separation of mathematics into Traditions I and II. At the source there was no dichotomy between number and magnitude. A split occurred, one side insisting on the exactness of constructions, leading to Tradition I and Pythagorean mathematics. The other expanded the arithmetic element in the source, especially upon discovery (or invention) of square root, leading to Tradition II and Old-Babylonian mathematics. I would argue in just the same way now relative to the Vedic and Chinese mathematics. It is merely that the Chinese-like mathematics stands closer to the common source than the Old-Babylonian.

Of course, the general characterization of these three mathematics and their overall ordering cannot tell us when a particular item entered the development. Thus in 1972 (*loc. cit.*, p. 184) I wrote: "If one were to find on some newly recovered cuneiform tablet any geometric problem occurring in the *Nine Books* (except possibly the one on the sphere) no-one would be in the least surprised." I was especially taken by the problem on the sphere: where, or rather when, I wondered, did problem IV 24 on the sphere enter into the development of the Chinese-like mathematics? This is the question I will try to answer below.

6. How far back do infinitesimals go ?

A knowledge that $\pi_1 = \pi_2$ requires the use of infinitesimals, and the *Nine Books*, the Old-Babylonians, and the Egyptians of the Middle Kingdom knew that $\pi_1 = \pi_2$, so infinitesimals go back at least to 1700 B.C.

Now, however, I would still like to consider infinitesimals of the kind occurring in CAVALIERI-like arguments, and first let us go to Greece and to the tetrahedron. One might convince oneself by a CAVALIERI-like argument that two tetrahedra of equal height and having bases of equal area have equal volume (from which with *The Elements* XII 7, which says that any prism with triangular base can be divided into three tetrahedra of equal volume, one gets $V = \frac{1}{3} \text{ base} \times \text{height}$),

but this, of course, is not the proof found in *The Elements*, Book XII, which meets the standards of modern rigor (though one can envision a CAVALIERI-like argument behind the rigorous proof). The proof has been ascribed to EUDOXUS (c. 370 B.C.). Now already eighty years earlier DEMOCRITUS knew about infinitesimals, for according to PLUTARCH, DEMOCRITUS raised the following question: "If a cone be cut by surfaces parallel to the base, then how are the sections, equal or unequal? If they were unequal, the cone would have the shape of a staircase, but if they were equal, then all sections will be equal, and the cone will look like a cylinder, made up of equal circles; but this is entirely nonsensical" (cf. VAN DER WAERDEN, *Science Awakening*, p. 137f.). Obviously, DEMOCRITUS was becoming critical of CAVALIERI-like (or *Method*-like) arguments. One might then conjecture that DEMOCRITUS used infinitesimals to get the volume of a tetrahedron. But one hardly has to do that, for ARCHIMEDES tells us as much in *The Method*: for having first said in *The Sphere and Cylinder I* that no-one before EUDOXUS had "observed" the formula for the volume of a tetrahedron, in *The Method* he corrects himself, saying that DEMOCRITUS "was the first to make the assertion ... though he did not prove it."

Now let us go to China. LIU HUI in his commentary on the *Nine Books* tells us how to get the formula $V = \frac{1}{3} \text{ base} \times \text{height}$ for the volume of a pyramid. His argument is not quite the same as EUCLID'S (and not quite as rigorous), but it is more EUDOXUS-like than CAVALIERI-like. Still LIU HUI did know infinitesimals, since he got the formula $S = \frac{\pi}{4} B$ comparing sphere and box-lid by a CAVALIERI-like argument (cf. VAN DER WAERDEN, *Geometry and Algebra*, p. 200ff.).

