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SCIENCE AWAKENING

I

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by

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with additions of the author

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GREEK MATHEMATICS

CHAPTER IV

THE AGE OF THALES AND PYTHAGORAS

Chronological Summary

General History	Philosophers and Historians	Mathematicians and Astronomers
610 B.C. The beginning of the New Babylonian empire 540 B.C. The beginning of the Persian empire	Milesian school: Thales Anaximander Anaximenes	585 B.C. Thales 550 B.C. Pythagoras Anaximander
500 B.C. Ionian revolt 480 B.C. Persian wars 450 B.C. Pericles 420 B.C. Peloponnesian war	Heraclitus The Eleatics 450 B.C. Anaxagoras Herodotus 430 B.C. Atomists 410 B.C. Thucydides	500 B.C. Hippasus 500 B.C.—350 B.C. Pythagoreans Anaxagoras Oenopides 430 B.C. Hippocrates Democritus Theodorus
370 B.C. Epaminondas	Socrates †399 Plato Heraclides of Pontus Aristotle Eudemus	390 B.C. Archytas Theaetetus 370 B.C. Eudoxus Callippus Hicetas 350 B.C. Menaechmus Dinostratus Autolycus
333 B.C. Alexander the Great Hellenism	Stoics	300 B.C. Euclid 280 B.C. Aristarchus 250 B.C. Archimedes 240 B.C. Eratosthenes Nicomedes 210 B.C. Apollonius 150 B.C. Hipparchus
60 B.C. Julius Caesar 1 A.D. Augustus 400 A.D. Migration	Neo-Pythagoreans Neo-Platonists: Proclus	60 A.D. Heron 100 A.D. Menelaus 150 A.D. Ptolemy 250 A.D. Diophantus 320 A.D. Pappus

Hellas and the Orient.

In his posthumous dialogue *Epinomis*, Plato sketches very strikingly the relation of the Greeks to the old cultures of the Orient, as follows: "... whatever Greeks acquire from foreigners is finally turned by them into something nobler" (987 E).

This is also applicable to the exact sciences. The Greeks themselves declare unanimously that they found in Egypt and in Babylon the material for their geometry and their astronomy. Thales and Pythagoras, Democritus and Eudoxus, all of them are reported to have travelled to Egypt and to Babylonia. Even if one does not accept these reports of travels as historical facts, but looks upon them rather as an anecdotal way of saying that Oriental elements were recognized in their theories, even then they prove enough. Only a few of the modern philologists absolutely refuse to recognize that the Greeks have taken anything essential from the Orient. As if the Hellenes were so narrow-minded as not to recognize the elements of value in an alien culture!

It is certainly not accidental that the Ionians were the first torch-bearers of Greek civilization. Did they not dwell near the boundaries of the great oriental empires, had they not, for many years, been subjects of the kings of Lydia and Persia? They had abundant opportunity to get well acquainted with oriental culture.

The close ties which linked Ionia and Asia Minor, politically and economically, become evident to any one who has looked through the first book of the *Histories* of Herodotus. And the relations between Hellas and Egypt are within easy reach. Numerous Greeks lived in the Nile delta. The Greek city of Naucratis, founded during the reign of Psammetichus (663—609) even received under Amasis (569—525) a trade monopoly for the whole of Egypt.

Less obvious, but nevertheless unmistakable, are the connections with the Assyrian empire. In 709, Sargon II, who expanded this empire to Syria, received presents from 7 city-kings on Cyprus. His successor Sanherib defeated the Ionians after they had landed in Cilicia. Later on, they succeeded however in establishing a trade-center there.

It was a serious blow to Ionian commerce, when Gyges, the founder of the Lydian empire (680—652), put a barrier across the Ionians' access to the hinterland. They attempted to find another route for commercial communication with Babylon. In the middle of the 7th century, they established the commercial towns of Sinope and Trapezus, on the Black Sea, at the terminals of old trade routes with Mesopotamia. Moreover, after the later kings of Lydia, Alyattes and Croesus, had annexed most of the Ionian coastal towns, the route through Asia Minor became again accessible to the Ionians.

In the mean time, important political changes had taken place. The powerful military empire of the Assyrians, which had oppressed the eastern peoples for so many years (consult the Bible!), had crumbled. In alliance with the Medes,

Nabopolassar, king of the Chaldeans, had liberated Babylon from Assyrian domination. In 612 the Medes destroyed Nineveh. When they attempted to advance further to the West, Alyattes of Lydia marched to meet them with Ionian soldiers. The battle at the Halys was brought to a sudden stop by the solar eclipse of 585, which had been predicted by Thales of Miletus. "Day turned into night", writes Herodotus. The soldiers were frightened to such an extent, that they were unwilling to continue the fight. A peace was concluded between the contending forces, in which Cilicia and Babylonia were included.

Since that time three great empires maintained a balance of power: Lydia, Media and the "New-Babylonian empire", founded by the kings of Chaldea. Cultural and commercial communications were opened in both directions; the entire hinterland became accessible to the Greeks. The powerful Chaldean Nebukadnezar, king of Babylon (604—562) was not only a great general, but he also nurtured the cultural development of his country. Greek noblemen, for instance the brother of the poet Alcibiades, took part in battles under his leadership. A cuneiform text from Uruk, bearing the date 551, mentions copper and iron from Ionia.

A statement of Herodotus (II 109) shows that there was also a cultural exchange with Greece; the Gnomon and the Polos, and the 12 hours of the day came to Greece from Babylon. The Polos was probably a sundial in the form of a hemisphere. The Gnomon is also a sun-dial, consisting of a vertical bar, which throws its shadow on a horizontal disc. There is no doubt as to the correctness of the statement, as far as it relates to the Gnomon, for a cuneiform text exists which contains a table for the lengths of the shadows of the bar at the various times of day.¹ The statement concerning the 12 hours, in which the Greeks divided the day from sunrise to sunset, is also correct, for another text from Nineveh, gives a table for the duration of $\frac{1}{12}$, $\frac{2}{12}$, $\frac{3}{12}$, . . . , $\frac{12}{12}$ of the period of daylight for various times of the year, expressed in terms of astronomical time-units (bêru and uš).² The names of the signs of the Zodiac, with which the Greeks became acquainted around the year 550 through Cleostratus of Tenedos, are also derived from Babylon.³

The political equilibrium was disturbed again, when Cyrus subjected the entire Orient to Persian domination about the year 540. The Ionian cities, which had come to the aid of his opponent Croesus, had to pay heavy tributes (plate 11). Many Ionians left the country; the Phocaeans, for example, established the town of Elea, in Italy, which was destined to play such an important part in the history of philosophy. It was also at this time the Pythagoras migrated from Samos to Croton. The center of gravity of the world of mathematics and philosophy moved from Ionia to Italy.

¹ See E. F. Weidner, *Ein babylonisches Kompendium der Himmelskunde*, American Journal for Semitic Languages, 40 (1924), p. 198.

² See van der Waerden, Babylonian Astronomy III, Journal of Near Eastern Studies 10 (1951) p. 25.

³ See Weidner, loc. cit., p. 192.

But it was not long before the Persian empire reestablished economic and cultural links with the Greeks. Ionian artisans and artists took part in the construction of the palace of Darius.¹ The sculptor Telephanes of Phocia worked for Darius and for Xerxes. The Greek physician Democedes from Croton and, later on, Histiaeus, the tyrant of Miletus, lived at the court of Darius, although not entirely voluntarily.

