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1978

APPENDIX

HOW THE PYTHAGOREANS DISCOVERED

PROPOSITION II. 5 OF THE *ELEMENTS**

1 THE PREVAILING VIEW

Nowadays most scholars would agree that Proposition II.5 forms part of *ancient Pythagorean mathematics* or, more precisely, that it is a theorem of the so-called '*geometrical algebra of the Pythagoreans*'. Since it is usually discussed together with II. 6, let me begin by quoting both of these propositions:

Proposition II. 5: "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half."

Proposition II. 6: "If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line."

According to the standard historical account of Greek mathematics, II. 5 can be regarded as a geometric equivalent of the algebraic formula

$$a^2 - b^2 = (a - b) \cdot (a + b)$$

(or $a^2 = (a - b) \cdot (a + b) + b^2$).¹ Since this can be said of II. 6 as well, it would appear that both propositions amount to the same thing. The explanation offered for this curious fact² is that II. 5 and II. 6 are not really propositions, but rather *the solutions to certain problems*. II. 5

* This paper was written (after the rest of the book had been completed) at the invitation of Professor Ph. M. Vassiliou, who presented it at a meeting of the Greek Academy of Sciences in Athens on 16 May 1968. It is reprinted here because it continues the work begun in the present book and points out a direction for future research.

¹ B. L. van der Waerden, *Erwachende Wissenschaft*, p. 196.

² Ibid., p. 198. This explanation goes back to Zeuthen and even to Tannery (see n. 6 below).

is concerned with the construction of two straight lines whose *sum and product* are given, whereas in II. 6 it is the *difference and product* of the two lines which is given. These problems are supposed to be algebraic in nature³ and to have originated amongst the Babylonians who also devised procedures for solving them.⁴ It is claimed that the Pythagoreans subsequently gave a geometrical formulation of this part of Babylonian mathematics and also supplied the necessary proofs.⁵

I disagree with the above interpretation on the following grounds:

(1) Even if we are convinced by Neugebauer's arguments and accept that there was such a thing as 'Babylonian algebra', this does not mean that the Greeks in pre-Euclidean times actually knew about it, let alone that they took it over and put it into geometric form; in fact, no one has yet succeeded in producing any concrete evidence to support the view that they did. (After all the Greeks never adopted the place-value system of notation for numbers from the Babylonians, even though it would have been much easier for them to have done this than to have reworked the whole of Babylonian algebra.)

(2) Ever since the time of Tannery⁶ it has been customary to regard certain of Euclid's propositions as 'algebraic theorems in geometric form'. However, these propositions are algebraic only in the sense that we can very easily find algebraic results which are equivalent to them. It certainly cannot be maintained that they started out as *algebraic* theorems or as the solutions to *algebraic* problems, for all of them have purely *geometric* origins. One such proposition is II. 5 which, when interpreted from a modern point of view, can be compared to the solu-

³ Heath (*Euclid's Elements*, Vol. 1, pp. 382ff) also points out that II.5 can be interpreted as a "geometrical solution of a quadratic equation". The difference between van der Waerden and Heath is that the latter makes no mention of 'Babylonian science'; Heath's translation of the *Elements* appeared before Neugebauer's research was published.

⁴ The standard reference for this is O. Neugebauer, 'Zur geometrischen Algebra. Studien zur Geschichte der antiken Algebra III'. *Quellen und Studien* . . . B 3 (1936) 245-59.

⁵ B. L. van der Waerden, *Erwachende Wissenschaft*, p. 203.

⁶ P. Tannery, 'De la solution géométrique des problèmes du second degré avant Euclide', 1882 (reprinted in *Mémoires scientifiques* 1, 254-80). Neugebauer regarded Zeuthen as the discoverer of 'geometrical algebra', whereas it was in fact Tannery who originated this idea.

tion of an algebraic problem. Nonetheless, it is a proposition which originally had a geometric meaning, and we should not allow ourselves to forget this fact.

2 MY OWN VIEW

In my opinion, II. 5 is a purely geometrical lemma needed for the solution of a purely geometrical problem (or to prove the correctness of this solution). The problem is stated as Proposition II. 14; it is "to construct a square equal to a given rectilinear figure".

It is obvious that II. 14 depends upon II. 5, both from modern commentaries and translations which almost always refer back to the latter proposition in their discussions of the former, and even more so from the original Greek text. To a large extent II. 5 is repeated word for word in the proof of II. 14, a sure sign that whoever composed the text intended to refer back to this proposition.

Indeed, the complicated and awkward language of II. 5 suggests that it was tailored especially to fit the needs of II. 14. The author appears to have formulated it in such a complicated way because he knew from the start that this seemingly awkward form of the proposition was the one best suited for use in the proof of II. 14. I shall show that this was in fact the case.

First, however, let me take this opportunity to point out that the kind of explanation offered above is equally convincing for other Euclidean propositions which up to now have been regarded as part of a 'geometrical algebra'. II. 6, for example, is a purely geometrical lemma which bears exactly the same relationship to Proposition II. 11 as II. 5 does to II. 14. The reason why it appears to be a special case of II. 5, is that II. 11 is basically nothing more than a special case of II. 14.

