

THEAETETUS AND THE HISTORY OF THE THEORY OF NUMBERS

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This famous passage has given rise to much discussion and some perplexity. Theodorus the mathematician is represented by Theaetetus as proving the irrationality of the square roots of the (non-square) numbers from 3 to 17:

'He took the separate cases up to the root of 17 square feet; and there, for some reason, he stopped.' (Transl. Cornford.)

The passage is of great importance in the history of Greek mathematics for more than one reason. Theaetetus is said to have generalized the proof of the irrationality of square roots of non-square integers; and thus his connexion with this passage is important because Plato here obviously implies that Theodorus was not giving a generalized proof—otherwise, why should he go up to 17? If Theodorus did not know the generalized proof, he clearly had to proceed by enumeration and proof of particular cases. But, if so, why stop at 17? The simplest explanation seems to be that he had to stop somewhere, and 17 was as good a place as any.¹ However, more interesting questions remain: what was Theodorus' proof? And how was this generalized by Theaetetus? And, perhaps most interesting of all, why did the generalization have to wait for Theaetetus? Did the Pythagoreans, who seem to have been very excited² about the discovery of the irrationality of the square root of 2, never try to extend their inquiries into a field of such great interest to them? Did Theodorus, who knew proofs applicable to particular cases, never try to formulate the proof in a general way?

Theodorus (in Plato's dialogue) started from $\sqrt{3}$, not from $\sqrt{2}$. Why? Obviously, because that case had been dealt with before, by Pythagoras or the Pythagoreans.³ By the time of Theodorus it was known. There is indeed no

¹ See Hardy and Wright, *Introduction to the Theory of Numbers* (3rd ed.), p. 43. See also Appendix below.

² Their excitement was probably caused not so much by joy at the great discovery, as by its disturbing implications for their metaphysical doctrine: here was a case where no number relationship could be established. The legend invented by the Pythagoreans to the effect that the man who first divulged the discovery died in a shipwreck is a sufficient indication of the disturbance that the 'irrational' caused in the Pythagorean school. See Schol. in Eucl. *Elem.* 10, p. 417 Heiberg: τῶν γὰρ Πυθαγορείων λόγος τὸν πρῶτον τὴν περὶ τούτων θεωρίαν εἰς τοῦμφανὲς ἐξαγαγόντα ναυαγίῳ περιπεσεῖν. The Scholiast's suggestion concerning the meaning of this story is not without interest: ἴσως ἤνιπτοντο ὅτι πᾶν

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³ See Cantor, *Vorlesungen z. Gesch. d. Mathem.* i. 154-5. The claim made by K. von Fritz ('The Discovery of Incommensurability by Hippasus of Metapontum', *Annals of Mathematics*, xlv [1945], 245) that 'the tradition is unanimous in attributing the discovery to a Pythagorean philosopher by the name of Hippasus of Metapontum' seems to me to be devoid of all foundation. So far from being unanimous, the tradition is, I believe, non-existent. I know of no single ancient author

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need to insist that this was all that was known. It could be argued (see, e.g., Allman, *Greek Geometry from Thales to Euclid*, pp. 213–14, and Zeuthen, *Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger*, 1910, p. 418, and 1915, p. 339) that the Pythagoreans knew more than the irrationality of $\sqrt{2}$; that if they knew the case of $\sqrt{2}$ they must have seen or at least suspected that there were other cases of irrational square roots; that perhaps even the proofs given by Theodorus of the particular instances up to $\sqrt{17}$ may have been Pythagorean doctrines rather than his own personal contributions. On the other hand (as is pointed out by Zeuthen, loc. cit.), the Pythagoreans were perhaps content with $\sqrt{2}$: it was enough to establish the existence of irrationals. Or perhaps we should rather say that they were shocked; and that that shock may have inhibited further inquiry. However that may be, we need not decide this question. For, as I shall try to show, the originality of Theodorus may have consisted not in the extension of the theory of irrationality from $\sqrt{2}$ to other particular cases (even if that is to be ascribed to him), but in the realization that the traditional method did not lend itself to automatic generalization, and in the determination of the preconditions for a generalized proof on the basis of the traditional method.

It is, then, tolerably certain that at least $\sqrt{2}$ and the proof of its irrationality was known. And if the Pythagoreans knew or suspected the existence of more than that single case, such extensions may not have been so well known outside the Pythagorean school. There is this further point to be noted: scholars¹ have argued as if Plato represented Theodorus as having made a new and very exciting discovery; but an unprejudiced reader of the dialogue will not, I think, be immediately convinced of the truth of that impression. It is at least conceivable that Plato means no more than that Theodorus was demonstrating not a new discovery of his own, but something which though known to professional mathematicians might be new and interesting to his young hearers.

From a passage in Aristotle we know what the traditional proof was (*Prior Analytics* I. 23. 41^a23–30) πάντες γὰρ οἱ διὰ τοῦ ἀδυνάτου περαίνοντες τὸ μὲν attributing the discovery to Hippasus. We have indeed the legend mentioned by Iamblichus (*Vit. Pyth.* 18. 88, p. 52 Deubner; cf. *De comm. math. sc.* 25, p. 77 Festa) to the effect that Hippasus was drowned at sea as a punishment for divulging the Pythagorean secret of how to inscribe a dodecahedron in a sphere; and this legend is obviously confused with the similar legend connected with the divulgation of the discovery of irrationality (see preceding note). But that confusion does not afford any reason for suggesting that Hippasus, because he is said to have suffered the same fate as the man who published the secret of irrational numbers, must have been identical with that man. And, of course, there is even less justification for identifying Hippasus not only with the divulger but also with the discoverer of irrationals. It may be of interest that this confusion is one of which Iamblichus already knew: see *Vit. Pyth.* 34. 247, p. 132 Deubner, where we are told the story of the punishment of the man who had divulged the secret of the dodecahedron inscribed in the sphere. It is followed by the remark that ‘some say that it was the man who divulged the secret of irrationality and incommensurability who suffered this (punishment)’. It may perhaps be useful to mention here that what v. Fritz calls an ‘obviously . . . corrupt reading in some manuscripts’, namely ἀλόγων in the Eudemean Summary, in Proclus, in *Eucl.*, p. 65, seems in fact to be not only the right reading but also the unanimous tradition of all manuscripts. There does not seem to be any manuscript evidence at all for ἀναλόγων or ἀναλογίων which are apparently based on no more than a note in August’s *Euclid* (i. 290): ‘alii ἀναλόγων’. See Heath, *The Thirteen Books of Euclid’s Elements*, i. 351.