The great Persian kings Cyrus and Darius (plate 12) were very tolerant; they did not interfere with the cultures and the religions of subject peoples (a reference to the Bible is again in order here). Babylonian stellar rituals continued to exist. The observation of the moon and the planets of the Babylonian priest-astronomers, were continued systematically during the Persian regime. Without these carefully dated observations, the later flowering of Babylonian theoretical astronomy, during the era of the Seleucids, the successors of Alexander the Great², would have been impossible. The Greeks also showed much interest in these observations; Callisthenes, a pupil of Aristotle, who accompanied Alexander the Great to Babylon, sent his uncle Aristotle, upon his request, Babylonian observations.³ Hypsicles, a Greek astronomer of the 3rd century calculates the times of rising and setting of the signs in the Babylonian manner, because Greek geometry of the sphere was not yet able to solve this problem.⁴ And in his *Isagoge*, Geminus discusses a method of the Chaldeans for the calculation of the velocity of the moon. Hipparchus (150 B.C.) makes use of Babylonian observations and periods of the moon, which Ptolemy could use 300 years later, practically without corrections.⁵

From all this we see that even during the period of flowering of their own astronomy, the Greeks were glad to learn from the Babylonians in any respect in which the latter had advanced beyond them. Might this not be applicable as well to the initial period of Greek mathematics, when the Babylonians were already in possession of a highly developed algebra and geometry, which the Greeks had not yet acquired?

The natural point at which fruitful contacts between East and West could take place at the beginning of the 6th century, was the flourishing commercial town of Miletus on the coast of Asia Minor, the most important center of Ionian culture. And the first Ionian natural philosopher, the first mathematician and also astronomer, is

Thales of Miletus.

Thales was the first of the "seven sages" (plates 10 and 12). In general, these sages were not scholars, but statesmen, lawgivers and moralists. Sayings, such as the celebrated Delphic "Know thyself" were ascribed to them. It was they who,

¹ F. W. König, *Der Burgbau zu Susa*, Mitteilungen Vorderasiatische Gesellschaft, 35 (1930) 1.

² Compare F. X. Kugler, *Babylonische Mondrechnung* (1900) and *Sternkunde und Sternendienst in Babel I*; O. Neugebauer, *Quellen und Studien B 4* (1938), p. 193 and p. 407; also A. Pannekoek and B. L. v. d. Waerden in *Eudemus I* (1940).

³ Simplicii in Arist. *De Caelo Comment.* II 12, p. 506 (Heiberg).

⁴ Hypsicles, *Anaphoricus*, ed. Manitius, Programm Gymnasium Heil. Kreuz, Dresden 1888, Compare O. Neugebauer, *Trans. Am. Phil. Soc.*, 32 (1942), p. 251.

after the collapse of the old feudal system, laid the foundations for the new political organization of the Greek cities; it is true that this was still aristocratic, but it was no longer feudal.

But the wisdom of Thales had a more universal, a more philosophical character. He asked for the origin of things. It is well known that he taught that everything originates in water.

Plato relates that Thales fell into a well while looking at the stars, and that a comely Thracian slave-girl laughed at him, saying: "he wanted to know what happens in the heavens, but he did not observe what was in front of his own feet".

It would be a mistake to draw from this anecdote the conclusion that Thales was a scholar, alien to this world, some sort of absent-minded professor. On the contrary, he was at the very center of the intense Ionian life of his time. It is reported that he made a great deal of money in an oil speculation and that he had a new river bed made to facilitate the crossing of Croesus' army. But he advised his fellow citizens against an alliance with Croesus, and later on he attempted to establish a federation of Ionian towns with Teos as the capital.

Another example of his practical wisdom is furnished by his advice to navigators to depend upon Ursa Minor rather than upon Ursa Major. Thus he was not only a theoretician and a philosopher, but also a statesman and a man of practical judgment, a "sage" in the complete ancient sense.

Prediction of a solar eclipse.

Herodotus reports (see p. 84) that, during the battle on the Halys, day was suddenly turned into night and that Thales had predicted this event to the Delians for that year. According to Diogenes Laertius, Xenophanes voiced his admiration of Thales for this prediction. Thus, besides Herodotus, we have the older witness Xenophanes for this accomplishment. At present it is generally agreed that this event refers to the solar eclipse of 585 B.C.

How was it possible for Thales, who, according to all our sources, is the first Greek astronomer, to predict a solar eclipse? Such a feat requires the experience of more than forty years, no matter how one proceeds. It is not possible for one man alone to gather this experience. But Thales had no Greek predecessors. The conclusion is inescapable that he must have drawn upon the experience of Oriental astronomers.

This points in the first place to the astronomers of Mesopotamia. For we know from the letters of Assyrian court astrologers of about 700 that these had predicted solar and lunar eclipses (with varying success)¹.

The prediction of Thales fits in very well with this series of predictions. It is a fairy tale of modern times that, in making it, Thales had used the "Saros", the 18-year period which had been known to the Babylonians since about 400 B.C.

¹ R. C. Thompson, *The reports of the magicians and astrologers of Nineveh and Babylon* (1900).

I prefer to think that Thales, as well as the ancient Babylonians, start from the approximate relation:

$$51 \text{ draconitic lunar periods} = 47 \text{ synodic months.}$$

According to this relation, the possibility for the repetition of a lunar eclipse exists 47 months after a total lunar eclipse, while the chance of a solar eclipse occurs $23\frac{1}{2}$ months after a total lunar eclipse. Indeed, a considerable lunar eclipse could be seen $23\frac{1}{2}$ months before the eclipse of Thales.¹ Whatever may actually have happened, the prediction of Thales indicates that he was acquainted with Babylonian astronomy.

The geometry of Thales.

Did Thales also know Babylonian mathematics?

The following information concerning Thales is given by Proclus, the commentator of the first book of Euclid's *Elements*,² who obtained it from the *History of Mathematics* of Eudemos, itself unfortunately lost:

1. He was the first to prove that a circle is divided into two equal parts by its diameter (Proclus, p. 275).

2. Besides several other theorems, he had obtained the equality of the base angles in an isosceles triangle; in ancient fashion, he called these angles not equal, but similar (Proclus, p. 341).

3. According to Eudemos, he discovered that when two straight lines intersect, angles are equal, (Proclus, p. 374).

4. The congruence proposition concerning two triangles, in which a side and two angles are equal, was ascribed by Eudemos to Thales, with the remark that, in order to demonstrate the validity of his method for determining the distance between two ships at sea, Thales had to make use of this congruence theorem (Proclus, p. 409).

How might Thales have determined the distance between ships at sea? According to Tannery, the most ancient method that has been brought down to us is that of the Roman surveyor Marcus Junius Nipsius, who gives the following, indeed very primitive, rule:

In order to find the distance from *A* to the inaccessible point *B*, one erects in the plane a perpendicular *AC* to *AB*, of arbitrary length and determines its midpoint *D*. In *C* one constructs a line *CE* perpendicular to *CA*, in a direction opposite to that of *AB*, and one extends it to a point *E*, collinear with *D* and *B*. Then *CE* has the same length as *AB*.

The congruence theorem (4), referred to by Eudemos, is indeed used in the

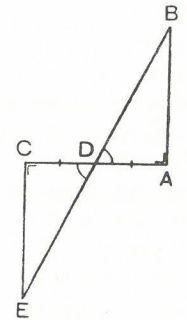


Fig. 24.

¹ See my paper *Voraussage von Finsternissen*, Ber. sächs. Akad. Wiss., Leipzig 92 (1940).

² Proclus *Diadochus*, *Euklidkommentar*, translated by P. L. Schönberger, edited by M. Steck, Halle 1945.

proof of this rule, as well as proposition (3), concerning the equality of vertical angles, also known to Thales. It is therefore possible that this was Thales' method.

Pamphile, as reported by Diogenes Laërtius, says that Thales was the first to construct a circle about a right triangle and that, in honor of this discovery, he sacrificed a bull. Hence, the proposition that an angle inscribed in a semicircle is a right angle, is ascribed to Thales. On the other hand, this proposition is related to certain calculations concerning chords and their apothems, which occur in Babylonian mathematics; but this relation is of course in no way sufficient to prove that Thales knew Babylonian mathematics. The matter must be looked at from a broader point of view, and the statements of Proclus and of Pamphile must be studied more closely.