Yet another purely geometrical lemma which has been viewed as part of a 'geometrical algebra', is Proposition II. 10.⁷ The writings of Proclus attest to the fact that it is in reality a lemma⁸ needed for the proof of an ancient Pythagorean theorem. Furthermore, they have

⁷ See B. L. van der Waerden, *Erwachende Wissenschaft*, p. 202.

⁸ In *Platonis Rem Publicam Commentarii* (ed. W. Kroll), 1901 Chapter II, pp. 23 and 27.

enabled modern researchers to reconstruct this theorem even though it was never incorporated into the *Elements*.⁹

I now want to give a more detailed account of the genesis of Proposition II. 5. This will require some discussion of the interesting way in which the Pythagoreans dealt with the areas of parallelograms. The next chapter is devoted to an outline of their theory.

3 ELEMENTS OF A PYTHAGOREAN THEORY ABOUT THE AREAS OF PARALLELOGRAMS

Our starting point is the scene in Plato's *Meno* (82b-85e) where Socrates asks an uneducated slave how to find a square having twice the area of one with sides two feet long. In other words, he wants the slave to find a number n such that the sides of the required square will

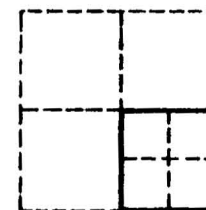


Fig. 12

have a length of n feet. The slave's first thought is that the way to double the area of a square is to double the length of its sides; so he answers that the required length is *four feet*. Socrates, however, proceeds to show him that this would yield a square having four times the original area. (The area of such a square would consist of sixteen unit squares, whereas the area of the original one consisted of only four; see Fig. 12.) The slave's next suggestion is that a square with *sides three feet long* might have twice the original area. Socrates then draws a diagram (see Fig. 13) showing that this square will not do because its area consists of nine unit squares. In recounting the slave's attempts to answer Socrates' question, Plato seems to be hinting that, once the sides of a given square have been *assigned some number as their length*, the same *cannot* be done for the sides of a square having twice the area

⁹ B. L. van der Waerden, *Erwachende Wissenschaft*, p. 206.

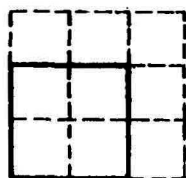


Fig. 13

of the first one. In fact, the two squares will have sides which are *incommensurable in length*. Strangely enough, however, the problem of incommensurability is never mentioned explicitly in this scene. Instead Socrates draws one last diagram (Fig. 14) which shows that a square is divided into two equal isosceles triangles by its *diagonal*, and hence that the square constructed upon the diagonal of the original one will have exactly twice its area.

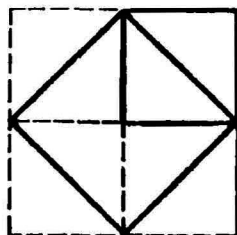


Fig. 14

I believe that this scene from the *Meno* gives us an idea of how the Greeks may have arrived at the discovery of linear incommensurability by way of an *arithmetical* problem which appeared in *geometry*. Given a square whose sides had been assigned a *number*, they wanted to find the *number* corresponding to the sides of a square with twice its area. Euclid's fundamental definition of 'number' (VII. 2) precluded the possibility of this problem having an *arithmetical* solution, nonetheless it remained of interest from a *geometric* point of view. The realization that any square could be doubled by constructing another one on its *diagonal*, together with their inability to ascertain the *numerical* ratio between the side of a square and its diagonal, must soon have led Greek mathematicians to conclude that these two segments were *incommensurable in length*. Of course, they must also have been aware that the two were *measurable in square* (since the square on the diagonal had

exactly twice the area of the square on the side). Hence the discovery of *linear incommensurability* went hand in hand with a knowledge of quadratic commensurability and the difficulties presented by the former could be circumvented by means of the latter.

The historical interest of Socrates' conversation with the slave is heightened by the fact that the *arithmetical problem of doubling the square* is equivalent to the problem of finding a *mean proportional between two numbers, one of which is twice as large as the other*.¹⁰ The Pythagoreans must have known from their study of arithmetic that the latter problem had no numerical solutions. This fact was certainly familiar to them, since it follows from a theorem (Proposition 3) in the *Secio Canonis* which states: "There does not exist a mean proportional number between two numbers in a ratio superparticularis (i.e. in a ratio of the form $(n + 1) : n$)."

Bearing in mind this proposition and the scene from the *Meno* discussed above, we can reconstruct three interesting stages in the development of pre-Euclidean mathematics.

(a) The first stage saw the emergence of certain arithmetical problems whose solutions could not be found. On closer inspection, it turned out that some of these were *completely insoluble*, whereas others had *solutions only under certain conditions*. Included among the former was the problem of finding a mean proportional number, x , between some number, a , and $2a$. The discovery that there was no such x for any a , represented of course a considerable achievement.

