

The Mystery Of The Aleph: Mathematics, the Kabbalah, and the Search for Infinity
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Chapter One Ancient Roots

Sometime between the fifth and sixth centuries B.C., the Greeks discovered infinity. The concept was so overwhelming, so bizarre, so contrary to every human intuition, that it confounded the ancient philosophers and mathematicians who discovered it, causing pain, insanity, and at least one murder. The consequences of the discovery would have profound affects on the worlds of science, mathematics, philosophy, and religion two-and-a-half millennia later.

We have evidence that the Greeks came upon the idea of infinity because of haunting paradoxes attributed to the philosopher Zeno of Elea (495-435 B.C.). The most well-known of these paradoxes is one in which Zeno described a race between Achilles, the fastest runner of antiquity, and a tortoise. Because he is much slower, the tortoise is given a head start. Zeno reasoned that by the time Achilles reaches the point at which the tortoise began the race, the tortoise will have advanced some distance. Then by the time Achilles travels that new distance to the tortoise, the tortoise will have advanced farther yet. And the argument continues in this way ad infinitum. Therefore, concluded Zeno, the fast Achilles can never beat the slow tortoise. Zeno inferred from his paradox that motion is impossible under the assumption that space and time can be subdivided infinitely many times.

Another of Zeno's paradoxes, the dichotomy, says that you can never leave the room in which you are right now. First you walk half the distance to the door, then half the remaining distance, then half of what still remains from where you are to the door, and so on. Even with infinitely many steps—each half the size of the previous one—you can never get past the door! Behind this paradox lies an important concept: even infinitely many steps can sometimes lead to a *finite* total distance. If each step you take measures half the size of the previous one, then even if you should take infinitely many steps, the total distance traveled measures twice your first distance:

$$0 \ 0 \ 1 \ 1/2 \ 2 \ 1+1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + \dots = 2 \text{ ----- } 1 \ 3/4$$

Zeno used this paradox to argue that under the assumption of infinite divisibility of space and time, motion can never even start.

These paradoxes are the first examples in history of the use of the concept of infinity. The surprising outcome that an infinite number of steps could still have a finite sum is called "convergence."

One could try to resolve the paradoxes by discarding the notion that Achilles, or the person trying to leave a room, must take smaller and smaller steps. Still, doubts remain, for if Achilles must take smaller and smaller steps, he can never win. These paradoxes point to disturbing properties of infinity and to the pitfalls that await us when we try to understand the meaning of infinite processes or phenomena. But the roots of infinity lie in the work done a century before Zeno by one of the most important mathematicians of antiquity, Pythagoras (c. 569-500 B.C.).

Pythagoras was born on the island of Samos, off the Anatolian coast. In his youth he traveled extensively throughout the ancient world. According to tradition, he visited Babylon and made a number of trips to Egypt, where he met the priests—keepers of Egypt's historical records dating from the dawn of civilization—and discussed with them Egyptian studies of number. Upon his return, he moved to Crotona, in the Italian boot, and established a school of philosophy dedicated to the study of numbers. Here he and his followers derived the famous Pythagorean theorem.

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Before Pythagoras, mathematicians did not understand that results, now called theorems, had to be proved. Pythagoras and his school, as well as other mathematicians of ancient Greece, introduced us to the world of rigorous mathematics, an edifice built level upon level from first principles using axioms and logic. Before Pythagoras, geometry had been a collection of rules derived by empirical measurement. Pythagoras discovered that a complete system of mathematics could be constructed, where geometric elements corresponded with numbers, and where integers and their ratios were all that was necessary to establish an entire system of logic and truth. But something shattered the elegant mathematical world built by Pythagoras and his followers. It was the discovery of irrational numbers.

The Pythagorean school at Crotona followed a strict code of conduct. The members believed in metempsychosis, the transmigration of souls. Therefore, animals could not be slaughtered for they might shelter the souls of deceased friends. The Pythagoreans were vegetarian and observed additional dietary restrictions.

The Pythagoreans pursued studies of mathematics and philosophy as the basis for a moral life. Pythagoras is believed to have coined the words *philosophy* (love of wisdom) and *mathematics* ("that which is learned"). Pythagoras gave two kinds of lectures: one restricted to members of his society, and the other designed for the wider community. The disturbing finding of the existence of irrational numbers was given in the first kind of lecture, and the members were sworn to complete secrecy.

The Pythagoreans had a symbol—a five-pointed star enclosed in a pentagon, inside of which was another pentagon, inside it another five-pointed star, and so on to infinity. In this figure, each diagonal is divided by the intersecting line into two unequal parts. The ratio of the larger section to the smaller one is the golden section, the mysterious ratio that appears in nature and in art. The golden section is the infinite limit of the ratio of two consecutive members of the Fibonacci series of the Middle Ages: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ... where each number is the sum of its two predecessors. The ratio of each two successive numbers approaches the golden section: 1.618.... This number is irrational. It has an infinite, nonrepeating decimal part. Irrational numbers would play a crucial role in the discovery of orders of infinity two and a half millennia after Pythagoras.