The absolute accuracy of statements (1) and (4) has been drawn into question, even though they come from the best source. It is thought that the structure of old Thales' mathematics can not have been so strictly logical, that he would undertake to *prove* such obvious things as the equality of the parts into which a diameter divides a circle. Heath, the eminent English historian of Greek mathematics, observes in this connection, that even in Euclid this proposition is not proved.¹ It has been held that the statements of Eudemos should be accounted for in this way, that he incorrectly assumed for the mathematics of Thales the same external structure as had been given to the mathematics of his own time (about 330—300), in which every proposition was derived by strictly logical steps from previous propositions, definitions and axioms. Thus Eudemos might, for instance, have reasoned as follows: Purely logically, the measurement of the distance between ships at sea, depends upon the congruence theorem mentioned in (4); therefore Thales must have known this theorem and have formulated it explicitly. It is thought that, in reality, Thales did perhaps apply this congruence proposition without being aware of it. Some people even believe that Thales did in no way prove his discoveries, but that he established them empirically, the very opposite of what Eudemos explicitly says in (1)!

The first objection to this view is that Eudemos not only knew the results of the mathematics of Thales, but also their external form, at least to a certain extent; he even knew the terminology which Thales used for the equality of angles. We are hardly justified therefore in simply ignoring his judgment that the geometry of Thales was constructed logically in the same way as that of the later mathematicians — and that evidently is his judgment because otherwise he would not have drawn the conclusion (4) — and certainly not his explicit statement that Thales had proved proposition (1). Only on the basis of better knowledge can one fairly correct an antique historian.

Furthermore, the entire criticism of the statements of Eudemos-Proclus, which has just occupied the stage, stands and falls with the view that Thales stands at

¹ T. Heath, *A History of Greek mathematics I* (Oxford), 1921, p. 131.

the beginning of ancient mathematics. The reasoning is as follows: since he was the first, he must have discovered the theorems empirically. But we know now that mathematics does not start with Thales, but, at least 1200 years earlier, in Babylon. This knocks out every reason for refusing Thales credit for the proofs and for the strictly logical structure which Eudemos evidently attributes to him.

A closer look at the propositions which are ascribed to Thales, shows that, instead of belonging to the early discoveries of mathematics, they form part of a systematic, logical exposition of mathematics. At the start, in the first excitement of discovery, one is occupied with questions such as these: how do I calculate the area of a quadrangle, of a circle, the volume of a pyramid, the length of a chord; how do I divide a trapezoid into two equal parts by means of a line parallel to the bases? These are indeed the questions with which the Egyptian and the Babylonian texts are concerned. It is only later on that the question arises: How do I prove all of this?

This question becomes a central one, especially when the results of the old mathematics, logically unconnected, partly right and partly wrong, are communicated to a younger generation of keenly interested foreigners. At the time of Thales, the Egyptian and the Babylonian mathematics had long been dead wisdom. The rules for computing could be deciphered and shown to Thales, but the train of thought which underlay them, was no longer known. From the Babylonians he might hear that the area of a circle is $3r^2$, while the Egyptians asserted that it is $(\frac{8}{9} \cdot 2r)^2$. How was Thales to discriminate between the exact, the correct recipes for computation, and the approximate, the incorrect ones? Obviously, by proving them, by fitting them into a logically connected system! This is exactly what he did, according to Eudemos and it is exactly at the beginning of such a logical system, that one may expect to find such Irish bulls as: vertical angles are equal, the base angles of an isosceles triangle are equal, a diameter divides a circle into two equal parts, etc.

It follows that we have to abandon the traditional belief that the oldest Greek mathematicians discovered geometry entirely by themselves and that they owed hardly anything to older cultures, a belief which was tenable only as long as nothing was known about Babylonian mathematics. This in no way diminishes the stature of Thales; on the contrary, his genius receives only now the honor that is due to it, the honor of having developed a logical structure for geometry, of having introduced proof into geometry.

Indeed, what is characteristic and absolutely new in Greek mathematics, is the advance by means of demonstration from theorem to theorem. Evidently, Greek geometry has had this character from the beginning, and it is Thales to whom it is due.

The material, from which Greek mathematics was constructed, was not new; the disiecta membra can be dug out of the remnants of the old civilizations. But

the style in which the edifice was erected, was new; it bears witness to the clear thinking of the Greeks, the thinking that does not tolerate obscurities, nor doubts as to the correctness of the conclusions that have been acquired.

From Thales to Euclid.

In his commentary on the first book of Euclid's *Elements* (Friedlein, pp. 65—68), the Neo-Platonist Proclus, who brought a late Indian summer to Plato's Academy, about 450 A.D., gives a rapid survey of the history of geometry from Thales (600 B.C.) to Euclid (300 B.C.):

Thales traveled to Egypt and brought geometry to Hellas; he made many discoveries himself, in many other things he showed his successors the road to the principles. Sometimes he treated questions in a more general manner, sometimes in a more intuitive way. Following him *Mamercus*, brother of the poet *Stesichorus*, occupied himself with geometry; and *Hippias* of Elis says that he acquired reputation as a geometer.

Pythagoras, who came after him, transformed this science into a free form of education; he examined this discipline from its first principles and he endeavoured to study the propositions, without concrete representation, by purely logical thinking. He also discovered the theory of irrationals (or of proportions) and the construction of the cosmic solids (i.e. of the regular polyhedra). After him *Anaxagoras* of Clazomenae dealt with many questions in geometry, and so did the somewhat younger *Oenopides* of Chios, also mentioned by Plato as famous mathematicians in the *Rivals*. Later on, *Hippocrates* of Chios, who discovered the quadrature of the lunules, and *Theodorus* of Cyrene, became famous geometers. Hippocrates was indeed the first of whom it is recorded that he compiled *Elements*.

Through his enthusiastic interest, *Plato*, who came after them, aided a great deal in the development of geometry and of the other mathematical disciplines; it is indeed well known that he filled his writings with mathematical arguments and that he used every opportunity to arouse admiration for mathematics among those who were devoting themselves to philosophy. In his time lived also *Leodamas* of Thasos, *Archytas* of Taras and *Theaetetus* of Athens, who increased the number of theorems and arranged them in a more scientific system.

Younger than *Leodamas* were *Neoclides* and his pupil *Leon*, who added many things to what was known before them. Thus *Leon* was able to develop *Elements*, better prepared as to the number of propositions and the use of proved propositions; and he could formulate restrictions as to the possibility or impossibility of solving a given problem.

Eudoxus of Cnidos, a little younger than *Leon*, on terms of friendship with Plato's circle, enlarged for the first time the number of so-called general theorems, joined three more to the three mean proportionals and continued the researches on the section, begun by Plato, making use of analysis. *Amyclas* of Heraclea, one of Plato's friends, *Menaechmus*, a pupil of *Eudoxus* and a member of Plato's circle, and his brother *Dinostratus*, perfected geometry still further. *Theudius* of Magnesia was reputed to be excellent in mathematics, as well as in the other sciences, because he put together *Elements* admirably and succeeded in generalizing special propositions. And *Athenaeus* of Cyzicus, who lived in the same period, became famous in other parts of mathematics, but especially in geometry. These men assembled in the Academy and conducted their investigations in common.

Hermotimus of Colophon continued the investigations, started by *Eudoxus* and *Theaetetus*; he discovered many of the propositions of the *Elements* and developed a part of the theory of geometrical loci. *Philippus* of Mende, one of Plato's pupils and led by him to an interest in mathematics, not only carried out researches in accordance with Plato's direc-

tions, but also undertook to do things which could, in his judgment, contribute to Plato's philosophy.

It is up to this point that the history of this science was carried by those who recorded events. Not much younger than these men is *Euclid* who composed the *Elements*, in which he collected many of the discoveries of *Eudoxus*, completed many of the results of *Theaetetus*, and supplied irrefutable proofs of the things which had not been proved strictly by his predecessors. This man lived in the days of Ptolemy I. For *Archimedes*, who came immediately after Ptolemy I, mentions *Euclid*. It is also reported that at one time, Ptolemy asked him whether there was not a shorter route through Geometry than by way of the *Elements*, and that he replied, that there is no royal road to geometry. Thus he was younger than the pupils of Plato, and older than *Eratosthenes* and *Archimedes*; for, as stated somewhere by *Eratosthenes*, these were contemporaries.

This "Catalogue of geometers" is obviously largely an extract from *Eudemus'* *History of Mathematics*. For who could have been meant, except *Eudemus*, by "those who recorded events", shortly after *Philippus* of Mende and before *Euclid*?