(b) Somewhat later it was realized that these same problems could easily be solved if they were given geometrical interpretations. For example, the mean proportional between a and $2a$ was found by taking a to be an arbitrary straight line instead of a number, and constructing a square on it. The diagonal, d , of this square was the required mean proportional since $a : d = d : 2a$. This equation can be proved with the

¹⁰ It was Hippocrates of Chios who suggested that the problem of doubling the cube could be solved by finding two mean proportionals between a number and its double. (Cf. O. Becker, *Das mathematische Denken der Antike*, Göttingen 1957, p. 75). It is interesting to consider how he might have come upon this idea. He knew, of course, that the *planimetric problem* of doubling the square could be solved by finding one mean proportional between a number and its double. In my opinion, this probably led him to conclude that the related *stereometric problem* of doubling the cube would be solved if one could find two such mean proportionals.

help of similar right-angled triangles, provided that the notion of 'being in the same ratio' is defined for incommensurable magnitudes, i.e. provided that something like the so-called Eudoxian definition of proportionality (V. 5 in the *Elements*) is available.

So we see that at this stage of development, problems which had seemed to be insoluble from the arithmetical viewpoint of stage (a) were given geometrical solutions which depended both on a knowledge of incommensurability and on Eudoxus' definition of proportionality.

(c) It was found at an 'intermediate stage'¹¹ that the original problems could be transformed into essentially equivalent ones which were capable of being solved geometrically without taking problems of commensurability (or incommensurability) into account and without the use of proportions. For example, the problem treated above (to find the mean proportional between some number or magnitude, a , and $2a$) was transformed into one of finding the straight line required to double a square with sides of length a . The significance of this intermediate stage lies in the fact that questions of proportionality and incommensurability were, in a manner of speaking, eliminated. It was probably the discovery that this could be accomplished by a transformation of the problems concerned, which led the Pythagoreans to develop an interesting 'geometry of areas'. This theory, which earlier scholars interpreted as 'geometrical algebra', will be discussed below.

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First of all it should be noted that there are a great many examples to be found in the *Elements*, which illustrate the three stages of development outlined above in (a), (b) and (c). I will quote three of these here.

Example 1

Stage (a) Consider the following arithmetical propositions from the *Elements*:

¹¹ I do not want to claim that stage (c) must have preceded stage (b) in time. Questions of chronology are not my prime concern here. What does seem to me to be important is that stage (c) shows how the problems raised at stage (a) could be given solutions which made no mention of proportions or the problem of incommensurability.

IX. 16: "If two numbers be prime to one another, the second will not be to any other number as the first is to the second." (I.e. if a and b are relatively prime, then there is no x such that $a : b = b : x$.)

IX. 18: "Given two numbers, to investigate whether it is possible to find a third proportional to them." (I.e. to find conditions on a and b which are necessary and sufficient to ensure the existence of an x such that $a : b = b : x$.)

IX. 19: "Given three numbers, to investigate whether it is possible to find a fourth proportional to them." (I.e. to find conditions on a , b and c which are necessary and sufficient to ensure the existence of an x such that $a : b = c : x$.)

These propositions were obviously written down at a time when it was known that the problem of finding a third proportional to two given numbers (or a fourth proportional to three given ones) could only be solved under certain conditions.

Stage (b) Now if a , b and c are taken to be arbitrary straight lines instead of numbers, geometrical solutions to the above problems can always be found. This is proved by Euclid in Propositions VI. 11 and VI. 12 where he shows how to construct the required third (or fourth proportional). His constructions, however, frequently yield incommensurable magnitudes. Hence it cannot be claimed that they always provide correct solutions, unless the Eudoxian definition of proportionality is assumed.

Stage (c) It is possible to solve these same problems without considering proportions or incommensurability, if they are first transformed in the following way. The problem of finding the third proportional (i.e. the x such that $a : b = b : x$) becomes that of finding the rectangle (ax) which has one side of length a and the same area as a given square, b^2 . Similarly, instead of trying to find the fourth proportional (i.e. the x such that $a : b = c : x$), one looks for the rectangle which has one side of length a and the same area as a given rectangle, bc . Euclid's Proposition I. 44 provides the means for solving both problems. It runs as follows: "To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle."

(Euclid speaks of 'a given triangle' and a 'parallelogram in a given rectilineal angle', instead of dealing with rectangles, for the sake of greater generality.) The proof of this proposition uses 'applications of areas', a method developed by the Pythagoreans which has nothing to do with incommensurability or proportions.

Example 2

Stage (a) The next two of Euclid's propositions which we want to examine are:¹²

VIII. 18: "Between two similar plane numbers there is one mean proportional number."

VIII. 20: "If one mean proportional number falls between two numbers, the numbers will be similar plane numbers."

(Both propositions assume Definitions VII. 16 and VII. 21.)