Recall that the Egyptians of the Middle Kingdom had a correct formula for the volume of a truncated pyramid ($V = (a^2 + ab + b^2) \cdot h/3$), and (as mentioned) I think the Old-Babylonians did too, though VAN DER WAERDEN holds the opposite opinion (cf. *Science Awakening*, p. 75f. and *Geometry and Algebra*, p. 44). Let us start by comparing Book V of the *Nine Chapters* with the Old-Babylonian text BM 85194 (cf. NEUGEBAUER, *Mathematische Keilschrifttexte I*, pp. 142–193). Both documents begin with problems about volumes of dams and walls and the number of workmen needed to build them. In many of the problems, both Chinese and Babylonian, the cross-section of the dam or wall is a trapezoid and the volume is calculated as length times area of cross-section. In a comparative study the arbitrary elements in a phenomenon may be of more importance than the substantial or logical elements themselves, for polemical purposes anyway, as it might be argued that the logical results are due to logical constraints, whereas this could not be said for the arbitrary elements, in our case the dams and the walls, etc. One might argue that the volume is computed as area of cross-section times length because that *is* the right way of doing it, but one will hardly be able to account for the other coincidences without assuming a common source.

Now, however, BM 85194 calculates the volume of a truncated circular cone of height $h = 6$ with lower circumference $u = 4$ and upper circumference $v = 2$. The areas of these circles are computed, as usual, as $A = u^2/12$ and $B = v^2/12$; and then the volume as $V = \frac{1}{2} (A + B) h$, which is wrong. Perhaps the Old-Babylonian wrote down $\frac{A + B}{2} \cdot h$ by false analogy with a formula for the

area of a trapezoid. On the other hand, Book V computes the volume as $V = \frac{u^2 + uv + v^2}{36} \cdot h$, which, modulo the value of π , is correct. A correct idea may, of course, go wrong in the course of transmission, though one will hardly suppose that an incorrect one would thus straighten itself out. It looks as if the common source had the correct idea.

Moreover in the same text BM 85194, the volume of a truncated pyramid of given height 18 and square bases of sides $a = 10$ and $b = 7$ is worked out as follows: the text begins by computing $\left(\frac{a+b}{2}\right)^2 = 1, 12; 15$ and $a - b = 3$. From this by an operation which is not entirely clear the text comes to a number 45, "which might be interpreted as 0; 45" (i.e., $\frac{3}{4}$). NEUGEBAUER has explained the 0; 45 as coming from $\frac{1}{3}\left(\frac{a-b}{2}\right)^2$; and, indeed, $\frac{1}{3}\left(\frac{10-7}{2}\right)^2 = 0;45$. Thus according to NEUGEBAUER the calculation is based on

$$V = \left\{ \left(\frac{a+b}{2} \right)^2 + \frac{1}{3} \left(\frac{a-b}{2} \right)^2 \right\} h,$$

which is a correct formula. THUREAU-DANGIN considers the 0; 45 to be $\frac{a-b}{4}$, so that according to him the calculation is based on

$$V = \left\{ \left(\frac{a+b}{2} \right)^2 + \left(\frac{a-b}{2} \right) \right\} h.$$

VAN DER WAERDEN has still another explanation. First he finds difficulties in NEUGEBAUER's explanation by pointing to the "closely related" texts BM 85196 and BM 85210 in which the wrong formula

$$V = \frac{1}{2}(a^2 + b^2)h \quad (3)$$

is used (not to mention the formula for a truncated cone in BM 85194 itself). And he notes, following NEUGEBAUER, that the "space in the text which precedes the number 45, and which contains a few illegible symbols, is too small for the calculation of $\frac{1}{3}\left(\frac{a-b}{2}\right)^2$." "Both difficulties", he continues, "disappear if we sup-

pose the 0; 45 is an error of calculation and should be replaced by $\left(\frac{a-b}{2}\right)^2 = 2;15$. This would mean that the work is based on the formula

$$V = \left\{ \left(\frac{a+b}{2} \right)^2 + \left(\frac{a-b}{2} \right)^2 \right\} h, \quad (4)$$

which is indeed wrong but agrees with (3)."