Eudemus is an excellent source. But unfortunately, the excerpt contains important additions which can not have been taken from *Eudemus*. For example, the passage at the end, about *Euclid*, does certainly not come from *Eudemus*. It is also striking that the entire catalogue bears the stamp of a Platonist, who is anxious to give fullest possible recognition to the merits of Plato and of his school. What is said about Plato and his pupil *Philippus* of Mende, obviously does not come from the shop of *Eudemus*, the pupil of *Aristotle*, who was not as fervent an admirer of Plato as the Neo-Platonist *Proclus*. On the other hand, *Democritus* is not mentioned at all, although he also played an important role in the development of geometry (he wrote several mathematical works and, according to *Archimedes*, he found the volume of the pyramid). This is probably due to the fact that Plato considered the influence of *Democritus* pernicious and would best have liked to burn his works. This indicates that, as a whole, the Catalogue serves a special interest.

The statements about *Pythagoras* must also be viewed in that light. It is possible that what *Proclus* (or whoever wrote the Catalogue before him) found in *Eudemus* concerning *Pythagoras*, did not satisfy him and that he went to other, more doubtful sources (e.g. to the Neo-Pythagorean *Iamblichus*, a fanciful and muddle-headed writer) for supplementary material, in order to be sure not to undervalue the famous and honored *Pythagoras*. In the absence of knowledge of the sources, the value of these statements is very doubtful.

In summary, we can say: What the Catalogue says about *Pythagoras* is unreliable, what it tells about Plato is not new; but the references to real mathematicians before *Euclid*, who were not also philosophers or legendary figures, must stem from *Eudemus* and therefore merits our confidence.

Unfortunately, not much is said about these mathematicians. We must therefore try to find better sources of information concerning the development of geometry after *Thales*.

Concerning the sixth century, we are restricted to scattered notices in various later writers. For the fifth century, there is moreover an important supplementary fragment of Eudemus, concerning the lunules of Hippocrates, which will be discussed when we come to Hippocrates of Chios. Relative to the fourth century, there is another fragment of Eudemus about the duplication of the cube by Archytas, and, furthermore, the mathematical passages in the writings of Plato and Aristotle.

Moreover, the Elements of Euclid are an extremely important source, especially for the mathematics of the Pythagoreans and of the contemporaries of Plato. It will be seen that the manner in which these Elements have been brought together from a variety of separate fragments, leads to important conclusions concerning the origin of these fragments. The reader who wishes to become acquainted with the special character of Greek mathematics, is strongly advised to read the Elements for himself, in Greek or in one of the excellent modern translations.¹

The other sources will now be discussed in chronological order.

Pythagoras of Samos.

At the present time Pythagoras is thought of primarily as a mathematician. In Amsterdam a street is named after him, in the neighbourhood of Archimedes, Newton and Copernicus; his name immediately makes one think of the famous "Theorem of Pythagoras". (plate 13)

It was quite different in antiquity. Herodotus calls him "an important sophist", i.e. teacher of wisdom; and he relates that Zalmoxis, the saint of the Getae, who had, according to a legend, risen from the dead, had been a pupil of Pythagoras. He also tells that the Pythagoreans did not bury their dead in woollen clothing.² This looks more like religious ritual than like mathematics. The Pythagoreans, who were held up to ridicule on the stage, were presented as superstitious, as filthy vegetarians,³ but not as mathematicians.

Pythagoras himself was looked upon by his contemporaries in the very first place as a religious prophet. The much-traveled poet-singer Xenophanes, for example, pokes fun at the Pythagorean doctrine of the transmigration of souls. He tells a story, in which Pythagoras, coming across a scene in which a little dog was being thrashed, said: "Stop the beating, for in this dog lives the soul of my friend; I recognize him by his voice".⁴

Pythagoras was also known as a performer of miracles. All kinds of wonderful tales concerning him were in circulation, as, e.g., that the calf of one of his legs was

¹ In English, with extensive commentaries: T. Heath, *The thirteen books of Euclid's Elements*, Cambridge, 1908 (2nd edition, 1926). In German: C. Thaer, *Ostwald's Klassiker der exakten Wissenschaften* 235, 236, 240, 241 and 243 (1933–37). The two volumes in Dutch of E. J. Dijksterhuis, *De Elementen van Euclides* (Hist. Bibl. ex. wet., I and III 1929–30) give a summary of the text and important commentaries.

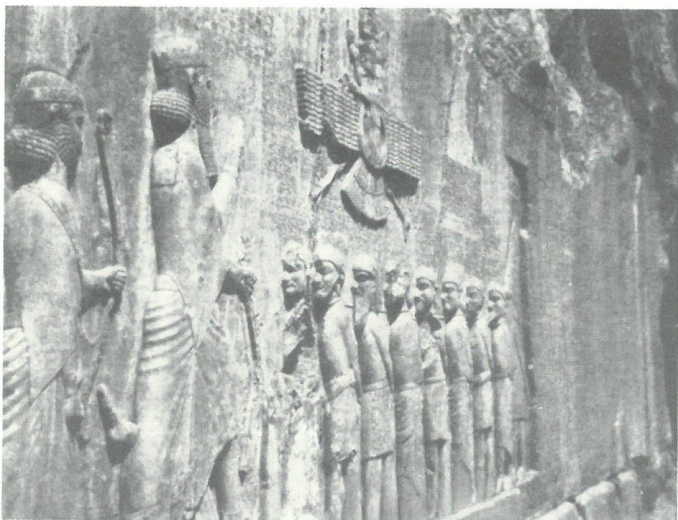
² See e.g., Diels, *Fragmente der Vorsokratiker*, Pythagoras A 1–2.

³ See, e.g., Diels, *Fragmente der Vorsokratiker*, Pythagoreische Schule E.

⁴ H. Diels, *Fragmente der Vorsokratiker*, Xenophanes B 7.

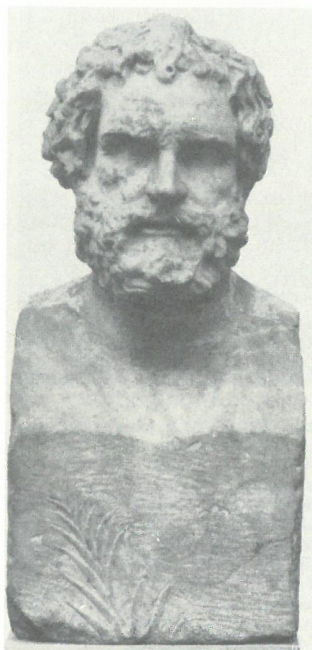


Pl. 11. Croesus on the pyre. Attic Amphora, with figures in red, (500–475 B.C.), from Vulci, Italy; Louvre, Paris. The names of the king (*Κροεσος*, sic!) and of his slave (*Ευθυμος*: Cheerful) are written on the vase. The slave holds two burning torches, with which he lights the pyre. While the light red-brown figures and pyre represent the original surface, being surrounded by the shiny black background, and the inner drawing and the contours have been indicated with black lines, the artist painted the flames blazing up from the pyre and also the burning torches with thin paint over red and black. Croesus, the last king of Lydia (560–546 B.C.) was ruined by the oncoming power of the Persians. According to Herodotus, his life was spared by Cyrus, the king of the Persians; in one of his odes, Bacchylides relates that Croesus ended his own life. This is the version of the saga followed by the painting on this vase. The technique of representing figures in red began about 530 B.C. (cf. Plates 7, 17). The vase is a splendid specimen of the severe style which dominated Greek art during the first half of the 5th century B.C. The archaic profuseness of ornamentation has been abandoned, but the very decorative effect of the ornaments and of the graceful folds recall the archaic art.



PL. 12a. King Darius of Persia as conqueror. Relief on the rock of Behistun (520/19 B.C.). In the center above the nine conquered liar-kings (the nine leaders of the revolt after the death of Cyrus) stand forth the symbol of Ahuramazda.