¹ e.g. Zeuthen, ‘Sur la constitution des livres arithmétiques des Éléments d’Euclide et leur rapport à la question de l’irrationalité’, in *Oversigt over det Kgl. Danske Videnskabernes Selskabs Forhandlinger*, 1915. See also Heath, *Greek Mathematics*, i. 206.

ψεῦδος συλλογίζονται, τὸ δ' ἐξ ἀρχῆς ἐξ ὑποθέσεως δεικνύουσιν, ὅταν ἀδύνατόν τι συμβαίνει τῆς ἀντιφάσεως τεθείσης, οἷον ὅτι ἀσύμμετρος ἢ διάμετρος διὰ τὸ γίνεσθαι τὰ περιττὰ ἴσα τοῖς ἀρτίοις συμμέτρου τεθείσης. τὸ μὲν οὖν ἴσα γίνεσθαι τὰ περιττὰ τοῖς ἀρτίοις συλλογίζεται, τὸ δ' ἀσύμμετρον εἶναι τὴν διάμετρον ἐξ ὑποθέσεως δεικνυσιν, ἐπεὶ ψεῦδος συμβαίνει διὰ τὴν ἀντίφασιν. See also 50^a37 οἷον τεθείσης τῆς διαμέτρου συμμέτρου τὸ τὰ περιττὰ ἴσα εἶναι τοῖς ἀρτίοις. The proof here alluded to is given in Euclid 10. 117 (which may be an interpolation). See Heiberg, *Euclid*. iii. 408, app. 27.¹ It may be represented as follows:

Theorem: $\sqrt{2}$ is not a rational number.

Proof: Suppose $\sqrt{2}$ is rational. Then it must be of the form m/n , this fraction being in its lowest terms. Then, if $m/n = \sqrt{2}$,

$$\frac{m^2}{n^2} = 2, \text{ and so } m^2 = 2n^2.$$

Therefore m^2 is divisible by 2.

If m^2 is divisible by 2, m is also divisible by 2. Take

$$m = 2p;$$

then

$$m^2 = 4p^2;$$

but $m^2 = 2n^2$, therefore $4p^2 = 2n^2$, and $2p^2 = n^2$. Therefore n^2 is divisible by 2, and, if so, n is also divisible by 2.

But if both m and n are divisible by 2, then this contradicts our initial assumption that the fraction m/n is in its lowest terms, or, in other words, that the numerator and the denominator have no factor other than 1 in common. (It will be observed that where Aristotle speaks of odd and even, we, while using what is basically, in this case, the same criterion, call it divisibility or indivisibility by 2.)

Now, it is clear that the proof of the irrationality of the square root of 3 is practically the same as that of the irrationality of the square root of 2, and that it can be used without any change whatsoever in method for the proof of the irrationality of the square root of any prime number. It can also be used in the same way for proving the irrationality of the square root of a composite number that is not a multiple of a square number, or, in other words, that does not contain any prime factor more than once.

But it is equally obvious that the proof will have to be modified in the case of numbers that, though not square numbers themselves, are multiples of square numbers. It is easy to show, for example, that the proof employed for 2 and 3 will do for 5, 7, 11, 13, 17, and so on; that it will also do for 6, 10, 14, 15, and so on, that is to say, composite numbers not being multiples of square numbers; but that it will not do, in its unchanged form, for non-square composite numbers that are multiples of square numbers, such as, for example, the numbers 8, 12, or 18. To show that this is so one need point out only that one of the essential steps in the proof, namely the inference that if m^2 is divisible by a given number, then m must be divisible by that number too, is not a valid inference in the case of numbers that are multiples of square numbers. Thus, for instance, if m^2 is divisible by any of the numbers 2, 3, 5, 7, or for that matter, by 6, 10,

¹ See also the note by Alexander Aphrodisiensis on *An. Pr.* 1. 41^a26. (It is to be observed that Alexander in this note [p. 260,

30, of the Berlin edition] uses the verb *πλασιάζειν* = to square, a sense which is unknown to L.S.J.)

14, then it is obvious that m is also divisible by that number. But if m^2 is divisible by a number like 8 or 12 or 18, then it does not at all follow that m must also be divisible by that number. To take only one example: take $m^2 = 6^2 = 36$. This is divisible by 18 (i.e. 2×3^2) and also by 12 (i.e. 3×2^2). But $m = 6$ is not divisible by either of these two numbers.

The proof employed hitherto can of course be very easily adapted to numbers that are multiples of square numbers; but the fact that it has to be so adapted vitiates the attempt at generalization of the proof suggested by Heath (*Gk. Mathematics*, i. 205). It is there argued that Theodorus is not likely to have employed the traditional proof, because that is capable of very easy generalization, that is to say, the generality of the proof would have become apparent long before in the enumeration of cases $\sqrt{17}$ had been reached. Since he did go on to $\sqrt{17}$, that, so argues Heath, suggests that his proof was not the traditional one, but a proof that was not capable of such easy and almost self-evident generalization. Unfortunately Heath's argument is weakened not only by the obvious objection that it would be hard to believe that Theodorus was not conversant with the traditional proof,¹ but by the graver difficulty that what Heath represents as a very simple generalization is not a valid generalization at all. He writes:

'We can put the proof quite generally thus. Suppose that N is a non-square number such as 3, 5, ..., and, if possible, let $\sqrt{N} = m/n$, where m, n , are integers prime to one another. Therefore $m^2 = N \cdot n^2$; therefore m^2 is divisible by N , so that m also is a multiple of N ' (my italics).