Thus for VAN DER WAERDEN the obscure term is $\left(\frac{a-b}{2}\right)^2$. VAN DER WAERDEN calls THUREAU-DANGIN's explanation "nonsensical" since it isn't even dimensionally

correct. His own explanation is, of course, not nonsensical in this sense. Since, however, I can see no motive for going over from the program $\frac{a^2 + b^2}{2}$ to the program $\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2$, I think his explanation is not much better. And his assumption that the 0;45, which is correct, is a miscalculation for $\left(\frac{a-b}{2}\right)^2$ depends on the presumption that the obscure term was $\left(\frac{a-b}{2}\right)^2$, which was to be proved. I go along with NEUGEBAUER and hold that the Old-Babylonians had the correct formula $V = \left\{ \left(\frac{a+b}{2}\right)^2 + \frac{1}{3} \left(\frac{a-b}{2}\right)^2 \right\} h$ for a truncated pyramid.

How did the Old-Babylonians come to the formula $V = \left\{ \left(\frac{a+b}{2}\right)^2 + \frac{1}{3} \left(\frac{a-b}{2}\right)^2 \right\} h$, which is not the same as the Chinese formula $C = \frac{1}{3} (a^2 + ab + b^2) h$? LIU HUI in his commentary on the *Nine Books* explains the Chinese formula as follows (see Figure 3), assuming as already known the formula $V = \frac{1}{3} a^2 h$ for a full pyramid:

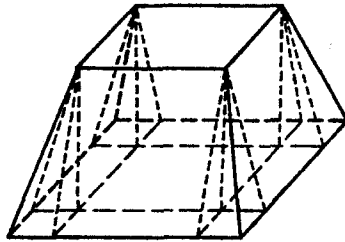


Fig. 3. Division of a truncated pyramid.

One divides the truncated pyramid by vertical planes as indicated. The parts are: one central block, four wedges, and four pyramids. LIU HUI assumes that the top base is symmetrically placed relative to the bottom base, though the reduction of the general case to this case would be easy; he also deals only with the case $h = 1$, $a = 3$, $b = 1$. Now LIU HUI brings the western wedge over to the north, inverting it, and the eastern wedge over to the south, inverting it, and thus gets a block of volume abh . With a slight variation on his argument, one now gets that $V = abh + \frac{1}{3} (a-b)^2 h$ and $V = \frac{1}{3} (a^2 + ab + b^2) h$. Now instead of bringing the western wedge to the north, let us bring it to the east, inverting it; and the northern wedge to the south, inverting it. Then instead of a block we get a prism built on a gnomon, whose volume, in conformity with Chinese and Old-Babylonian thought, is $\left\{ \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 \right\} h$; and adding the volume $\frac{4}{3} \left(\frac{a-b}{2}\right)^2$ of the four pyramids, we get $V = \left\{ \left(\frac{a+b}{2}\right)^2 + \frac{1}{3} \left(\frac{a-b}{2}\right)^2 \right\} h$. Thus it looks as though the Old-Babylonians (or a predecessor) knew this argument. Now the *Nine Books* also know the volume of a truncated pyramid-like body having rectangular but

dissimilar bottom and top. With a symmetry assumption of the kind mentioned (and assuming that the top face can be orthogonally projected into the bottom), the second of the above arguments applies, but the first doesn't. Hence I suppose the Chinese-like mathematics knew the second argument (or something like it). I conclude that the common source of Old-Babylonian and Chinese-like mathematics already knew the volume of a pyramid.