(British Museum)



PL. 12b. Portrait of a philosopher, probably Thales of Miletus (632–546 B.C.). Marble herma (Ny Carlsberg Glyptothek, Copenhagen). Roman copy from the 2nd century A.D. of a famous original, difficult to date. The name Thales is merely a plausible hypothesis. According to K. Schefold, the palm branch on the herma is an indication for Thales, because the poet Callimachus said of him: "The victory is to Thales". It can certainly not be said that we have here an authentic portrait of the philosopher. It is an "image" of the sage of which the style indicates that is not earlier than the 4th century B.C.

of gold, and that he was seen at two places at the same time. When he crossed a small stream, the river rose out of its bed, greeted him and said: "Hail, Pythagoras."¹

Was Pythagoras a mathematician, a philosopher, a prophet, a saint or a charlatan? He had some of the qualities of each of these. For his adorers he was the personification of the highest divine wisdom, but Heraclitus spoke of "a lot of knowledge without intellect".² He preached the immortality of the soul, he established a strict regime for his followers and founded a brotherhood of believers, the order of Pythagoreans, which later on spread from Croton to a number of Greek cities in Italy and which seems to have played an important role in the political life of these cities.³ After a testing period and after rigorous selection, the initiates of this order were allowed to hear the voice of the Master behind a curtain; but only after some years, when their souls had been further purified by music and by living in purity in accordance with the regulations, were they allowed to see him. This purification and the initiation into the mysteries of harmony and of numbers, would enable the soul to approach the Divine and thus to escape the circular chain of rebirths.⁴

In this period there were many mystery-rites, which promised their followers eternal life. Orphic prophets roamed through Italy and Greece. The cult of Dionysus swept men and women into its wild ecstasy. A more tranquil way to attain immortality existed through initiation into the mysteries of Demeter and of Persephone in Eleusis.

All such initiations began with ritual purifications. After the soul had been freed from earthly blemishes, it could aspire to unity with the Divine. There was a rebirth in God, and thus eternal life was attained.

The Pythagoreans thus have purification and initiation in common with several other mystery-rites. Ascetic, monastic living, vegetarianism and common ownership of goods occur also in other sects. But, what distinguishes the Pythagoreans from all others, is the road along which they believe the elevation of the soul and the union with God to take place, namely by means of mathematics. Mathematics formed a part of their religion. Their doctrine proclaims that God has ordered the universe by means of numbers. God is unity, the world is plurality and it consists of contrasting elements. It is harmony which restores unity to the contrasting parts and which moulds them into a cosmos. Harmony is divine, it consists of numerical ratios. Whosoever acquires full understanding of this number-harmony, he becomes himself divine and immortal.

Music, harmony and numbers — these three are indissolubly united according

¹ For these legends, compare especially Is. Lévy, *Les sources de la légende de Pythagore*, Paris, 1926.

² Diels, *Fragmente der Vorsokratiker*, Heraclitus B 40.

³ A. Delatte, *Essai sur la politique Pythagoricienne*, Liège 1922; K. v. Fritz, *Pythagorean Politics in Southern Italy*, New York, 1940.

⁴ For the religious philosophy of Pythagoras and his followers, see P. Boyancé, *Le culte des muses chez les philosophes grecs*, Paris 1936.

to the doctrine of the Pythagoreans. All three are among the essential elements of the Pythagorean system of education and of its path for the elevation of the soul. According to Heraclides of Pontus, Pythagoras said that "Beatitude is the knowledge of the perfection of the numbers of the soul". Mathematics and number mysticism mingle fantastically in the Pythagorean doctrine. Nevertheless, it was from this mystical doctrine that the exact science of the later Pythagoreans developed.

*The travels of Pythagoras.*¹

Mystery-rituals and number mysticism come from the orient. For this reason it is not surprising that, even in antiquity, it was believed that Pythagoras had made extensive journeys to practically all Oriental countries. The orator Isocrates, whom many later writers copy, says that he journeyed to Egypt. The story then relates that, in Egypt, Cambyses, the Persian conqueror, made him a prisoner and carried him to Babylon. Here the priests initiated him into the mysteries; according to Iamblichus, he remained there for seven years, during which time he learned from the Magi the theory of numbers, the theory of music and the other sciences. Iamblichus (Introductio in Nicomachi Arithm.) also reports that he learned from the Babylonians the "golden proportionality"

$$A : H = R : B,$$

in which H and R are, respectively, the harmonic and the arithmetic mean of A and B .² Older writers merely mention, that he came in contact with "Zaratas, the Chaldean" (i.e. Zarathustra) or that he received lessons from the Chaldeans.

Of how much value are these various statements? There was a time when everything was believed without any hesitation. This was followed by a hypercritical tendency, which pushed all these later tales aside as being entirely unreliable fantasies.

At the present time, since the work of Delatte and Rostagni, the late-antique tradition is again given greater credit. It has been recognized that these accounts rest, to a large extent, upon an old tradition recorded by Aristotle and his pupil Aristoxenus, as well as by historians like Dicaearchus and Timaeus of Tauromenion towards the close of the fourth century. At all accounts, the travel records show that, already in antiquity, Pythagorean and Oriental wisdom were considered as being related.³

¹ A good survey of the travel accounts, with indication of the sources, is found in Th. Höpfner, *Orientalische und griechische Philosophie, Beihefte zur alten Orient* 4 (1925).

² This proportionality plays an important role in the Pythagorean theory of music (see my article in *Hermes* 78). The two means may be defined (in modern notation) by

$$R = \frac{A+B}{2}, \quad H = \frac{2AB}{A+B},$$

or (in classical form) by:

$$R - A = B - R, \quad (H - A) : A = (B - H) : B.$$

It is easy to prove the golden proportionality $A : H = R : B$ by use of either of these definitions.

³ Frequently Pythagoras is represented wearing an oriental turban.

In our further account we shall have occasion to point out a number of instances of the connections between Babylonian mathematics and the science of the Pythagoreans. I am convinced that it was Pythagoras himself who transmitted Babylonian scholarship; did he, as an Ionian, not stand much closer to the source of this old wisdom than the Pythagoreans in Italy?

Not only in pure mathematics, but also in the theory of music and in astronomy, Pythagoras must have learned quite a few things from the Babylonians. Did Heraclitus not speak of "a lot of knowledge without intelligence"? This contemptuous remark cannot refer to a logically constructed theory of numbers and a geometry such as we find in the writings of the later Pythagoreans. But, if Pythagoras gathered into one lump, all kinds of half-assimilated learning about the gods and the stars, about musical scales, sacred numbers and geometrical calculations, and proclaimed such an omnium-gatherum to his followers as divine wisdom in a prophetic manner, then Heraclitus' ridicule, as well as the veneration of mystics, such as Empedocles, become entirely understandable.

*Pythagoras and the theory of harmony.*¹

When a string or a flute is shortened to half its length, the tone produced is an octave higher. Similarly, the ratios 3 : 2 and 4 : 3 correspond to the intervals of the fifth and the fourth. It was of eminent importance for the Pythagoreans to have learned that the most important consonant intervals could be obtained in this manner by ratios of the numbers 1, 2, 3, 4; it confirmed their general thesis "Everything is number", or "Everything is ordered by numbers". The numbers 1, 2, 3, 4 themselves constituted the famous "tetractys". A very old saying runs: "What is the oracle of Delphi? The tetractys! For it is the scale of the sirens".

Geometrically the tetractys was represented by the "perfect triangle", arithmetically by the "triangular number" $1 + 2 + 3 + 4 = 10$. According to Lucian, Pythagoras asked some one to count; when he had said 1, 2, 3, 4, Pythagoras interrupted him as follows: "Do you see? What you take to be 4, is 10, a perfect triangle, and our oath". For the Pythagoreans pledged themselves by oath to "him, who had entrusted to our soul the tetractys, the source and the root of eternal nature". All these sayings and these forms of oaths are characteristic of antiquity; it seems therefore necessary to ascribe to Pythagoras himself the Tetractys, the triangular numbers and the numerical ratios of the consonant intervals.



Fig. 25.

On his deathbed, Pythagoras urged his followers to practice on "the monochord". Gaudentius records the following history of this instrument: Pythagoras stretched a string over a straight edge and divided it into twelve parts. When he shortened the string from 12 to 6, or to 8 or 9, i.e. in the ratios 2 : 1, 3 : 2 or 4 : 3, he obtained tones which were an octave, a fifth or a fourth higher.