After this the proof proceeds in the traditional way so that the matter is disposed of in a few more steps. But everything that comes after this is of no immediate interest to us. For the words italicized above represent an *invalid* step in the argument. The divisibility of a square number m^2 by a number N implies the divisibility of m by that number N only where N is not itself the multiple of a square number. Where N is e.g. 8 or 12 or 18 or any such number that contains the same prime factor more than once, the inference 'if m^2 is divisible

¹ That the traditional proof is indeed 'traditional' (i.e. known before Theaetetus) is accepted by most scholars; see, e.g., Cantor, *Vorlesungen zur Gesch. der Mathematik*, i. 155; Heiberg, 'Mathematisches zu Aristoteles', *Abhandl. Gesch. Math. Wiss.*, 1904, p. 24: 'dieser alte pythagoräische Beweis'. And the case is well argued by Zeuthen (1915, pp. 357-62). The arithmetical character of the proof and its reliance on the distinction between odd and even (in the case of the square root of 2) seem to stamp the proof as Pythagorean. A further argument in favour of its Pythagorean origin is its obvious relationship with the so-called Pythagorean theorem (Euclid 1. 47) which is attested as Pythagorean by many, though admittedly late, authorities. For reference see Heath, *The Thirteen Books of Euclid's Elements*, i. 351. He also quotes Heron, who ascribes to Pythagoras a general rule for the formation of right-angled triangles with rational integers as sides. And the well-known Eudemian Summary in

Proclus (ed. Friedlein, p. 65) attributes the discovery of the theory of irrationals to Pythagoras; the scholiast on Euclid 10. 1, Heiberg, vol. v, pp. 415-16, attributes the discovery of incommensurability to the Pythagoreans. See Heath, op. cit., p. 353: 'The investigation (of 1. 47) from the arithmetical point of view would ultimately lead Pythagoras to the other momentous discovery of the irrationality of the length of the diagonal of a square expressed in terms of its side.'

The use of *ἔγγραφει* in the Platonic dialogue does not, of course, refer to the geometrical character of the proof. At most it means that Theodorus drew a figure to demonstrate the *existence* of such lengths; that they were irrational needed a further, arithmetical, proof with which the 'Existenzbeweis' to which *ἔγγραφει* in that case refers, had nothing to do. Heath in fact gives good reasons for thinking that *ἔγγραφει* means no more than 'he was proving'. See *Gk. Math.* i. 303.

by N , then m also is divisible by N is invalid. Thus the possibility of generalization assumed by Heath does not really exist in that very simple form.¹ And this seriously diminishes the cogency of his conclusion that because Theodorus apparently did not hit upon the generalization he is unlikely to have used the traditional proof.

In fact the valid generalization is slightly more complicated than that assumed by Heath. It could be put as follows:

N not being a square number has no rational square root. For, suppose it has: then it must be of the form m/n , this fraction being in its lowest terms. Then, if

$$\frac{m}{n} = \sqrt{N}, \quad \frac{m^2}{n^2} = N, \quad m^2 = N \cdot n^2;$$

therefore

m^2 is divisible by N .

Now, although it does not follow that m too is divisible by N (that would follow only if N is a square-free number) we can at any rate say that if m^2 is divisible by N then m is divisible by the square-free factors of N . Call them a . Then

$$m = ar$$

and
$$\frac{m^2}{n^2} = N = \frac{a^2 r^2}{n^2}, \quad a^2 r^2 = N \cdot n^2, \quad \frac{a^2 r^2}{N} = n^2.$$

But N is either (1) $= a$ or (2) a multiple of a , say ap (where p is of course a square number).

Then either (1) where $N = a$

$$\frac{a^2 r^2}{N} = ar^2 = n^2;$$

therefore a divides n^2 and since a is square-free (by hypothesis) it must also divide n .

Or (2) where $N = ap$ (a being square-free, p being square)

$$\frac{a^2 r^2}{N} = \frac{a^2 r^2}{ap} = \frac{ar^2}{p} = n^2 = a \frac{r^2}{p} = n^2.$$

Therefore n^2 is divisible by a , and since a is square-free, n also is divisible by a . But if a divides both m and n , this contradicts the initial assumption that m/n was in its lowest terms.

It will be seen at once that this generalization of the traditional proof is somewhat more complicated than one might think at first sight and that it is necessarily more complicated than that assumed by Heath. But of course the complication would have presented no difficulty to a mathematician like Theodorus, and we must still ask ourselves why he and others before him did not hit upon the almost self-evident generality of a method that they already applied in particular cases.

We have seen that the generalization presents a few more complications than the particular cases. It is here, I think, that we must still look for an answer

¹ Zeuthen (1915, p. 341) makes the same mistake as Heath: '... l'application de la démonstration en question à ces différentes

racines ne présenterait que cette seule différence qui résulte de la substitution d'un nouveau nombre à celui dont on part.'

to our question. For while the complications would not in themselves present any great difficulty to a mathematician (though even here we must not underestimate the difficulty of reasoning that proceeds from step to step with algebraic symbols, or rather, with their equivalent verbal expressions, a procedure not as intimately familiar and easy to Greek as to modern mathematicians), yet the presence of these particular complications may have made clear to the rigorous mathematician a difficulty so fundamental and so elementary that it might remain unnoticed in the application of the method to particular cases as long as no attempt at generalization was made. I refer to the doubt that a rigorous mathematician must have felt when, or if, he tried to generalize the traditional proof in the manner indicated above, about the cogency of an argument like: if m^2 is divisible by a then m also is divisible by a . Obviously the necessity for this step in the argument would make the mathematician consider the different possible cases. And as we have done above he would find that the argument does not hold in the case where a is the multiple of a square number. But even when the possibilities are narrowed down to $a =$ any *prime* number, the statement though apparently true would still seem to the rigorous mathematician to require *proof*. Needless to say he would also think about cases where a is neither prime nor a multiple of square numbers, i.e. where a is a non-square, square-free, composite number. In brief the rigorous mathematician who tried to generalize the traditional proof would have thought hard about such matters as prime numbers, composite numbers of various forms, and in particular, about the cogency of the step in our argument that we have discussed above; in other words, he would see that the above proof really depends on something else that has to be proved first.

Now, such a preliminary proof does indeed exist: Euclid, 7. 30, proves that if c divides ab , and c is prime, then c must divide either a or b .

Our case, that of a square number, is no more than a special example of this; that is to say: the product ab is such that $a = b$; and thus, the prime that divides a or b in Euclid's theorem will divide both a and b ; i.e. where we are dealing with a number that is not just a product of two numbers, but a product of two equal numbers, in other words: a square number, Euclid's theorem can be read like this: if c is a prime and divides a square number it will also divide the square root of that number. And that is what is needed for the generalization above discussed.