The problem remains of how the Old-Babylonians got the volume ($= \frac{1}{3}$ base times altitude) of a tetrahedron. Now in a famous problem HILBERT posed the question of whether on the basis of his axioms for geometry one could find the volume of a tetrahedron by a finitistic argument; or, somewhat more exactly put (though not entirely so), whether two tetrahedra having the same altitude and bases of equal area can be cut up each into the same (finite) number of polyhedra which shall be two by two congruent. For an exact formulation see HILBERT [1902]. Shortly afterwards M. DEHN [1902] showed that the answer is negative. On the basis of this NEUGEBAUER [1934, p. 126] argued that the ancients of about 1800 B.C. already employed infinite processes. Strictly speaking this argument is wrong, or incomplete, since I can show by a simple argument accessible to the Old-Babylonians, and based on plausible hypotheses and not using infinite processes, how to get the formula $V = \frac{1}{3}$ base times altitude. Still, wherever we see the ancients getting this formula, in Greece and in China, they do use infinite processes, so it looks as though this was always so anciently. In China, in the third century A.D., the argument is Eudoxian, but even in Greece the Eudoxian argument doesn't show up before the fourth century B.C., so it is plausible to suppose that LIU HUI's argument is derivative of a CAVALIERI-like argument. My conclusion is that the common source of Chinese-like and Old-Babylonian mathematics had the volume of a tetrahedron (or pyramid) by a CAVALIERI-like argument, and hence that infinitesimals of such arguments go back to a pre-Babylonian source.

7. How old is the problem on the volume of a sphere?

As we have seen, the *Nine Books* teaches that the volume V of a sphere of diameter d is given by $V = \frac{9}{16} d^3$. The circumscribed right circular cylinder of height d , which I will call a can, has volume $\frac{3}{4} d^2 \cdot d$ (i.e., $\frac{\pi}{4} d^2 \cdot d$ with $\pi = 3$), so we would get the $\frac{9}{16}$ if we could explain why the sphere was considered to have volume $= \frac{3}{4}$ of the can. In 1972 (*loc. cit.*, p. 182, n. 20) I conjectured the following. Let a plane be taken through the axis of the cylinder and rotated about the axis. Then the plane cuts the sphere in a great circle and the can in a circumscribed square.

The circle is $\frac{\pi}{4}$ of the square, so it might have been thought that the sphere is $\frac{\pi}{4}$ of the can. This is wrong, of course, but it at least makes the $\frac{9}{16}$ intelligible. It also indicates that the volume was obtained by an erroneous use of infinitesimals.

LIU HUI in his commentary on the *Nine Books* mentions the can; and thinks that the author of the formula $V = \frac{9}{16} d^3$ held that sphere: can $= \pi:4$ (or $3:4$).

In this, says LIU HUI, he was wrong, though whether LIU HUI explained the $\frac{3}{4}$ as I did, I cannot say. He does not have it in the relevant part of his commentary; see WAGNER [1978b]. Anyway, the question remains as to how LIU HUI saw that

$V = \frac{\pi}{4} C_a$ and $V = \frac{9}{16} d^3$ are wrong, where C_a is the volume of the can. We saw

that LIU HUI knew that $V = \frac{\pi}{4} B$, where B is the volume of the box-lid. Now

$B < C_a$ since the box-lid is the intersection of two distinct congruent cans, so

$V = \frac{\pi}{4} B < \frac{\pi}{4} C_a = \frac{\pi^2}{16} d^3$. Probably the foregoing is LIU HUI's explanation,

and in any event LIU HUI brings in the box-lid in explaining that $V = \frac{9}{16} d^3$ is wrong (cf. VAN DER WAERDEN, *Geometry and Algebra*, p. 204f.).

LIU HUI remarks that the $\frac{9}{16}$ is about right: as we have just seen, $\pi^2/16$ is too large; on the other hand 3 (for π) is too small; so the two errors partly cancel each other. However, LIU HUI adds, the $\frac{9}{16}$ is still too large. How he would have seen this unless he knew more about the sphere than comes out from the foregoing I cannot say. (This observation may fit in with WAGNER's doubts about the authorship of the commentary.)