¹ See B. L. v. d. Waerden, *Die Harmonielehre der Pythagoreer*, *Hermes* 78 (1943).

The same 4 numbers 6, 8, 9, 12 are found again in practically all Pythagorean and Neo-Pythagorean writers on music. All these writers obtain the middle terms 9 and 8 as the arithmetic and the harmonic means of the extreme terms 12 and 6. Usually the number 12 was assigned to the highest note and 6 to the lowest note, these numbers being not proportional to the lengths of the strings, but inversely proportional. What was the empirical significance of these numbers? Apparently the Pythagoreans did not care very much whether they represented the lengths of the strings, or their tensions or velocities. The most important thing was that the correct ratios of the harmonic intervals appeared, such as $12 : 9 = 8 : 6$ for the fourth and $12 : 8 = 9 : 6$ for the fifth, as the Master had taught.

The tradition which credits Pythagoras with the calculation of the intervals of the diatonic scale, the whole tone ($9 : 8$) and the major semi-tone or the "leimma" (256 : 243), also merits confidence, for these ratios can be obtained by successive division from those for the octave ($2 : 1$), the fifth ($3 : 2$) and the fourth ($4 : 3$):

$$\begin{aligned} 3/2 : 4/3 &= 9/8, \\ 4/3 : 9/8 &= 32/27, \\ 32/27 : 9/8 &= 256/243. \end{aligned}$$

Pythagoras and the Theory of Numbers.

As magic and number magic belong together, so do mysticism and number mysticism. Every magician utilizes the magic power of words and of numbers, every superstitious person knows sacred numbers, lucky numbers, etc. These things had of old played an important role among the Babylonians and the Magi, and among the Pythagoreans as well. For example, they called 10 a perfect number, they looked upon even and odd as the roots of all things; and, as Aristotle informs us¹, abstract concepts such as "justice" were identified with definite numbers. The even numbers were called feminine, the odd ones masculine, and the number 5, the sum of the first masculine and the first feminine number, was taken as a symbol of marriage, and so forth.

The following example, taken from Plutarch (Isis and Osiris 42) has some mathematical interest:

The Pythagoreans also have a horror for the number 17. For 17 lies exactly halfway between 16, which is a square, and the number 18, which is the double of a square, these two being the only two numbers representing areas, for which the perimeter (of the rectangle) equals the area.

Interpretation: Let x and y be integers which measure the sides of a rectangle and let the area, the "plane-number" xy be equal to the perimeter

$$xy = 2x + 2y.$$

Then one can express the unknown y in terms of x , as is done in the Babylonian

¹ Aristotle, *Metaphysics* A 5; see also Diehl, *Fragmente, Pythagoreische Schule* B 4.

lesson text AO 6770 (see p. 74):

$$y = \frac{2x}{x-2} = 2 + \frac{4}{x-2}.$$

In order that y be an integer, $x-2$ must be a divisor of 4, so that we must have

$$\begin{array}{llll} \text{or} & x-2=1, & x=3, & y=6, & xy=18, \\ & x-2=2, & x=4, & y=4, & xy=16, \\ \text{or} & x-2=4, & x=6, & y=3, & xy=18. \end{array}$$

This gives the two possibilities mentioned by Plutarch.

Neo-Pythagoreans, like Nicomachus of Gerasa (100 A.D.) and Iamblichus (300 A.D.) revel in this kind of number-mysticism. In his *Arithmetic Theology*, Iamblichus expatiates broadly on the mystical and divine significance of numbers. The purpose of the layman's Introduction to Arithmetic of Nicomachus¹, is to explain in a manner intelligible to every one, the wonderful and divine properties of numbers. In a very entertaining way he tells about triangular numbers, about square, rectangular and polygonal numbers, about gnomonic numbers and spatial numbers, about ratios and the distinction between multiple and epimoric² ratios, etc., about prime numbers, about geometric progressions, all illustrated by numerous examples, but never accompanied by proofs. He knew his public; he was well aware of the fact that his readers wanted to be initiated into the mystery of numbers, but that they would be bored by prosaic proofs, which would moreover deprive these things of a large part of their mystery.

Another source for the Pythagorean theory of numbers is found in the three arithmetical books of Euclid (Books 7, 8 and 9 of the Elements). But these are purely scientific, there is nothing left of the mystery, everything is carefully and neatly proved. Although Nicomachus lived four centuries after Euclid, he makes nevertheless a much more primitive impression. He is much closer to the original number-mysticism of Pythagoras and his school. It looks to me as if Pythagoras, the Prophet, clothed his wisdom in a mysterious oracular form, and that only much later, the Pythagoreans made the theory of numbers into an exact science.

For this reason we use Nicomachus as our source for the theory of numbers of Pythagoras and of his immediate followers. In the next chapter, in which we shall be concerned with the mathematics of the later Pythagoreans, we shall draw chiefly on Euclid.

Perfect numbers.

It was considered as something very remarkable by the Pythagoreans, when a number equals the sum of its proper divisors, such as

$$6 = 1 + 2 + 3.$$

¹ English translation with extensive commentary by Martin Luther d'Ooge, New York 1926.

² The word "epimoric" is introduced as the English equivalent of the Greek *ἐπιμόριος* (superparticularis). Two numbers, A and B , are in "epimoric ratio", if the difference $B-A$ is a part of A and of B , i.e., if A and B are multiples of $B-A$. E.g., the ratio of $n+1$ to n is an epimoric ratio. In the discussion of proportionalities in the next Chapter (see p. 110), we shall see that Archytas proved that every epimoric ratio can be reduced to the form $(n+1) : n$.

They called such numbers "perfect". Nicomachus gives the four instances 6, 28, 496 and 8128. He also gives the following general rule, which is proved in Euclid (Elements IX 36): When the sum

$$1 + 2 + 2^2 + \dots + 2^n = p$$

is a prime number, then $2^n p$ is a perfect number. For example, $1 + 2 + 4 = 7$ is a prime number, hence $4 \times 7 = 28$ is a perfect number.

The proof makes use of the formula for the sum of a geometric progression

$$1 + 2 + \dots + 2^{n-1} = 2^n - 1,$$

which is already found in Pythagorean texts. It is therefore very probable that Pythagoras knew this formula.

The numbers mentioned by Nicomachus are $2(2^2 - 1)$, $2^2(2^3 - 1)$, $2^4(2^5 - 1)$ and $2^6(2^7 - 1)$. The next perfect number is $2^{12}(2^{13} - 1)$, the largest known one is $2^{126}(2^{127} - 1)$. For further details, see L. E. Dickson, History of the Theory of Numbers, I.

Similar interest is attached to

Amicable numbers

like 220 and 284, each of which equals the sum of the proper divisors of the other:

$$\begin{aligned} 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 &= 284, \\ 1 + 2 + 4 + 71 + 142 &= 220. \end{aligned}$$

When Pythagoras was asked what a friend is, he said "a second I" and he mentioned the amicable numbers 284 and 220. The Babylonians also enjoyed such tricks based on the interchange of numbers.

In Nicomachus one finds many more links with Babylonian arithmetic than in Euclid. He pays especial attention, just as the Babylonians did, to the squares n^2 and the cubes n^3 , and he considers, among the paralleloiped numbers, abc , especially those of the form $n^2(n \pm 1)$, of which the Babylonians had constructed tables.

This leads us to one of the favorite topics of Nicomachus:

Figurate numbers.

He knew triangular numbers, square numbers n^2 , rectangular numbers $n(n+1)$,

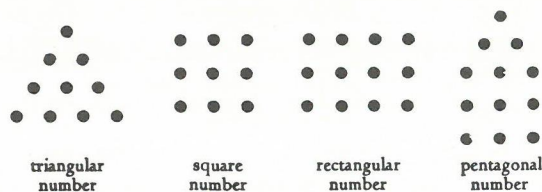


Fig. 26.

pentagonal numbers, etc.