The connexion between Theaetetus and the arithmetical books of Euclid seems well established. It is suggested by the evidence and the matter is examined by Zeuthen who comes to the conclusion that the mathematics of Book 7 of Euclid must be attributed in its essentials to Theaetetus.¹ Yet in the same article Zeuthen argues (as does Heath after him) that the particular

¹ Zeuthen, 1910, p. 421; he gives examples of terminological parallels between Euclid (7, Def. 16 and 18) and the passage in the *Theaetetus*. As regards Book 10 we have the statement of the scholiast on 10. 9 who expressly says that that theorem was the discovery of Theaetetus (see Heiberg, *Euclid*, v. 450). On 13. 1 the scholiast tells us (Heiberg, op. cit. v. 654) that part of the subject-matter of that book was derived from Theaetetus. Generally, we are told by Proclus (*in Eucl.*

Elem., ed. Friedlein, p. 68) that *Εὐκλείδης ὁ τὰ στοιχεῖα συναγαγὼν . . . πολλὰ . . . τῶν Θεαιτήτου τελεωσάμενος . . .*: 'Euclid put together the Elements and perfected many of the theorems of Theaetetus'. The practically certain case for pre-Euclidean provenance of part at any rate of the contents of the arithmetical books is supported by a letter of Eratosthenes given by Eutocius (Archimedes, ed. Heiberg, iii. 102 ff.).

proofs employed by Theodorus cannot have been adaptations of the traditional proof: for, he argues, these would have led to the generalization; and also there would have been no need for Plato to single out Theodorus' discovery for special praise. However, we have seen that the generalized formulation of the proof requires rather more than a mechanical substitution of algebraical symbols for particular numbers: that the attempt to generalize leads to complications which suggest a preoccupation with the fundamentals of the theory of numbers. On the other hand, contrary to what one might expect, the fundamental difficulty, namely the need first to prove what we now have in Euclid 7. 30,¹ does not necessarily enter into the proof for *particular* cases like the square roots of 2, 3, 5, 6, etc., since for each particular number we can substitute for the preliminary general proof of the critical inference ' c dividing a^2 implies c dividing a ' a special proof by enumeration of cases, without relying on Euclid 7. 30.²

I think therefore that the most economical use of the evidence will still make it possible to assume that Theodorus' proof was in fact the traditional proof; that Theodorus did not proceed to the generalization, because he was well aware of the need (in the generalization) of a more fundamental treatment of such subjects as prime and composite numbers, factorization, etc.; and that it was precisely this refusal of the rigorous mathematician to enunciate a general theory based on doubtful foundations that led his pupil Theaetetus to investigate not only the problem of irrationality but also the more fundamental arithmetical questions; thus the generalized theory of irrationality will not only depend on fundamental number-theory: it will be seen to have led directly to a preoccupation with the latter. We should in this way have (1) a *logical* sequence: preoccupation with irrationality leading to work on fundamental arithmetic of the kind found in Euclid 7-9; and (2) we could fix *historically* the point where this kind of arithmetic was systematized. In saying that Theodorus did not and could not generalize a proof of which he used the particular applications, one need not of course assume that he could not enunciate conjecturally the general theorem whose general proof he did not know. In fact he may have wanted by the enumeration of the cases up to 17 to suggest the general theorem; that may be the reason why he went so far as 17. But there is a limit to everything, even a mathematics lesson; and that is why he went no farther.

Thus we may disagree with Heath and Zeuthen on both counts. First, we need not account for anything particularly original in Theodorus' proof to make it worthy of Plato's mention; indeed, there is nothing in Plato's text to suggest that Theodorus had made a new discovery; and so we can, on that count at any rate, assume that his was the traditional proof. Yet though his proof was not original, we may still think of him as an original mathematician:

¹ It is of course 7. 30, not as v. Fritz (in P.W. s.v. Theodorus, c. 1821) says, 7. 27. The latter is the enunciation of the theorem that if two numbers are prime to one another, then their squares will be prime to one another; and so also with their cubes. But it is clear that what we need for our proof is 7. 30, that is to say, the theorem 'if c divides ab , and c is prime, then c divides either a or b ', with a simple substitution of a square number for ab , that is to say, taking $a = b$. Else-

where v. Fritz (s.v. Theaetetus, cc. 1356 and 1359) derives what we need here from 7. 25. But that, too, is unnecessary: for there we are told that if two numbers are prime to one another the square of one of them will be prime to the other: what we need does not follow from it. Whereas at 7. 30 the enunciation of $c|ab \rightarrow c|a$ or $c|b$ obviously includes, as a special case, $c|a^2 \rightarrow c|a$.

² See on this Hardy and Wright, *Introduction to the Theory of Numbers*, p. 41.

there must have been originality in him if he had an inkling of the insufficiency of a mechanical generalization of the traditional proof without previously proving what we now have in Euclid 7. 30.

Secondly: nor yet is the generalization of the traditional proof quite so easy as Heath and Zeuthen thought. If it is easy enough to become apparent by a mere enumeration of cases, it yet brings in its train complications that lead to the realization of fundamental difficulties without the solution of which the generalized proof, as distinct from the conjectural enunciation, of the theorem is impossible.

We conclude therefore:

1. The proof employed by Theodorus may well have been, indeed is extremely likely to have been, the traditional proof.
2. The fact that this was not generalized, so far from leading us to think that the traditional proof could not have been used, fits exactly into the picture of the rigorous mathematician who refuses to jump his fences before he has come to them.
3. The intimate connexion of Theodorus' pupil Theaetetus with the theory of irrationality *and with fundamental arithmetical theory* suggests that it was the realization of difficulties inherent in the former that led to a pre-occupation with parts at least of the latter.

APPENDIX

An ancient commentator on the *Theaetetus* (*Anonymer Kommentar zu Platons Theaetet*, ed. H. Diels and W. Schubart, *Berliner Klassikertexte*, ii, 1905) mentions some interesting suggestions concerning the two questions: (a) Why did Theodorus start with $\sqrt{3}$, not with $\sqrt{2}$? (b) Is there anything about $\sqrt{17}$ that made it appropriate for Theodorus to stop there?

On the first question he says *inter alia* that it had been suggested that Plato made Theodorus begin with $\sqrt{3}$ because he had already shown in the *Meno* that the square on the diagonal (of a square) was twice as great as the original square; see col. 28, 37 to col. 29, 1:

... ἔστιν δὲ καὶ τὸ δίπουν τετράγωνον ἀσύμμετρον τῷ ποδιείῳ (sic) κατὰ τὴν πλευράν, ἀλλὰ παρήλθεν, φασίν, αὐτὸ διότι ἐν τῷ Μένωνι ἔδειξεν ὅτι τὸ ἀπὸ τῆς διαγωνίου τετράγωνον διπλάσιόν ἐστιν τοῦ ἀπὸ τῆς πλευρᾶς τετραγώνου.