The box-lid occurs in ARCHIMEDES' *Method*. A. YOUSCHKEVITCH reminded VAN DER WAERDEN of this, who adds (*Geometry and Algebra*, p. 205): "Youschkevitch feels that this fact furnishes a strong argument in favour of the hypothesis of a Greek influence on Liu Hui." Notice that VAN DER WAERDEN does not say he agrees; nor does he say that he disagrees. Apparently VAN DER WAERDEN is applying "criteria of quantity and quality." A few pages earlier he was comparing EUCLID's proof of *The Elements*, XII 5, and LIU HUI's development of a formula for the volume of a pyramid (both of which come essentially to saying that two pyramids of the same height and having bases of equal area have equal volume). He notes that both, as one says, *exhaust* the pyramid with prisms, that both use the same division of a tetrahedron into two prisms and two smaller tetrahedra, and in a word that both use the method "of EUDOXUS". But still, he adds, these resemblances are not sufficient to conclude that LIU HUI was influenced by Greek sources. He seeks more evidence, and considers that he finds it in LIU HUI's approximation to π .

In his commentary on the *Nine Books* LIU HUI starts with a regular inscribed hexagon, as does ARCHIMEDES in his *Measurement of a Circle*. Doubling the number of sides, LIU HUI considers a regularly inscribed dodecagon; so does ARCHIMEDES. Then LIU HUI considers successively regular inscribed polygons of 24, 48, and 96 sides, as does ARCHIMEDES. In Proposition 1 ARCHIMEDES proved that $\pi_1 = \pi_2$; and LIU HUI also proves that $\pi_1 = \pi_2$. ARCHIMEDES gets bounds above and below π ; so does LIU HUI. LIU HUI computes the perimeter C_{96} and area A_{96} of the regular inscribed 96-gon. ARCHIMEDES computes C_{96} , though not A_{96} .

There are also some striking differences in the treatments of ARCHIMEDES and LIU HUI. ARCHIMEDES is really computing π_2 , and except in Proposition 1 computes no area; whereas LIU HUI computes, rather, π_1 . ARCHIMEDES also considers circumscribed regular polygons; and shows that the perimeter of the regular circumscribed 96-gon is less than $3\frac{1}{7}d$, and that C_{96} is greater than

$3\frac{10}{71}d$; so, in sum, $3\frac{10}{71} < \pi < 3\frac{1}{7}$. LIU HUI on the other hand computes A_{96} and C_{96} , and then using the relation $A_{2n} = \frac{1}{2} r C_n$, where A_m and C_m are the area and perimeter of a regular inscribed m -gon, computes A_{192} . He observes that the area of the 96 smaller segments having a side of the regular 96-gon as chord is less than $2(A_{192} - A_{96})$, see Figure 4, so $A_{192} < A < A_{96} + 2(A_{192} - A_{96})$; and, in sum, that $3.141024 < \pi < 3.142704$, whence the approximation 3.14 (or, more exactly said, $314/100$, since LIU HUI did not have decimal fractions).

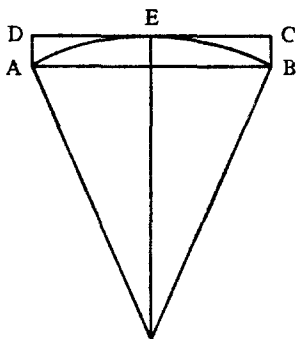


Fig. 4. LIU HUI's upper bound.

However, VAN DER WAERDEN considers that LIU HUI was not only following ARCHIMEDES but also APOLLONIUS (fl. c. 210 B.C.). APOLLONIUS is known to have written a work *Rapid Delivery*, now lost. HERON (fl. c. 62 A.D.), however, in his *Metrica* has a *rapid* method for obtaining an approximation to π , and VAN DER WAERDEN explains why he thinks this was due to APOLLONIUS. The idea is that, having computed A_n and a side of the regular inscribed n -gon, AB in Figure 4, one will add n times an approximation of the associated segment, AEB in Figure 4, to get A approximately. Now one can pass a parabola with vertex at E through A and B (and only one), and according to a theorem of ARCHIMEDES, the segment of the parabola, has area $= \frac{4}{3}$ of the area of triangle AEB in Figure 4. For AB small in comparison with the radius, the parabola fits the circle along EA quite well. In this way one gets a close approximation to the area of the circular segment.