The marvellous properties of these numbers may be read in Nicomachus (in the translation of d'Ooge, see p. 97). They follow from the summation of simple arithmetical progressions such as

- | | | |
|-----|---|-----------------------|
| (1) | $1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1)$ | (triangular number), |
| (2) | $1 + 3 + 5 + \dots + (2n-1) = n^2$ | (square number), |
| (3) | $2 + 4 + 6 + \dots + 2n = n(n+1)$ | (rectangular number), |
| (4) | $1 + 4 + 7 + \dots + (3n-2) = \frac{1}{2}n(3n-1)$ | (pentagonal number). |

For example, to deduce (2) from a diagram, one observes that a square may be divided into a smaller square and a carpenter's square or "gnomon". By repeating this division, one concludes that the number of dots in a square is the sum of "gnomonic numbers", i.e. of odd numbers $1 + 3 + 5 + \dots$

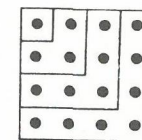


Fig. 27.

Less immediate, but nevertheless also a consequence of the formula for the sum of an arithmetic progression, is the following rule for the sums of one, two, three... successive odd numbers, found in Nicomachus:

$$\begin{aligned} 1 &= 1^3 \\ 3 + 5 &= 2^3 \\ 7 + 9 + 11 &= 3^3 \end{aligned}$$

From this, one obtains readily a rule for the sum of cubes, known to the Roman surveyors¹ (undoubtedly from a Greek source)

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left\{ \frac{1}{2}n(n+1) \right\}^2.$$

An analogous formula for $1^2 + 2^2 + \dots + n^2$ we have already encountered in a Babylonian text.

Pythagoras has been given credit for a rule for determining numerical solutions of the indeterminate equation

$$(5) \quad x^2 + y^2 = z^2,$$

in the form

$$x = \frac{1}{2}(m^2 - 1), \quad y = m, \quad z = \frac{1}{2}(m^2 + 1),$$

where m is an odd integer (Proclus in Euclid I, p. 487).

It can be verified by means of a diagram that these values do indeed satisfy (5). To a square whose side is $x = \frac{1}{2}(m^2 - 1)$, we add a gnomon of width 1, thus obtaining a square of side $z = \frac{1}{2}(m^2 + 1)$. The area of the first square, of the gnomon and of the final square are equal exactly to x^2 , y^2 and z^2 , so that (5) follows. The special case $m = 3$ leads to the right triangle whose sides are 3, 4, 5, discovered by Pythagoras, according to Vitruvius. We have already observed that the Babylonians knew a more general formula for the determination of right triangles with integral sides.

It is hardly thinkable that all these coincidences between the Babylonian and the

¹ See M. Cantor, *Geschichte der Mathematik* I, p. 559.

Pythagorean mathematics are only a matter of chance. Apparently Pythagoras has taken from the Babylonians various remarkable things about numbers and about their mystical significance. As we will see in the next chapter, his disciples continued the investigation in a more systematic manner and built them into a logically consistent system.

Pythagoras and Geometry.

Who does not think, when he hears the name Pythagoras, of the famous theorem about the square of the hypotenuse? Alas, the proofs of the connection between the two are extremely doubtful.¹ In his commentary on this proposition (Euclid I 47), Proclus says, very indefinitely "If we listen to those who wish to recount ancient history, we may find some of them referring this theorem to Pythagoras, and saying that he sacrificed an ox in honor of the discovery". Plutarch quotes a distich: "When Pythagoras discovered his famous figure, for which he sacrificed a bull", and he says that the figure in question is either that of the square on the hypotenuse or that of the adaptation of areas.² In another place, the same Plutarch says however, that the bull was sacrificed in connection with the problem of constructing a figure of the same area as another figure and similar to a third one.³ But Vitruvius is of the opinion that the bull fell victim to the discovery of the right triangle whose sides are 3, 4, 5.

But the entire story is an impossible one, because Pythagoras was strongly opposed to the killing and sacrificing of animals, of cattle especially.

It has been thought, that the theorem can not have been known in the days of Pythagoras, during the first stages of the development of geometry. But this objection loses force now that we have found it applied even 1200 years earlier in the cuneiform texts. It is quite possible that Pythagoras became acquainted with the theorem in Babylon. That is about all we can properly say about it.

Proclus' catalogue also ascribes to Pythagoras the construction of the regular polyhedra. But a scholium in the thirteenth book of the *Elements* says that the Pythagoreans knew only three regular polyhedra, viz. the cube, the tetrahedron and the dodecahedron, and that it was Theaetetus who first discovered the other two. Precisely because this scholium directly contradicts the tradition which used to ascribe to Pythagoras anything that came along, it is given greater credit nowadays than the catalogue. An Etruscan dodecahedron, made of soapstone, was found near Padua, dating from before 500 B.C.⁴ It is therefore quite possible that Pythagoras was acquainted with the cube, the tetrahedron and the dodecahedron.

The faces of a dodecahedron are regular pentagons. The diagonals of such a pentagon form a star-pentagon.

¹ Collected by Allman, *Greek geometry from Thales to Euclid* (1889), p. 26.

² Non posse suaviter vivi sec. Epicurum IX.

³ Plutarch, *Quaestiones Convivii*, VIII, Quaest. 2, 4.

⁴ See F. Lindemann, *Sitzungsber. Bay. Akad. Wiss.* 26 (1897), pp. 625–768.

This figure, the symbol of health, served as a distinctive mark among the Pythagoreans. One of them, lying on his deathbed in a foreign country and unable to pay the man who had taken care of him until the end, advised this man to paint the star-pentagon on the outside wall of his house, so that any Pythagorean who might ever pass the house, would make inquiries. As a matter of fact, a Pythagorean did come past many years later, and the man was richly rewarded.

For the construction of the star-pentagon, the Pythagoreans could make use of the fact, that each of these 5 lines divides every other one in mean and extreme ratio, i.e. that the shorter piece $AK = a - x$ is to the larger one $KB = x$ as the larger piece is to the whole segment $AB = a$. The correctness of this statement follows at once from the similar triangles AIB and KFB . The proportionality $BF : BK = BI : BA$, or

$$(a - x) : x = x : a,$$

leads to the quadratic equation

$$x^2 = a(a - x),$$

which the Pythagoreans knew well how to solve by their method of "adaptation of areas", to be discussed in the next chapter; we are already acquainted with the fact that the Babylonians also knew how to solve this equation. Again we do not know whether the Pythagoreans actually constructed the star-pentagon in such a way; they certainly had the knowledge for doing it.

The star-pentagon is also found on Babylonian drawings, another point of contact between Babylon and the most ancient Pythagorean mathematics.

Little more is to be said about Pythagorean geometry. We know still less about

The astronomy of the Pythagoreans.

According to Alexander Polyhistor¹, Pythagoras assumed the earth to be spherical in shape, placed at the center of the cosmos, and he knew the proper motions of sun, moon and planets to be opposite to the diurnal motion of the fixed stars.

By no means can we put unlimited faith in the compilations of Alexander, which he derives from certain "Pythagorean hypomnemata". They contain among other things, the statement that the numbers are produced from the unit and the indefinite duality. But Aristotle asserts explicitly, that this idea is Platonic and not Pythagorean. The Pythagoreans generated the numbers from the unit and the unlimited, and Plato replaced the unlimited by the indefinite duality of "large and

¹ See Diogenes Laërtius VIII 25. Compare J. E. Raven, *Pythagoreans and Eleatics* (Cambridge 1948), p. 160, concerning this much-discussed fragment.

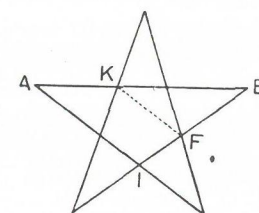


Fig. 28. Pentagramma.

small" (Aristotle, *Metaphysics* A 6). The compilation on which Polyhistor draws, was therefore written after Plato and by some one who was not able to discriminate between the doctrine of Pythagoras and that of Plato. Furthermore, as becomes clear from his terminology, the writer was also influenced by the ideas of the Stoa. Certainly not a reliable witness!

Nonetheless, the statement concerning the proper motions can be credited, particularly since Alcmaeon, a little younger than Pythagoras, with ideas closely related to those of Pythagoras, also speaks of the motion of the planets from West to East, contrary to that of the fixed stars.¹ Indeed, this view is already found among the Babylonians.