On the second point we read that the question had been asked why Theodorus stopped when he reached 17. One suggested answer,¹ he says, was that

Θεόδωρος γεωμέτρης ὦν καὶ μουσικὸς ἔμειξεν γεωμετρικὸν καὶ μουσικὸν θεώρημα· γεωμετρικὸν μὲν οὖν τὸ κατὰ τὰς δυνάμεις, μουσικὸν δὲ τὸ τῆς ἑπτακαίδεκάποδος. Οὗτος γὰρ ὁ ὅρος ἐνέχει (ἐνίσχει?) ὅτι οὐ διαιρεῖται ὁ τόνος εἰς ἴσα ἡμιτόνια. Ἐπεὶ γὰρ ὁ τόνος ἐστὶν ἐν ἐπογδόῳ λόγῳ, εἰς διπλασιάσεως τὸν ὀκτῶ καὶ τὸν ἐννέα, γίνονται (sic) ἑκκαίδεκα καὶ ὀκτωκαίδεκα, ὧν μέσος ἐστὶν ὁ ἑπτακαίδεκα εἰς ἄνισα διαιρῶν τοὺς ἄκρους, ὡς δέδεικται ἐν τοῖς εἰς τὸν Τίμαιον ὑπομνήμασιν.

¹ The commentator does not seem to accept this answer. He himself suggests, 35, 21–36, 35, that 17 may have been an appropriate place to stop because it is the first number after 16; and 16 is the number of the only

square in which the number denoting the sum of the sides is equal to the number denoting the area of the square, since

$$4+4+4+4 = 4 \times 4.$$

'Theodorus was both a geometer and a musician.¹ He therefore combined here a geometrical with a musical theorem. The geometrical one was concerned with the roots (or squares), the musical with the number 17. For this term . . . ?² because the tone cannot be divided into equal semitones. The tone interval has the ratio 9/8. Now if one doubles the number 8, and also the number 9, one obtains the results 16 and 18, between which the number 17 is the (arithmetic) mean dividing the extremes into (geometrically) unequal ratios, as has been shown in the commentary to the *Timaeus*.'

What does this mean? The musical interval of a full tone cannot be exactly divided into two semitones. The reason for that is that the ratio of the tone is 9/8. To find the semitone we would have to find the *geometric* mean between 9 and 8, or, what essentially comes to the same, between 9/8 and 1. The geometric mean between 9/8 and 1 would be a number such that its square = 9/8. It would therefore be $\sqrt{(9/8)}$. But there is no such rational number; the square root of a fraction of the form $a + 1/a$ is irrational; there are no adjacent squares in the series of integers.³ Now, since a geometric mean representing the interval-ratio of the semitone cannot be found, what is one to do? Perhaps the next best thing to do is to take the *arithmetic* mean instead, knowing that it will not give an *exact* division into *equal* semitones. That clearly is the meaning of ἐὰν διπλασιάσῃς . . . τοὺς ἄκρους. In doing so we find the number 17 (which suggests 17/16, the arithmetic mean between 1 and 9/8); and thus, the commentator reasons, the number 17 may have had a special significance for Theodorus. For if we use it once as the denominator and once as the numerator, we obtain two ratios, 18/17 and 17/16, both of which are approximations to the geometric mean; and they have the additional virtue of resulting, when multiplied with each other, exactly in 9/8, the ratio of the full tone.

But what our author gives us here is clearly a garbled, or at any rate a shortened, version of an explanation that must have been clearer and more explicit than this. And indeed he goes on to refer to an explanation to be found elsewhere: ὡς δέδεικται ἐν τοῖς εἰς τὸν Τίμαιον ὑπομνήμασιν. I can indeed not be certain that this refers to a commentary of his own (so Diels and Schubart in their Index, s.v. 'Kommentare'); one might in that case perhaps have expected the explanation here to have been a little clearer. On the other hand, two other references in this commentary, at 48. 10 and 70. 12, quite clearly refer to commentaries of his own on the *Phaedo* and the *Symposium*, so that it is at least possible that here too he means a commentary of his own. However that may be, there certainly was a fuller treatment elsewhere of the matter adumbrated in this short passage.

Now, we read the following passage in Proclus, *Comm. in Tim.* 195 a (on Plato, *Tim.* 35 b):

. . . περὶ . . . τοῦ λείμματος ἰστέον, ὅτι, ἐπειδὴ περ οὐδεὶς ἐπιμόριος εἰς λόγους ἴσους τέμνεσθαι δύναται, τὸ ἡμιτόνιον ἐν ἀριθμοῖς οὐκ ἔστι λαβεῖν, ἀλλὰ τοὺς

¹ i.e. concerned with musical theory. Cf. Plato, *Theaet.* 145 d.

² I do not know what ἐνέχει means here. It has been suggested to me that it may perhaps mean 'comes in', 'enters into the argument'.

³ Cf. Euclid, *Sectio Canonis*, Mus. scr. gr. ed. Jan, prop. 3, p. 152; also Archytas ap. Boeth. de mus. iii. 11 = Vorsokratiker⁶, 47 A 19, p. 429. See Heath, *The Thirteen Books of Euclid's Elements*, ii. 295.