So, according to these two propositions, a mean proportional number exists between two numbers (a and b) if, and only if, $a = cd$ and $b = ef$ for some c, d, e and f such that $c : e = d : f$. In other words, this condition is necessary and sufficient for the problem of finding a mean proportional to have an *arithmetical solution*.

Stage (b) Now Proposition VI. 13 shows how to construct a mean proportional to any two straight lines, a and b . As in the previous example, however, the construction is one which depends upon *Eudoxus'* definition of proportionality because of the fact that it yields an *incommensurable magnitude* in many cases.

Stage (c) The above problem (to find an x such that $a : x = x : b$) is equivalent to that of finding a square (x^2) which has the same area as a given rectangle, ab . Once it has been reformulated in this way, it can be solved *without using proportions and without bothering about incommensurability*. This is exactly what Euclid does in Proposition II. 14.

Example 3

Stage (a) As my third example I would like to take the so-called "Theorem of Pythagoras". It is well known that this proposition can be verified *arithmetically* only for right-angled triangles with rational sides. The pioneers of Greek mathematics seem to have been interested in this particular case. (They even went so far as to devise rules for generating triples of numbers which could be the sides of right triangles.¹³) So the verification of Pythagoras' theorem is yet another

¹² For the sake of simplicity, Proposition VIII. 18 is quoted in an abbreviated form.

¹³ Cf. O. Becker, *Das mathematische Denken der Antike*, Göttingen 1957, pp. 52ff.

problem which has an *arithmetical solution only under very restricted conditions*.

Stage (b) The usual proofs of the theorem make use of *proportions*, for they rely on the fact that the perpendicular from the right angle to the hypotenuse divides a right triangle into two smaller ones which are similar both to each other and to the original triangle. This means that the corresponding sides of these triangles are proportional, from which it is easy to derive the formula $a^2 + b^2 = c^2$. It must be admitted that no proof of this kind is to be found in Euclid. Nonetheless, it is reasonable to conjecture that one was known in pre-Euclidean times, since he himself uses *proportions* to prove a more general form of Pythagoras' theorem (VI. 31). Of course such a proof is not convincing unless proportionality has somehow been defined for incommensurable magnitudes. (This is accomplished by Definition V. 5 as far as Euclid is concerned.)

Stage (c) A kind of 'geometry of areas' is used to prove Pythagoras' theorem (I. 47) in the *Elements*. The proof, like those in our previous examples of stage (c), is distinguished by the fact that it *avoids entirely both proportions and the problem of incommensurability*. Figures 15 and 16 are intended to illustrate Euclid's argument, which runs roughly as follows:

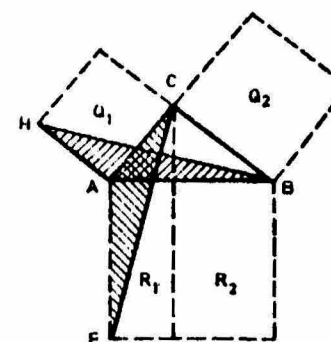


Fig. 15

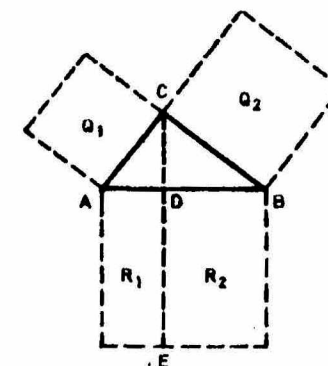


Fig. 16

The perpendicular (CD) from the right angle to the hypotenuse of an arbitrary right triangle (ACB) can be extended (by DE) so as to divide the square on the hypotenuse into two rectangles, R_1 and R_2 (see Fig. 15). Now $Q_1 = R_1$, and $Q_2 = R_2$. The shaded triangles CAF and HAB (see Fig. 16) are obviously congruent. Furthermore, HAB has

half the area of Q_1 , and CAF has half the area of R_1 (by Proposition I. 41). Hence R_1 and Q_1 must have the same area. In a similar manner one can prove that $R_2 = Q_2$. The proof is completed by combining these two equations.

The preceding examples together with my discussion of Socrates' mathematical demonstration in the *Meno* have, I hope, lent some plausibility to the claim that the kind of 'geometry of areas' used at stage (c) was designed to eliminate the application of proportions and the problem of incommensurability.

The most elementary portions of this Pythagorean 'geometry of areas' (or 'geometrical algebra', as Tannery called it)¹⁴ are surveyed in (1), (2), and (3) below.

(1) Once again our starting point is that part of the *Meno* which deals with squares and their areas. The passage in question describes how Socrates, having drawn a square with sides two feet long, bisects

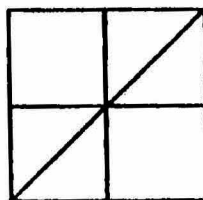


Fig. 17

these sides to show that it is made up of four unit squares, and how he goes on to indicate that the square's area is *halved* by its *diagonal*. (See Fig. 17. I should mention here the trivial fact that the diagonal and the lines which bisect the sides intersect at a point. This point of intersection will play an important role in the sequel.) In short, Socrates demonstrates two ways of splitting up the area of a square. Both of these are treated in a more general form by Euclid. For his Proposition I. 34, "In parallelogrammic areas . . . the diameter bisects the areas", is just Socrates' remark about the diagonal of a square generalized to the case of parallelograms; whereas Proposition II. 4 discusses the way in which the area of a square can be split up when its side, instead of being bisected, is divided into two arbitrary line segments. This latter proposi-

¹⁴ See n. 6 above.