Number mysticism did not originate with the Pythagoreans. But the Pythagoreans carried number-worship to a high level, both mathematically and religiously. The Pythagoreans considered one as the generator of all numbers. This assumption makes it clear that they had some understanding of the idea of infinity, since given any number—no matter how large—they could generate a larger number by simply adding one to it. Two was the first even number, and represented opinion. The Pythagoreans considered even numbers female, and odd numbers male. Three was the first true odd number, representing harmony. Four, the first square, was seen as a symbol of justice and the squaring of accounts. Five represented marriage: the joining of the first female and male numbers. Six was the number of creation. The number seven held special awe for the Pythagoreans: it was the number of the seven planets, or "wandering stars."

The holiest number of all was ten, *tetractys*. It represented the number of the universe and the sum of all generators of geometric dimensions: $10=1+2+3+4$, where 1 element determines a point (dimension 0), 2 elements determine a line (dimension 1), 3 determine a plane (dimension 2), and 4 determine a tetrahedron (3 dimensions). A great tribute to the Pythagoreans' intellectual achievements is the fact that they deduced the special place of the number 10 from an abstract mathematical argument rather than from counting the fingers on two hands. Incidentally, the number 20, the sum of all fingers and toes, held no special place in their world, while the relics of a counting system based on 20 can still be found in the French language. This strengthens the argument that the Pythagoreans made inferences based on abstract mathematical reasoning rather than common anatomical features.

Ten is a *triangular* number. Here again we see the strong connection the Pythagoreans saw between geometry and arithmetic. Triangular numbers are numbers whose elements, when drawn, form triangles. Smaller triangular numbers are three and six. The next triangular number after ten is fifteen.

A later Pythagorean, Philolaos (4th c. B.C.) wrote about the veneration of the triangular numbers, especially the *tetractys*. Philolaos described the holy *tetractys* as all-powerful, all-producing, the

beginning and the guide to divine and terrestrial life. Much of what we know about the Pythagoreans comes to us from the writings of Philolaos and other scholars who lived after Pythagoras.

The Pythagoreans discovered that there are numbers that cannot be written as the ratio of two whole numbers. Numbers that cannot be written as the ratio of two integers are called *irrational* numbers. The Pythagoreans deduced the existence of irrational numbers from their famous theorem, which says that the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides, $a^2 + b^2 = c^2$. This is demonstrated in the figure below.

When the Pythagorean theorem is applied to a triangle with two sides equal to one, the result is that the hypotenuse is given by the equation $c^2 = 1^2 + 1^2 = 2$, so that $c = [\text{square root of } 2]$. The Pythagoreans realized that this new number could not possibly be written as the ratio of two integers, or whole numbers. Rational numbers, which are of the form a/b where both a and b are integers, have decimals that either become zero eventually, or have a pattern that repeats itself indefinitely. For example, $1/2 = 0.50000 \dots$; $2/3 = 0.6666666 \dots$; $6/11 = 0.54545454 \dots$. Irrational numbers, on the other hand, have decimals that do not repeat the same pattern. Thus to write them exactly one would need to write infinitely many decimals.

The irrationals were a devastating discovery for Pythagoras and his followers because numbers had become the Pythagoreans' religion. *God is number* was the cult's motto. And by *number* they meant whole numbers and their ratios. The existence of the square root of two, a number that could not possibly be expressed as the ratio of two of God's creations, thus jeopardized the cult's entire belief system. By the time this shattering discovery was made, the Pythagoreans had become a well-established society dedicated to the study of the power and mystery of numbers.

Hippasus, one of the members of the Pythagorean order, is believed to have committed the ultimate crime by divulging to the outside world the secret of the existence of irrational numbers. A number of legends record the aftermath of the affair. Some claim that Hippasus was expelled from the society. Others tell how he died. One story says that Pythagoras himself strangled or drowned the traitor, while another describes how the Pythagoreans dug a grave for Hippasus while he was still alive and then mysteriously caused him to die. Yet another legend has it that Hippasus was set afloat on a boat that was then sunk by members of the society.

In a sense, the Pythagoreans' idea of the divinity of the integers died with Hippasus, to be replaced by the richer concept of the continuum. For it was after the world learned the secret of the irrational numbers that Greek geometry was born. Geometry deals with lines and planes and angles, all of which are continuous. The irrational numbers are the natural inhabitants of the world of the continuum—although rational numbers live in that realm as well—since they constitute the majority of numbers in the continuum. A rational number can be stated in a finite number of terms, while an irrational number, such as $[\pi]$ (the ratio of the circumference of a circle to its diameter), is intrinsically infinite in its representation: to identify it completely, one would have to specify an infinite number of digits. (With irrational numbers there is no possibility of saying: "repeat the decimals 17342 forever," since irrational numbers have no patterns that repeat forever.)