σύνεγγυς ἀλλήλων λαβόντες τὸν ἑφεπτακαιδέκατον λόγον καὶ τὸν ἑφεκκαιδέκατον καὶ δείξαντες τὸν ἑφεπτακαιδέκατον μείζονα τοῦ καλουμένου λείμματος, ὃς ἦν ἐλάσσων τοῦ ἀκριβῶς ἡμιτονίου, συνάγουσιν, ὅτι καὶ τοῦ ἡμιτονίου ἐλάσσων ἐξ ἀνάγκης ἐστὶν ὁ τοῦ λείμματος λόγος. ὅτι δ' οὖν ἐλάσσων ἐστὶν ἢ ἑφεπτακαιδέκατος, οὗτος δὲ ἐλάσσων (ἐλασσον MSS.) ἢ ἡμιτόνιον, δέικνυται οὕτως. ἐκκείσθω γὰρ ὁ $\iota\varsigma$, καὶ τούτου ἐπόγδοος ὁ $\iota\eta'$ · τούτων δὴ μεταξὺ τεθεῖς ὁ $\iota\zeta$ εἰς ἀνίσους διαιρεῖ τὸν ἐπόγδοον λόγους σύνεγγυς ὄντας τοῦ ἡμιτονίου διαστήματος, μονάδι διαφέρων τῶν ἄκρων. καὶ δῆλον ὅτι τὸν πρὸς τῷ ἐλάσσονι μείζων ποιήσει λόγον· ἐν πάσῃ γὰρ ἀριθμητικῇ μεσότητι μείζων ἐστὶν ὁ λόγος ἐν τοῖς ἐλάττωσιν ὅροις, ὥστε ὁ ἑφεπτακαιδέκατος ἐλάσσων ἐστὶν ἡμιτονίου. ἀλλὰ μὴν καὶ τὸ λείμμα ἐλασσον ἐστὶν ἢ ἑφεπτακαιδέκατον, ὥς ἀπὸ τῶν παρὰ Πλάτωνι κειμένων ὅρων δῆλον. ὁ γὰρ $\sigma\nu\varsigma$ πρὸς τὸν $\sigma\mu\gamma$ τὸν τοῦ λείμματος ἔχων λόγον, ὥς καὶ τοῦτο δείξομεν πυθμενικὸν ἀποδείξαντες τὸν ἐν τούτοις τοῖς ἀριθμοῖς τοῦ λείμματος λόγον, ἐλάσσων ἐστὶ τοῦ ἑφεπτακαιδεκάτου [πρὸς τὸν $\sigma\mu\gamma$]· ὑπερέχει μὲν γὰρ αὐτοῦ $\iota\gamma$ μονάσιν, τὸ δὲ ἑπτακαιδέκατον τοῦ $\sigma\mu\gamma$ πλειόνων ἐστὶν ἢ $\iota\gamma$ μονάδων. πολλῶν ἄρα μᾶλλον ὁ τοῦ λείμματος λόγος ἐλάσσων ἐστὶ τοῦ ἡμιτονίου διαστήματος, ὥστε καὶ ὁ λοιπὸς εἰς τὸν τόνον, ὃς ἐστὶ τῆς ἀποτομῆς λόγος, ἐξ ἀνάγκης ἡμιτονίου μείζων ἐστίν.

'About the *leimma* one must know this: since no superparticular ratio can be cut (divided) into equal ratios it is impossible to comprehend (i.e. to express) the semitone in numbers (i.e. as numerical ratios). Instead one takes the ratios that are proximate to each other, viz. $18/17$ and $17/16$ and, showing that $18/17$ is greater than the so-called *leimma* ($18/17$ being smaller than the exact semitone) one infers that the ratio of the *leimma* is therefore also necessarily smaller than the semitone (i.e. the exact semitone). Now, that it (i.e. the ratio of the *leimma*) is smaller than $18/17$, and that this (i.e. the ratio $18/17$) is smaller than the semitone, is shown in the following way: take the number 16, and take the number that stands to 16 in the ratio of $9/8$, namely 18. Now the number 17 if put between these divides the ratio $9/8$ into unequal ratios that are near to the semitone ratio, and it differs by a unit from the extreme terms.

'Now it is clear that it will make the ratio to the smaller term greater. For in every arithmetical mean the ratio is greater in the smaller terms, so that $18/17$ is smaller than the semitone. But the *leimma* is smaller than $18/17$ as is clear from Plato's terms: for $256/243$, the ratio of the *leimma* (that this ratio is in its lowest terms I shall demonstrate when I have shown that the *leimma*-ratio is in fact the ratio of these numbers, i.e. $256/243$) is smaller than $18/17$ [to 243]: for it exceeds by 13 units: while $1/17$ of 243 consists of more than 13 units. All the more therefore the ratio of the *leimma* is smaller than the semitone interval, so that what is left to (make up) the full tone, i.e. the ratio of the *apotome*, is necessarily greater than the semitone.'

περὶ . . . τοῦ λείμματος:

The *leimma* is the 'residue' that is 'left' when two full tones (each of the ratio $9/8$) are cut off from the fourth (διὰ τεσσάρων). For example, the ratio of the tone C (for the sake of simplicity I am assuming here the ordinary modern major scale moving upwards) to the tone F, i.e. διὰ τεσσάρων, is $4/3$; the ratio C to D is $9/8$; D to E is $9/8$; what is the ratio of E to F? This ratio is obtained by dividing $4/3$ twice by $9/8$, or, what comes to the same, once by $81/64$. The

result is $256/243$. To make this clearer: we know

$$C:D = 9:8$$

$$D:E = 9:8$$

therefore

$$C:E = 81:64.$$

Now, if the ratio of C to E is known and likewise the ratio of C to F, then there is a ratio E to F which we must find: call it X . Then

$$81/64 \text{ multiplied by } X = 4/3,$$

$$81X = 256/3$$

and

$$X = 256/243.$$

See Plato, *Tim.* 36 b; Theon Sm. p. 86 H.; Gaudentius 12. 342. 7 ff. Jan: τὸ δὲ ἡμιτόνιον καλούμενον οὐκ ἔστιν ἀκριβῶς ἡμιτόνιον. λέγεται δὲ κοινῶς μὲν ἡμιτόνιον ἰδίως δὲ λεῖμμα, καὶ ἔχει λόγον ὃν τὰ $\sigma\mu\gamma$ πρὸς τὰ $\sigma\nu\varsigma$. See also Ptol. *Harm.* i. 10. The name *leimma* is said by Proclus (p. 211 c) to have been used by Plato.

ἐπειδήπερ οὐδεὶς ἐπιμόριος εἰς λόγους ἴσους τέμνεσθαι δύναται, τὸ ἡμιτόνιον ἐν ἀριθμοῖς οὐκ ἔστι λαβεῖν:

The *ἐπιμόριος* is a ratio of the form $\frac{a+1}{a} \left(= 1 + \frac{1}{a} \right)$. No such (superparticular)

ratio can be 'cut' into equal ratios, i.e. one cannot find a (rational) geometric mean between a and $a+1$. To find the exact semitone ratio one would have to find the length of a chord such that the length of the chord producing the lower tone has to the semitone chord the same ratio as the latter has to the length of the chord producing the higher tone. For instance, $C:D = 9:8$; what length of chord will produce the *exact* semitone between C and D? Obviously the length of C will be to its length (i.e. to the length of the chord producing C sharp) as the latter is to the length of D. But, if C equals 9, D equals 8; therefore

$$g:C \text{ sharp} = C \text{ sharp}:8,$$

i.e. C sharp is the geometric mean between 9 and 8. But there is no rational geometric mean¹ between a and $a+1$, or, in this case, between 8 and 9. For if

$$9:X = X:8$$

then $X^2 = 72$, $X = \sqrt{72} = 6\sqrt{2}$, which is irrational.