Using the trick of APOLLONIUS, or of HERON, one gets the estimate $\pi = 3.1416$, which is correct to four places. LIU HUI had this estimate, and so did ARYABHATA (fl. c. 510 A.D.) and BHASCARA II (fl. c. 1150). VAN DER WAERDEN has given reasons for thinking that all these estimates go back to APOLLONIUS (cf. *Geometry and Algebra*, pp. 211–214).

In LIU HUI's calculations based on A_{96} and C_{96} the trick would call for adding $35/625 \times 10^{-2}$ ($= .00056$) to the 3.141024 (to get 3.141584), whereas LIU HUI adds $36/625 \times 10^{-2}$ ($= .000576$) to get 3.1416. It is not clear to me what LIU HUI intends in adding the $36/625 \times 10^{-2}$; still he is saying that one adds a fraction of the 96 triangles AEB to get the answer. Thus the treatments of LIU HUI and APOLLONIUS both seek π_1 (not π_2); and both calculate π as A_{192} plus a fraction of the 96 triangles AEB .

(The procedure described, which gives a lower bound for π , can reasonably be called a good guess, but I don't see how APOLLONIUS himself could

assert the accuracy without further computation or using more theorems. Let $D_n = A_{2n} - A_n = n$ times area triangle AEB in Figure 4 $= n(1 - \cos \theta) \sin \theta$ for $\theta = \text{arc } AE$, and

$$\frac{D_{2n}}{D_n} = \frac{2(1 - \cos \frac{1}{2} \theta) \sin \frac{1}{2} \theta}{(1 - \cos \theta) \sin \theta} = \frac{2(1 + \cos \theta) \sin^3 \frac{1}{2} \theta}{(1 + \cos \frac{1}{2} \theta) \sin^3 \theta} = \frac{1}{2(1 + \cos \frac{1}{2} \theta) \cos \frac{1}{2} \theta} > \frac{1}{4}.$$

Now $A = A_n + D_n + D_{2n} + D_{4n} + \dots = A_n + D_n \left(1 + \frac{D_{2n}}{D_n} + \frac{D_{4n}}{D_{2n}} \cdot \frac{D_{2n}}{D_n} + \dots \right) > A_n + D_n(1 + \frac{1}{4} + (\frac{1}{4})^2 + \dots) = A_n + \frac{4}{3} D_n$. This is, in essence, HERON'S argument. Now D_{2n}/D_n is monotonically decreasing as $n \rightarrow \infty$, so

$$\begin{aligned} A &= A_n + D_n(1 + \dots) < A_n + D_n \left(1 + \frac{D_{2n}}{D_n} + \left(\frac{D_{2n}}{D_n} \right)^2 + \dots \right) \\ &= A_n + D_n / \left(1 - \frac{D_{2n}}{D_n} \right), \end{aligned}$$

and this will be a desired upper bound (for $n = 48$). This is entirely in the spirit of HERON'S argument, and I would conjecture that it is due to APOLLONIUS. Using a calculator showing 8 significant figures and giving square roots (but, for example, in no way using π), I find

$$3.1326278 < A_{48} < 3.1326291, \quad 3.1393467 < A_{96} < 3.1393505,$$

$$3.1410195 < A_{192} < 3.1410490, \quad 0.0016690 < D_{96} < 0.0017023,$$

$0.0067176 < D_{48} < 0.0067227, D_{96}/D_{48} < 0.25340896, A_{192} + \frac{1}{3} D_{96} > 3.1415758$
 $A_{48} + D_{48}/(1 - D_{96}/D_{48}) < 3.1416336$, allowing the conclusion that 3.1416 as an estimate for π is correct to 4 places but not that 3.14159 is correct to 5, in accordance with what has been credited to APOLLONIUS.)