Summary.

Taking it all in all, we know something about the theory of music of Pythagoras, next to nothing about his theory of numbers, still less about his astronomy and, properly speaking, nothing at all about his geometry. A sorry result!

Are there no other sources of information for the development of mathematics in these ancient days? Is it not necessary that the architects, who built their marvelous temples in Ionia and in Southern Italy — the temple of Artemis in Ephesus was one of the seven wonders of the world — were also skilled in geometry?

Speaking frankly: we do not know. It is possible to build beautifully and on a large scale even without mathematics — witness the accomplishments of the Romans. The Roman architect Vitruvius describes very carefully how to build a hall of columns, but mathematics is not involved.

There is one case however in which we know something about the mathematical preparation of a Greek building construction, namely in the case of

The tunnel on Samos.

In about 530 Eupalinus constructed an aqueduct, at the request of the powerful tyrant Polycrates straight across the lime-stone of the mountain Castro on the island of Samos.

Herodotus (III 60) describes this work as follows:

I have written thus at length of the Samians, because they are the makers of the three greatest works to be seen in any Greek land. First of these is the double-mouthed channel pierced for a hundred and fifty fathoms through the base of a high hill; the whole channel is seven furlongs long, eight feet high and eight feet wide; and throughout the whole of its length there runs another channel twenty cubits deep and three feet wide, wherethrough the water coming from an abundant spring is carried by its pipes to the city of Samos. The designer of this work was Eupalinus, son of Naustrophus, a Megarian...

When German archaeologists looked for antiquities on Samos in 1882, they found the tunnel, in a state of good preservation, exactly as described by Herodotus: 1 Kilometer in length, 7 feet high and wide, with a deep ditch con-

¹ H. Diels, *Fragmente, Alcmaeon* A 4.

taining the pipes, with vertical vents for changing the air and for cleaning away rubble, and with niches, used by the working men to place their lamps. The ditch has a depth of 7 feet at the upper end and of 27 feet at the lower end; it was probably dug because the drop which had originally been planned, turned out to be too small.¹ (Plate 14a)

But now we come to the most important point: apparently the digging of the tunnel proceeded from both ends. The working men approaching from two directions, met at the center with an error of about 30 feet horizontally and 10 feet vertically. A splendid accomplishment!

When king Ezechias of Judaea had a similar aqueduct constructed through the rocks near Jerusalem about the year 700, his workers had to keep check on the direction in which the work was proceeding in a very primitive way, by means of vertical shafts from the top; the result was a zigzag tunnel, twice as long as the distance between the ends.²

Eupalinus did much better: his tunnel was essentially a straight line. How could this be accomplished?

The answer is found in the "Dioptra" of Heron of Alexandria (60 A.D.). Heron first describes a dioptra, i.e. a horizontal bar mounted so as to rotate, with two sights, which made it possible e.g. to sight a right angle in a plane. Then he proposed the following problem (no. 15): "To cut through a mountain $AB\Gamma\Delta$ in a straight line, the openings B and Δ of the tunnel being given." He draws an arbitrary line BE in the plane, then, by means of the dioptra, the perpendicular EZ , next ZH perpendicular to EZ , and thus successively $H\Theta$, ΘK , $K\Lambda$. Now he moves the dioptra along the line $K\Lambda$, until the point Δ is sighted in a right angle. If this takes place at M , then $M\Delta$ will be perpendicular to $K\Lambda$.

Now drop the perpendicular ΔN from Δ on EB . Then ΔN can be determined from EZ , $H\Theta$ and KM ; similarly BN can be found from BE , ZH , ΘK and $M\Delta$ by addition and subtraction. Consequently the ratio $BN : \Delta N$ is known; Heron gives for this ratio the value 5 : 1. Now, he constructs right triangles $BO\Xi$ and $\Delta\Pi\Gamma$ whose legs have the same ratio 5 : 1. The hypotenuses of these triangles

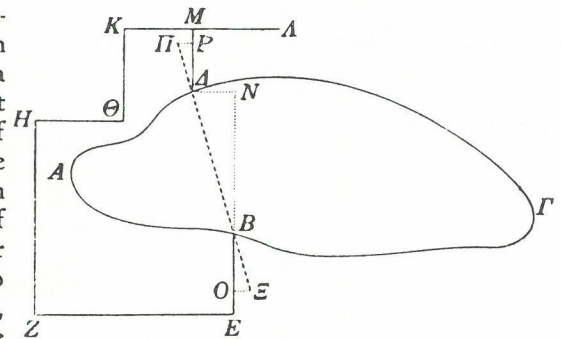


Fig. 29.

¹ See E. Fabricius, *Mitt. D. archäol. Inst. Athens* 9 (1884), p. 165.

² See J. Bidez, *Eos, Platon et l'Orient*, Brussels 1945, p. 12.

then give the directions in which the digging must take place. "If the tunnel is dug in this manner", Heron concludes, "then the laborers will meet."

In no. 16, Heron also deals with the problem of making vertical shafts, to meet the tunnel, supposed to be rectilinear. Such shafts are indeed present on Samos.

Antique measuring instruments.

Eupalinus must also have been able to determine differences in altitude. It is quite likely that he did this like Heron and as we still do it: proceeding from point

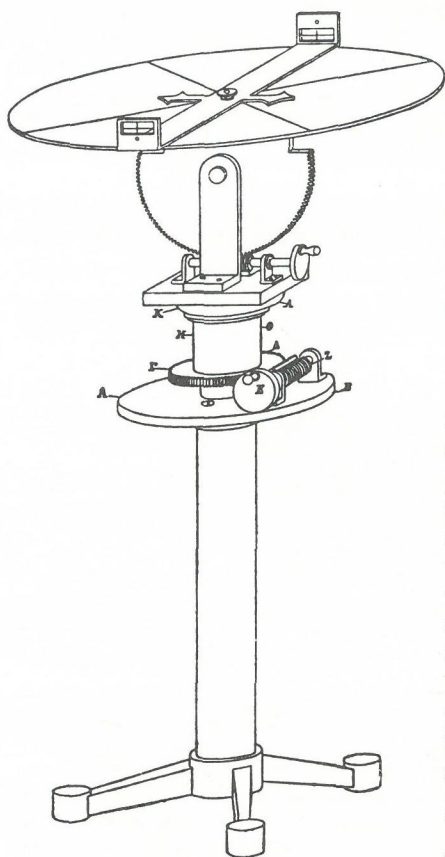


Fig. 30. Heron's dioptra, as reconstructed by Schöne from Heron's own description.

to point with vertical measuring rods and a horizontal sight. There were no telescopes; the sighting instrument must have been a dioptra. Heron obtains a level by means of communicating tubes.

The reader may think that this instrument is too clever for the era of Eupalinus; we are frequently inclined to think of the ancient Greeks as being more primitive than they actually were. We must remember that Anaximander of Milete, the pupil of Thales, had a workshop in which, among other things, wooden celestial spheres were manufactured. Anaximander also was commissioned to place a gnomon on the market place in Sparta, i.e. a vertical sundial, supplied of course with all the cleverly constructed hourlines and monthlines, which belonged there according to Vitruvius.¹ This happened around 560, about thirty years before the construction of the tunnel on Samos.

A knowledge of these things makes it possible to form a clearer picture of scientific life in the sixth century than is possible merely on the basis of the vague reports concerning Thales and Pythagoras.

We can also better appreciate now the Proclus tradition, which says that Pythagoras "transformed mathematics into a free education". "Free education"

¹ See Pauly-Wissowa, *Realencyklopaedie des klassischen Altertums*, article *Horelogium*.

man, in contrast with training for a trade. The Ionians cultivated mathematics not only for its inherent interest, but also for the sake of the practical applications. Surveyors and architects, such as Eupalinus, had to know something about geometry, and the training of an instrument maker in the workshop of Anaximander undoubtedly involved astronomy. But Pythagoras freed mathematics from these practical applications. The Pythagoreans pursued mathematics as a kind of religious contemplation, as a way to approach the eternal Truth.