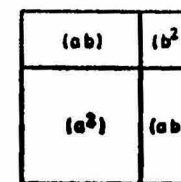


Fig. 18

tion, which is perhaps an even more interesting generalization than the former, runs as follows: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments."

So, when the side of a square is 'cut at random', its area is obtained as the sum of two distinct squares and two congruent rectangles (see Fig. 18). Although the above proposition is a geometric equivalent of the algebraic formula $(a + b)^2 = a^2 + 2ab + b^2$, this fact should not in my opinion be taken to mean that II. 4 is "a geometric version of what was originally an algebraic idea". In the first place there is no evidence that Greek mathematics contained any genuinely algebraic ideas either before or during Euclid's time, and in the second place such an interpretation would obliterate the fundamental difference between the algebraic equation and its geometric counterpart. In the final analysis the one is much more general than the other. (Euclid's proposition is one possible interpretation of the algebraic formula, but it is by no means the only one.)

§(2) It can be shown that the Pythagoreans also generalized Proposition II. 4 to *parallelograms*. They succeeded in proving a theorem something like the following (see Fig. 19):

If two lines which are parallel to the sides of a given parallelogram are drawn through an arbitrary point of its diagonal, they will split the area of the parallelogram into four parts. Two of these parts, the 'parallelograms about the diameter' (left unshaded in Fig. 19), will be similar



Fig. 19

to each other and to the original parallelogram, whereas the other two, the so-called *παραπληρώματα* (shaded in Fig. 19), will be equal.

I must admit that the above theorem is largely my own reconstruction. Euclid does not state it in this form, but treats it instead as two separate propositions. One, which discusses the *similarity* of the 'parallelograms about the diameter', does not appear until Book VI; the other deals with the *equality* of their 'complements' (*parapleromata*) and is included in Book I. This arrangement was forced upon Euclid by the fact that the proof of the former (Proposition VI. 24) depends in an essential way on *Eudoxus' definition* (V. 5) of *proportionality*. On the other hand, the latter (Proposition I. 43) is proved simply by subtracting areas; its proof has nothing to do with proportionality or incommensurability.

I should emphasize here that the theorem which I have reconstructed played a very important role in the Pythagorean 'geometry of areas' or, to use the unfortunate and misleading name which Tannery gave this theory, 'geometrical algebra'. If, for example, the Pythagoreans¹⁵ had not known that the *parapleromata* were equal, they would not have been able to develop their method of *application of areas*. This method was mentioned in *Example 1* above, where it was pointed out that finding a fourth proportional to three given *numbers* or *magnitudes* (i.e. an x such that $a : b = c : x$) could be construed as the problem of finding a rectangle with a given side (a) which has the same area as a given rectangle (bc).

Proposition I. 44 of the *Elements* provides the following solution to this problem (see Figs 20 and 21). If we think of the given rectangle (bc) as a *parapleroma* and the rectangle which the line a makes with one of its sides (c or b) as a 'parallelogram about the diagonal', then the second 'parallelogram about the diagonal' (bx or cx) can be constructed by extending the side (c or b) of the original rectangle until it meets the continuation of the diagonal of ac (Fig. 20) or of ab (Fig. 21). Now these three rectangles together determine uniquely the second *parapleroma* and, in particular, its side x which we set out to find. (This is sometimes called *parabolic application of areas*, because the given area bc is *applied* to the line a ; cf. the word *παράβállειν*.)

¹⁵ Proclus (ed. Friedlein), 419.15–420.23. See also T. L. Heath, *Euclid's Elements*, Vol. 1, pp. 343ff.

It seems to me, however, that the importance of our theorem for Pythagorean geometry can be illustrated even more convincingly. Let us look once more at the theorem itself. It states that two lines drawn parallel to the sides of a parallelogram through an arbitrary point on its diagonal will split the area of the parallelogram into four parts; two of these will be equal, and the other two will be similar (both to each

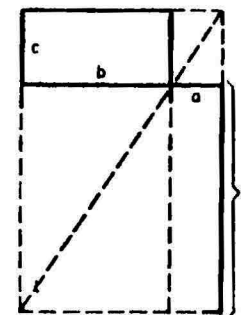


Fig. 20

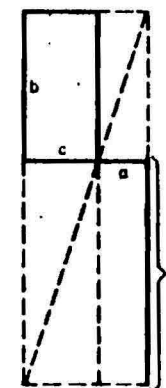


Fig. 21

other and to the original parallelogram). My claim is that this simple yet beautiful discovery led the way to a famous problem of the early Pythagoreans about which Plutarch wrote the following (*Symposium* VIII. 2.4):