Thus I think VAN DER WAERDEN'S view that the Greek mathematics of the third century B.C. influenced the Chinese mathematics of the third century A.D. is justified. Nor is this very surprising since the trade route from Persia to China was already in existence by the second century B.C. (cf. RAGLAN, *How Came Civilization?*, p. 186). Even so, I do not think the box-lid came to China from Greece (not after 250 B.C. anyway). Let me explain why.

I am agreeing, of course, with YOUSCHKEVITCH'S view that the box-lids in Greece and in China are historically related.

In China we first see the box-lid with LIU HUI in his commentary, where it is used to explain a misconception in the *Nine Books* relative to the volume of a sphere. Next we see it with TSU KENG-CHIH, who actually uses it in finding the volume of a sphere. In China wherever we see the box-lid, we see it in association with the sphere. In Greece there is no such association; at best the sphere and the box-lid both occur in *The Method*. Now we can easily imagine that the box-lid was associated with the sphere in Greece in pre-Archimedean days; but ARCHIMEDES had no use for the box-lid in his study of the sphere, and (we may suppose) kept the box-lid merely because it is an amusing surface. (In the prefatory letter of *The Method*, ARCHIMEDES makes a special remark on the box-lid and on an associated volume, which is a slight generalization of one-eighth of a box-lid.)

But the sphere and the box-lid having become separated by ARCHIMEDES (or predecessors), it is hard to imagine what might bring them together again in the course of transmission to China. Thus I consider it implausible that the box-lid came to China from Greece in post-Archimedean days.

Of course, the box-lid did not come from China to Greece in LIU HUI's day. But the box-lid in association with the sphere is well-ensconced in the Chinese mathematics, so we may try to visualize it as moving from somewhere in the Chinese tradition over to Greece. This would, of course, have had to be in pre-Archimedean days. The picture we are getting is first, that the problem of the volume of the sphere was already an old problem in ARCHIMEDES' day, and second, that in Greece, as in China, the problem of the sphere was associated with the box-lid in pre-Archimedean days.

How far back shall we put this common tradition? Shall we put it back to the common source of Pythagorean and Chinese-like mathematics? We may picture mathematics as flowing from a common source, then branching, with one branch headed for Greece, another for China. I know of no way absolutely to exclude communication between the two branches, but the two traditions must have kept pretty much to themselves for a long time, for otherwise they could not have built up the distinct characters we now perceive them to have. This suggests placing the problem of the volume of the sphere, and the box-lid, back to the common source of Pythagorean and Chinese-like mathematics, to the first part of the second millenium B.C. or earlier.

Of course, one would not put the problem before the existence of CAVALIERI-like infinitesimals, but we have already put such infinitesimals back to the common source.

I also do not know how absolutely to exclude that some third source inserted itself separately into the Pythagorean-like and Chinese-like traditions; but I have no candidates for such a source.

Thus I propose the hypothesis that *the problem of the volume of a sphere goes back to the common source of Pythagorean and Chinese-like mathematics*. A hypothesis, to be a good hypothesis, need not be proved. If it throws light on the evidence and is not in contradiction with any known fact, that is all that one requires of it.

Recall that ARCHIMEDES, so it is said, wished that the figure of a sphere and the circumscribing cylinder be engraved on his tombstone. What was it that made ARCHIMEDES' heart shine? Was it that he had solved the old, old problem of finding the volume of a sphere, having proved that this is $\frac{2}{3}$ of the volume of the cylinder? Or was it that he had invented and solved the brand new problem of finding the area of a sphere, having proved that this is the same as that of the cylinder (*sans* top and bottom)? Or both? I leave it to the reader to answer these questions.

References

- W. W. R. BALL 1901. *A Short Account of the History of Mathematics*, 3rd edition, London.
 O. BECKER & J. E. HOFFMAN 1951. *Geschichte der Mathematik*. Bonn.
 M. DEHN 1902. "Ueber den Rauminhalt," *Mathematische Annalen* **55**, 465-478.