That is why it is impossible to express the exact semitone in numbers, i.e. as a numerical ratio. Though, of course, physically, and geometrically, there is no such difficulty; think only of the progression of semitones on the *tempered* pianoforte.

τοὺς σύνεγγυς . . . ἑφεκκαίδέκατον:

To find a rational geometric mean between 9 and 8 is impossible. By doubling the two terms one obtains 18 and 16. The arithmetic mean between these is 17. This number is then used to obtain two new ratios viz. $18/17$ and $17/16$; these are rational approximations to the square root of $9/8$. $18/17 < \sqrt{(9/8)}$; $17/16 > \sqrt{(9/8)}$. And multiplied by each other they produce exactly $9/8$.

¹ See p. 172, n. 3.

δείξαντες τὸν ἑφепτακαιδέκατον μείζονα τοῦ καλουμένου λείμματος:

This, i.e. that $18/17$ is greater than $256/243$, will be shown below.

ὅς ἦν ἐλάσσων τοῦ ἀκριβῶς ἡμιτονίου:

The antecedent of $\delta\varsigma$ is, of course, τὸν ἑφепτακαιδέκατον. That this is smaller than the semitone will also be shown below.

συνάγουσιν . . . λόγος:

If the *leimma* is smaller than $18/17$ (δείξαντες τὸν ἑφепτακαιδέκατον μείζονα τοῦ . . . λείμματος), and if $18/17$ is smaller than the semitone ($\delta\varsigma$ ἦν ἐλάσσων τοῦ . . . ἡμιτονίου), then, *a fortiori*, the leimma is smaller than the semitone.

In what follows Proclus establishes the premisses of this conclusion one by one: (1) That $18/17$ is smaller than the semitone. (2) That the leimma is smaller than $18/17$.

ὅτι δ' οὖν ἐλάσσων ἐστὶν ἢ ἑφепτακαιδέκατος:

Scil. ὁ τοῦ λείμματος λόγος.

ἐκκείσθω γὰρ ὁ $\overline{15}$:

$16 =$ twice the denominator of the fraction $9/8$.

καὶ τούτου ἐπόγδοος ὁ $\overline{17}$:

$18:16 = 9:8$ ($18 =$ twice the numerator of the fraction $9/8$).

τούτων δὴ μεταξύ τεθεῖς ὁ $\overline{17}$ εἰς ἀνίσους διαιρεῖ τὸν ἐπόγδοον λόγους σύνεγγυς ὄντας τοῦ ἡμιτονίου διαστήματος μονάδι διαφέρων τῶν ἄκρων:

'The number 17 if put between these (i.e. between 16 and 18) divides the ratio $9/8$ into unequal ratios that are near approximations to the semitone ratio; it differs by a unit from the extreme terms.'

The two ratios $18/17$ and $17/16$ are near approximations to the ratio of the semitone, being respectively just smaller and just larger than the geometric mean between 1 and $9/8$. On the whole procedure adopted here see also Aristid. Quintil. iii. 114 Meibom; Plut. *Moral. Περὶ τῆς ἐν Τιμαίῳ ψυχογονίας* 1021 c-e; Boethius, *de Mus.* cap. 16; Exc. Neap. § 19 (Jan, p. 416); Ptol. *Harm.* i. 10; Theon Sm. p. 69 H.; Gaud. 14. 343. 1 ff. J.

καὶ δῆλον ὅτι τὸν πρὸς τῷ ἐλάσσονι ὄρω μείζω ποιήσει λόγον' ἐν πάσῃ γὰρ ἀριθμητικῇ μεσότητι μείζων ἐστὶν ὁ λόγος ἐν τοῖς ἐλάττοσιν ὄροις, ὥστε ὁ ἑφепτακαιδέκατος ἐλάσσων ἐστὶν ἡμιτονίου:

'Now it is clear that it will make the ratio to the smaller term greater. For in every arithmetical mean the ratio is greater in the smaller terms, so that $18/17$ is smaller than the semitone.'

Three terms are involved: 16, 18, and, the arithmetic mean, 17. It is clear that the ratio of 17 to the smaller term, 16, viz. $17/16$ is greater than the ratio of the larger term to the middle term, that is to say than $18/17$. Compare, for this, Archytas, Fragment 2 Diels-Kranz, pp. 435-6 (from Porph. in Ptol. *Harm.* p. 92): μέσαι δέ ἐντι τρῖς τᾷ μουσικᾷ . . . ἀριθμητικὰ μὲν, ὅκκα ἔωντι τρεῖς ὄροι κατὰ τὰν τοίαν ὑπεροχάν ἀνὰ λόγον' ᾧ πρᾶτος δευτέρου ὑπερέχει, τούτῳ δεύτερος τρίτου ὑπερέχει. καὶ ἐν ταῦτα τᾷ ἀναλογίᾳ συμπίπτει ἡμεν τὸ τῶν μειζόνων ὄρων διάστημα μείων, τὸ δὲ τῶν μειόνων μείζον.

ὥστε ὁ ἑφепτακαιδέκατος ἐλάσσων ἐστὶ ἡμιτονίου:

‘so that 18/17 is smaller than the semitone’. The ratio of the semitone is $\sqrt{(9/8)}$; multiplied by itself that gives 9/8. But 9/8 is also the product of 18/17 and 17/16. These are not equal to each other, and therefore one of them must be greater than the semitone, and one smaller. But 18/17 is smaller than 17/16. Therefore 18/17 is smaller than the semitone (and, of course, 17/16 is greater than the semitone). A simple test:

$$\left(\frac{18}{17}\right)^2 = \frac{324}{289} < \frac{9}{8}, \quad \left(\frac{17}{16}\right)^2 = \frac{289}{256} > \frac{9}{8}.$$

τὸ λείμμα ἔλασσόν ἐστὶν ἢ ἑφепτακαιδέκατον:

Having established the first premiss, viz. that 18/17 is smaller than the semitone, he now proceeds to establish the second premiss: that the leimma is smaller than 18/17.

ὁ γὰρ σνς πρὸς τὸν σμγ τὸν τοῦ λείμματος ἔχων λόγον . . . ἐλάσσων ἐστὶ τοῦ ἑφепτακαιδέκατου [πρὸς τὸν σμγ]:

‘The ratio of 256 to 243, which is the ratio of the leimma, is smaller than 18/17.’ (MSS. ‘smaller than 18/17 to 243’.) This is proved in the next sentence *ὑπερέχει*, etc.

πρὸς τὸν σμγ must be deleted. It makes no sense at all; and it is in fact omitted in one of the good manuscripts; see ed. Diehl (Teubner).