"Among the most geometrical theorems, or rather problems, is the following: given two figures, to apply a third equal to the one and similar to the other, on the strength of which discovery they say moreover that Pythagoras sacrificed. This is indeed unquestionably more subtle and scientific than the theorem which demonstrated that the square on the hypotenuse is equal to the squares on the sides about the right angle."¹⁶

(3) I cannot conclude this account of the elements of the Pythagorean 'geometry of areas' without mentioning Euclid's definition of the geometrical concept *gnomon*:

¹⁶ Quoted by Heath, loc. cit. (*Euclid's Elements*, Vol. 1, pp. 343ff).

Definition 11. 2: "In any parallelogrammic area let any one whatever of the parallelograms about its diagonal with the two complements be called a *gnomon*."

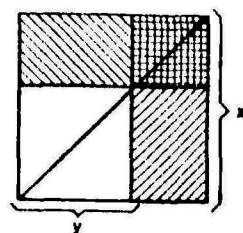


Fig. 22

As we can see, the definition of this concept presupposes that the area of the parallelogram has already been split into *four parts* in the manner described previously. A *gnomon* consists of *three* of these parts, namely the two *parapleromata* together with either one of the 'parallelograms about the diagonal'. The *gnomon* of a square is illustrated in Fig. 22, where its three components are indicated by different kinds of shading. The figure also shows that the original square (x^2) can be obtained as the sum of a smaller square (y^2) and the *gnomon*. (In other words, subtracting the smaller square from the greater leaves a *gnomon* as the remainder.)

This fact is important because the *gnomon* itself can be very easily transformed into a *rectangle*. There are two ways of carrying out the transformation; these are illustrated in Figs. 23 and 24 respectively.

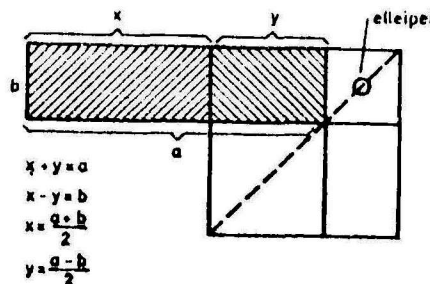


Fig. 23

In one case (Fig. 23) the area consisting of one of the 'parallelograms about the diagonal' (the smallest square in the illustration, which for convenience we will denote by 'o') and one of the *parapleromata* is, in a manner of speaking, joined onto the left-hand side of the second *parapleroma*; whereas in the other case (Fig. 24) it is a *parapleroma* which is joined onto the left-hand side of the area consisting of the smallest square and the other *parapleroma*. The resulting rectangle (the shaded part in Figs 23 and 24) is the same in both cases; it has sides $(x + y)$ and $(x - y)$ (these letters refer to the same lengths as in Fig. 22). The only difference between the two constructions has to do with the square o, a component of the original *gnomon*. In the first case, a rectangle

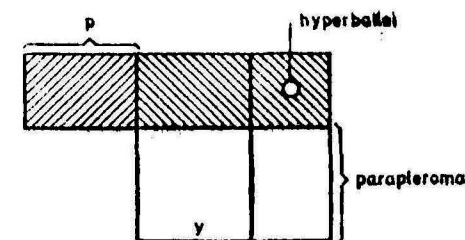


Fig. 24

having the same area as the *gnomon* is applied to the line $2x$ and *falls short* (*ἐλλείπειν*) by O (of the rectangle on this line); hence this is sometimes called *elliptic application of areas*. In the second case, the rectangle is applied to $2y$ and *exceeds* (*ὑπερβολή*) by O (the rectangle on this line); so this is sometimes referred to as *hyperbolic application of areas*. Now is not the time to go into any more detail about *elleipsis* and *hyperbola*,¹⁷ suffice it to say that Euclid employs *elliptic application* in the proof of Proposition II. 5 and *hyperbolic application* in II. 6.

4 HOW TO FIND A SQUARE WITH THE SAME AREA AS A GIVEN RECTANGLE

We are now in a position to return to II. 5, the proposition whose genesis we set out to explain.

¹⁷ See the discussion in Heath's edition of the *Elements* (Vol. 1, pp. 343-4)

I have already mentioned (in Chapter 2 above) that it is a lemma which is needed to prove another important proposition, namely II. 14, and that it is quoted word for word in the proof of the latter. Let us begin by taking a closer look at II. 14, which is concerned with the following problem:

"To construct a square equal to a given rectilineal figure." The proposition is really about *tetragonismos* the transformation of a rectangle into a square of the same area. It is stated for an arbitrary rectilinear figure only because Euclid always sought after the greatest possible generality. Therefore his proof begins with a reference to Proposition I. 45, which makes it possible for any such figure to be transformed into a rectangle.