- T. L. HEATH 1908. *The Thirteen Books of Euclid's Elements* (2nd edition, 1926). Cambridge.
- T. L. HEATH 1921. *A History of Greek Mathematics*. 2 vols. Oxford.
- T. L. HEATH 1912. *Works of Archimedes*. New York.
- D. HILBERT 1902. "Mathematical Problems," (translated by M. W. NEWSON). *Bulletin of the American Mathematical Society* 8, 437-479.
- M. S. MAHONEY 1971. "Babylonian Algebra: Form vs. Content," *Studies in History and Philosophy of Science* 1, 369-380.
- J. MATHEWS 1985. "A Neolithic Oral Tradition for the van der Waerden/Seidenberg Origin of Mathematics," *Archive for History of Exact Sciences* 34, 193-220.
- O. NEUGEBAUER 1928. "Zur Geschichte des Pythagoräischen Lehrsatzes," *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen* (Math.-Phys. Klasse).
- O. NEUGEBAUER 1934. *Vorlesungen über Geschichte der Antiken Mathematischen Wissenschaften*. Berlin.
- O. NEUGEBAUER 1935/37. "Mathematische Keilschrift-Texte," *Quellen u. Studien zur Geschichte d. Math. Astr. u. Phys. Abt. A.*, vol. 2.
- O. NEUGEBAUER 1962. *The Exact Sciences in Antiquity* (2nd edition). Princeton.
- O. NEUGEBAUER & A. SACHS 1945. *Mathematical Cuneiform Texts*. (Amer. Oriental Ser., vol. 29). New Haven.
- T. E. PEET 1931. "A Problem in Egyptian Geometry," *J. of Egyptian Archeology* 17, 100-106.
- F. R. S. RAGLAN 1939. *How Came Civilization?* London.
- A. SEIDENBERG 1962. "Ritual Origin of Geometry," *Archive for History of Exact Sciences* 1, 488-527.
- A. SEIDENBERG 1972. "On the Area of a Semi-Circle," *Archive for History of Exact Sciences* 9, 171-211.
- A. SEIDENBERG 1978. "The Origin of Mathematics," *Archive for History of Exact Sciences* 18, 301-342.
- A. SEIDENBERG 1983. "Geometry of the Vedic Rituals," in F. STAAL (ed.), *Agni, the Vedic Ritual of the Fire Altar*, 2 vols., Berkeley, vol. 2.
- A. J. E. M. SMEUR 1970. "On the Value Equivalent to π in Ancient Mathematical Texts. A New Interpretation," *Archive for History of Exact Sciences* 6, 249-270.
- W. W. STRUVE 1930. "Mathematische Papyrus des Staatlichen Museum der Schönen Künste in Moskau," *Quellen u. Studien z. Geschichte der Math. Abt. A.*, vol. 1.
- G. THIBAUT 1875. "On the Sulvasutras," *J. Asiatic Society Bengal*, vol. 44: 1.
- S. UNGURU 1975. "On the Need to Rewrite the History of Greek Mathematics," *Archive for History of Exact Sciences*, vol. 15, 67-114.
- K. VOGEL 1968. *Chiu Chang Suan Shu, Neun Bücher Arithmetischer Technik*. Braunschweig.
- B. L. VAN DER WAERDEN 1961. *Science Awakening, I* (2nd edition). Groningen. (Dutch original, 1950.)
- B. L. VAN DER WAERDEN 1983. *Geometry and Algebra in Ancient Civilizations*. Berlin.
- D. B. WAGNER 1978a. "Doubts Concerning the Attribution of Liu Hui's Commentary to the *Chiu-chang Suan-shu*," *Acta Orientalia* 39, 199-212.
- D. B. WAGNER 1978b. "Liu Hui and Tsu Keng-chih on the Volume of a Sphere," *Chinese Science* 3, 59-79. Philadelphia.

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