ὥς καὶ τοῦτο δείξομεν πυθμενικὸν ἀποδείξαντες τὸν ἐν τούτοις τοῖς ἀριθμοῖς τοῦ λείμματος λόγον:

‘. . . that this ratio (i.e. 256:243) is in its lowest terms I shall demonstrate when I have shown that the leimma ratio is in fact the ratio of these numbers (i.e. 256:243)’. (The order of the words here is a little odd; I have translated as if *πυθμενικόν* came first and *δείξομεν* second.)

πυθμενικόν:

This, like the noun *πυθμήν*, is the technical term for the lowest terms of a ratio or a fraction. *Πυθμήν* is found in this sense at least as early as Plato; cf. *Rep.* 546 c.

δείξομεν . . . ἀποδείξαντες:

(a) That the leimma has the ratio 256/243 is demonstrated on pp. 195 e–196 a (pp. 181–2 Diehl).

(b) That this ratio is in its lowest terms: p. 196 a (p. 182 Diehl).

ὑπερέχει μὲν γὰρ . . .:

‘(256/243 is smaller than 18/17;) for it exceeds by 13 units while 1/17 of 243 consists of more than 13 units’.

The reasoning here is as follows:

It is required to prove that $256/243 < 18/17$.

256/243 can be expressed as $1 + 13/243$ (as exceeding 1 by 13/243, *ὑπερέχει* . . . *ἢ μονάσιν*); similarly 18/17 can be expressed as $1 + 1/17$.

It is therefore now only required to prove that $13/243 < 1/17$. This is true if

$$243 \text{ times } 13/243 < 243 \text{ times } 1/17$$

that is, if $13 < 243/17$, or, as the Greek has it, τὸ ἑπτακαιδέκατον τοῦ $\overline{\sigma\mu\gamma}$ πλειόνων ἐστὶν ἢ $\overline{\iota\gamma}$ μονάδων.

πολλῶ ἄρα μᾶλλον ὁ τοῦ λείμματος λόγος ἐλάσσων ἐστὶ τοῦ ἡμιτονίου διαστήματος:

We have established

(a) $18/17 < \text{semitone}$.

(b) $\text{leimma} = 256/243 < 18/17$.

From this it follows that the leimma ratio is smaller than the exact ratio of the semitone.

ὥστε καὶ ὁ λοιπὸς εἰς τὸν τόνον, ὅς ἐστι τῆς ἀποτομῆς λόγος, ἐξ ἀνάγκης ἡμιτονίου μείζων ἐστίν:

The leimma would be the (near) semitone interval between, say, the Third and the Fourth of a (modern upward-moving) major scale, e.g. E:F = 256:243. Since this is not an exact semitone, another question now arises: what is the ratio of F to F sharp? Suppose F sharp is a full tone above E, that is, E:F sharp = 9:8. Then, if F had been the exact semitone between E and F sharp, the ratio of F to F sharp would have been the same as that of E to F. But F is not the exact semitone. And so F:F sharp is not equal to the ratio E:F. Since the leimma (E:F) is smaller than the semitone, the apotome (i.e. what is needed to make up the full tone, ὁ λοιπὸς εἰς τὸν τόνον) must be greater. It will in fact be obtained very simply by dividing 9/8 by 256/243 = 2187/2048; see Proclus, in *Tim.* p. 195 d (181 Diehl).

Clearly there is a connexion between the explanation given by the Theaetetus commentator, and referred by him to a Timaeus commentary, and the Proclus passage discussed above. The Theaetetus commentator not only explicitly says that his explanation is based on a Timaeus commentary but he obviously gives a version, though perhaps a garbled version, of something very much like the passage we have just examined. Here at once an interesting observation occurs: Proclus, of course, wrote much later than the author of the Theaetetus commentary (the latter is dated by Diels-Schubart in the second century A.D. or at any rate not later; this dating is confirmed, in a letter, by Professor E. G. Turner). Thus we can take this coincidence as valuable evidence for Proclus' practice of using older commentaries in writing his own.¹

More importantly the connexion which the Theaetetus commentator wishes to establish between musical work and work on irrational numbers leads to other interesting reflections: the coincidence of Pythagorean interest in irrational numbers and in scale formation was certainly not accidental. It is, for instance, easy to show that a first attempt at a mathematical description of the exact mid-point of the scale would result in the discovery of the irrationality of the square root of 2. For if the ratio of the octave is 2:1, then the mid-point, the geometric mean between 2 and 1, would be $\sqrt{2}$. And it would soon be discovered that no matter what lengths are originally ascribed to the two chords

¹ It is well known that Greek commentators were in the habit of literally repeating the views of their predecessors when they

agreed with them. See, e.g., L. Minio-Paluello in *J.H.S.* lxxvii (1957), 100.

producing the octave¹ there are no numbers which could express the ratio of, say, lower C:mid-scale = mid-scale:upper C. Thus musical theory, as much as geometry, may have led to the discovery of the irrationality of $\sqrt{2}$. But whether this was in fact so is not really important: the musical consideration would at any rate have provided confirmation of the geometrical and arithmetical considerations that had dealt such a shattering blow to the edifice of Pythagorean metaphysics.

But here a comforting thought may have occurred to the Pythagorean: true, the exact mid-point of the scale, or, for that matter, the exact semitone between the two extremes of a full tone interval, cannot be described mathematically in rational numerical expressions. But then there does not seem to be any need for such description. For in the natural scale the exact semitone and the exact mid-point of the scale simply do not occur. And not only do they not occur but if we introduced them that would at once destroy the basis of our scale formation. (For instance, if F were an *exact* semitone above E, the ratio C:F = 4:3 would disappear.) Thus the Pythagorean may have comforted himself by the thought that nature, in informing our ear, has shunned the irrational.

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¹ The numerical ratios of the Octave, the Fourth, the Fifth, and the full tone, i.e. the difference between the Fourth and the Fifth, were, of course known very early as being respectively 2:1; 4:3; 3:2; 9:8. The last ratio is obtained simply by taking the length of the chord producing the fundamental note

as 12; the octave will then be 6, the Fourth 9, the Fifth 8, and the ratio of the Fourth to the Fifth, 9:8; this was then defined as the typical full tone. (See on this also Aristides Quintil. iii. 113 Meibom; Nicom. *Harm.* i. 12 Meibom; Theon Smyrn. p. 66 H.; Ptol. *Harm.* i. 5.)