In example 2 (of Chapter 3) it was observed that the transformation of a rectangle into a square of the same area is equivalent to the problem of finding the mean proportional between two numbers or magnitudes. Euclid discusses the construction of the mean proportional to two straight lines in Proposition VI. 13. It turns out that II. 14 and VI. 13 lead to the same result.¹⁸ Our interest, however, is in the differences between these two propositions.

The surprising thing about the constructions in II. 14 and VI. 13 is that at first sight they appear to be identical in all important respects (see Fig. 25). In both cases the sides of the rectangle concerned (or the two lines, a and b , to which the mean proportional is being sought) are added together to form one straight line. This line is then bisected, and a semicircle with radius $\frac{a+b}{2}$ is drawn around it. Finally a perpendicular is raised from the point at which a and b meet to the circumference of the semicircle. This perpendicular, d , is the side of a square having the same area as ab (or the required mean proportional between a and b).

The steps outlined above make it seem as though the identical construction is used in both propositions. Yet the significance of these steps in the two cases is as different as it could be. Completely different ideas lie behind II. 14 and VI. 13. Let us first interpret the above from the point of view of the latter proposition.

¹⁸ See Aristotle, *De Anima* II.2.413a13-20 and *Metaphysics* B2. 996b18-21. These two passages convinced both Heiberg ('*Mathematisches bei Aristoteles*', *Abhandlungen zur Geschichte der mathematischen Wissenschaften* 18 Heft, Leipzig 1904) and Heath (*Mathematics in Aristotle*, Oxford 1949, pp. 191-3) that VI.13 came before II.14.

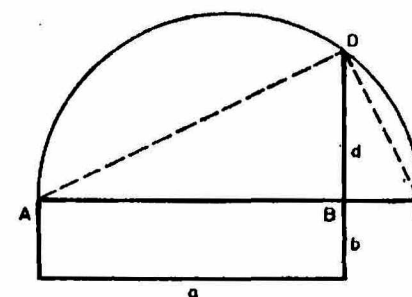


Fig. 25

The two segments a and b are added together in order to obtain the hypotenuse (AC in Fig. 25) of a right-angled triangle (with the thought already in mind that a will eventually form the longer side about the right angle of another triangle, and b the shorter side about the right angle of a third one). The segment AC is bisected and a semicircle is drawn around it, since this gives, by Thales' theorem, the locus of points which can be the third vertex of a right triangle with hypotenuse AC . Finally, d is the required mean proportional between a and b , because the triangles ABD and BCD are similar, and d is the shorter side about the right angle in the former and the longer side about the right angle in the latter. This is undoubtedly the argument upon which the construction in VI. 13 is based. What appears to be the same construction in II. 14, however, rests upon an entirely different chain of reasoning, which we shall now attempt to reproduce.¹⁹

In order to transform a given rectangle, ab , into a square of the same area, an argument which runs something like the following is required. There is, so to speak, quite a distance to cover. Our starting point, the rectangle ab , is given, and our aim is to find a square, d^2 , which has the same area. Since our goal is so far removed from our starting point, it would be desirable to find some way of bringing them closer together. If, for example, it could be shown that both the rectangle and the square can be obtained by performing certain geometrical operations, a link would have been established between them. We shall now take up the question of how this is to be accomplished in the case of the rectangle.

¹⁹ Cf. G. Polya, *How to solve it?* 2nd edn., New York 1957. This book was my guide in the reconstruction which follows in the text.

Consider the subtraction illustrated in Figs 22 and 23. A square, y^2 , is subtracted from a larger one, x^2 , leaving a *gnomon* as the remainder; this 'gnomon' can easily be transformed into the rectangle $(x + y) \cdot (x - y)$. Our present case differs from the preceding in that the sides (a and b) of the rectangle are given instead of the two squares. However, it is a simple matter to find x^2 and y^2 from a and b . Let $x^2 = \left(\frac{a+b}{2}\right)^2$ and $y^2 = \left(\frac{a-b}{2}\right)^2$ (see Fig. 23), then the rectangle ab can be regarded as the difference of two squares, since $ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$; hence it can be described as the result of performing a geometrical operation. (This is actually the transformation which is accomplished by the lemma II. 5.)

Similarly the square d^2 can be viewed as the result of performing another geometrical operation, the operation being *subtraction* in this case as well. Given the right-angled triangle shown in Fig. 26, we know by Pythagoras' Theorem that $x^2 = y^2 + d^2$. So the square on one of the sides about the right-angle can be obtained by subtracting the square on the other from the square on the hypotenuse, and in particular $d^2 = x^2 - y^2$. The solution to our problem is now almost at hand.

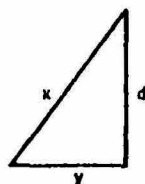


Fig. 26

Subtracting one square from another led us first to a *rectangle* and then to a *square*,²⁰ so we should be able to transform the rectangle ab into a square of the same area by combining these two results. $\left(\frac{a+b}{2}\right)^2$ will be the square on the hypotenuse of a right-angled triangle, whereas

²⁰ The lettering used in Fig. 23 is intended to make the discussion easier to understand.

$\left(\frac{a-b}{2}\right)^2$ will be the square on one of the sides about the right angle. Since we know that the *hypotenuse* equals $\frac{a+b}{2}$ and one of the other sides equals $\frac{a-b}{2}$, we should be able to obtain the third side, d , of the triangle (see Fig. 27). The square on d will then have the same area as the given rectangle ab . This is the argument which underlies the construction in Proposition II. 14.

So we see the *addition* of the two sides of the rectangle plays a different role in this construction (see Fig. 27) to the one which it plays in VI. 13 (for the latter see Fig. 25). The lines a and b are added together in VI. 13 because $a + b$ will form the hypotenuse of a right triangle; this hypotenuse then has to be *bisected* in order to apply Thales' theorem. In II. 14, on the other hand, a is *added* to b , and their sum *bisected* because $\frac{a+b}{2}$ will itself be the hypotenuse of a right triangle (see Fig. 27).

There is one other interesting fact which I should like to mention here. It concerns the side of the square $\left(\frac{a-b}{2}\right)^2$. No special construction is required to obtain this side in the proof of II. 14, for the line

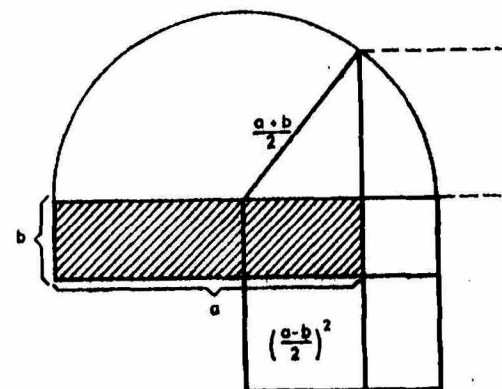


Fig. 27

between the midpoint of $a + b$ and the point at which a and b meet is exactly $\frac{a-b}{2} = y$.

In conclusion, I would like to say something more about my claim that II. 5 is a lemma which was especially tailored to suit the needs of Proposition II. 14.

First of all I should note that Euclid, in his explanation and proof of Proposition II. 5, makes use of a diagram which corresponds in all essential respects to my Fig. 23. This figure shows how the Pythagoreans obtained an arbitrary square as the sum of a *smaller square* and a *gnomon* (or as the sum of a *square* and a *rectangle*). Basically this is the subject matter of II. 5 as well. Of course Euclid does not start out with a square, but with a straight line which is *cut into equal and unequal segments*. Cutting the line into unequal segments is the reverse of *adding* the two sides of the rectangle in II. 14. The latter operation converts a rectangle into a straight line, whereas the former turns a straight line into a rectangle. *Bisecting* the line plays the same role in both propositions; it furnishes the side of the larger square (from which the smaller one is subtracted).

The following is even more interesting. As I have already emphasized, it is necessary to find *half the difference between the two sides of the rectangle* before Pythagoras' Theorem can be applied in II. 14. However, there is no need to construct this length since it is just *the line between the midpoint of $a + b$ and the point at which a and b meet*. In II. 5 this same segment is described as "*the straight line between the points of section*". Euclid's use of this clumsy and awkward phrase, in my opinion, proves conclusively that he had II. 14 in mind when he stated and proved Proposition II. 5.

5 CONCLUSION

It seems to me that the results of the preceding investigation are of some interest from the viewpoint of further research. They are summarized below.

(1) Proposition II. 5 is a purely geometrical lemma needed for the proof of II. 14. Although it is equivalent to 'the solution of an algebraic equation', it should not be interpreted in this way. Such an interpretation is misleading because it obscures the true geometric meaning of the proposition and suggests the false historical idea that the Greeks actually operated with algebraic equations in pre-Euclidean times.

(2) As was mentioned in Chapter II above, the other propositions which are usually regarded as part of 'Pythagorean geometrical algebra' can also be given a purely geometrical explanation. On the other hand, no traces of genuine algebraic ideas²¹ have yet been discovered in the mathematical tradition which culminated in Euclid's *Elements*.

(3) There is however one point on which I agree with Tannery, the scholar whose ideas gave rise to later speculation about the supposed 'geometrical algebra of the Greeks'.²² He was of the opinion that this kind of geometry owed its existence to the discovery of incommensurability. In fact, as I have tried to show, the 'Pythagorean geometry of areas' *eliminated* the problem of incommensurability and consequently *avoided* the use of proportions.

(4) The claim that the 'geometrical algebra of the Pythagoreans' resulted from the Greeks either taking over or developing further an idea of the Babylonians has no basis in fact. No connection has ever been established between this branch of mathematics and 'Babylonian science'. It seems much more likely that the Greeks were solely responsible for the creation of this 'geometry of areas'.

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²¹ Van der Waerden (in *Science Awakening*, p. 118) gives a genuinely algebraic interpretation of Proposition II.14, but this modern interpretation is foreign to the spirit of ancient mathematics.

²² See n. 6 above.