Pythagoras died in Metapontum in southern Italy around 500 B.C., but his ideas were perpetuated by many of his disciples who dispersed throughout the ancient world. The center at Crotona was abandoned after a rival mystical group called the Sybaris mounted a surprise attack on the Pythagoreans and murdered many of them. Among those who fled, carrying Pythagoras's flame, was a group that settled in Tarantum, farther inland in the Italian boot than Crotona. Here Philolaos was trained in the Pythagorean number mysticism in the following century. Philolaos's writings about the work of Pythagoras and his disciples brought this important body of work to the attention of Plato in Athens. While not himself a mathematician, the great philosopher was committed to the Pythagorean veneration of number. Plato's enthusiasm for the mathematics of Pythagoras made Athens the world's center for mathematics in the fourth century B.C. Plato became known as the "maker of mathematicians," and his academy had at least four members considered among the most prominent mathematicians in the ancient world. The most important one for our story was Eudoxus (408-355 B.C.).

Plato and his students understood the power of the continuum. In keeping with number worship—now brought to a new level—Plato wrote above the gates of his academy: "Let no one ignorant of geometry enter here." Plato's dialogues show that the discovery of the incommensurable magnitudes—the irrational numbers such as the square roots of two or five—stunned the Greek mathematical community and upset the religious basis of the Pythagoreans' number worship. If integers and their ratios could not describe the relationship of the diagonal of a square to one of its sides, what could one say about the perfection the sect had attributed to whole numbers?

The Pythagoreans represented magnitudes by pebbles or *calculi*. The words "calculus" and "calculation" come from the calculi of the Pythagoreans. Through the work of Plato's mathematicians and Euclid of Alexandria (c. 330-275 B.C.), author of the famous book *The Elements*, magnitudes became associated with line segments, as arithmetized geometry took the place of the calculi. The dichotomy between numbers and continuous magnitudes required a new approach to mathematics—as well as to philosophy and religion. In keeping with this new way of seeing things, Euclid's *Elements* discussed the solution of a quadratic equation, for example, not algebraically but as an application of areas of rectangles. Numbers still reigned in Plato's academy, but now they were viewed in the wider context of geometry.

In the *Republic*, Plato says "Arithmetic has a very great and elevating effect, compelling the mind to reason about abstract number." *Timaeus*, a book in which Plato writes about Atlantis, is named after a member of the Pythagorean order. Plato also refers to a number he calls "the lord of better and worse births," a number that through the centuries has become the subject of much speculation. But Plato's greatest contribution to the history of mathematics lies in having had disciples who advanced the understanding of infinity.

Zeno's idea of infinity was taken up by two of the greatest mathematicians of antiquity: Eudoxus of Cnidus (408-355 B.C.) and Archimedes of Syracuse (287-212 B.C.). These two Greek mathematicians made use of infinitesimal quantities—numbers that are infinitely small—in trying to find areas and volumes. They used the idea of dividing the area of a figure into small rectangles, then computing the areas of the rectangles and adding these up to an approximation of the unknown desired total area.

Eudoxus was born to a poor family, but had great ambition. As a young man, he moved to Athens to attend Plato's Academy. Too poor to afford life in the big city, he found lodgings in the port town of Piraeus, where the cost of living was low, and commuted daily to the academy in Athens. Eudoxus became Plato's star student and traveled with him to Egypt. Later in his life, Eudoxus became a physician and legislator and even contributed to the field of astronomy.

In mathematics, Eudoxus used the idea of a limit process. He found areas and volumes of curved surfaces by dividing the area or volume in question into a large number of rectangles or three-dimensional objects and then calculating their sum. Curvature is not easily understood, and to compute it, we need to view a curved surface as the sum of a large number of flat surfaces. Book V of Euclid's *Elements* describes this, Eudoxus's greatest achievement: the method of exhaustion, devised to compute areas and volumes. Eudoxus demonstrated that we do not have to assume the actual existence of *infinitely* many, infinitely small quantities used in such a computation of the total area or volume of a curved surface. All we have to assume is that there exist quantities "as small as we wish" by the continued division of any given total magnitude: a brilliant introduction of the concept of a *potential infinity*. Potential infinity enabled mathematicians to develop the concept of a limit, developed in the nineteenth century to establish the theory of calculus on a firm foundation.

The techniques first developed by Eudoxus were expanded a century later by the most famous mathematician of antiquity: Archimedes. Influenced in his work by ideas of Euclid and his school in Alexandria, Archimedes is credited with many inventions. Among his discoveries is the famous law determining how much weight an item loses when it is immersed in a liquid. His work on catapults and other mechanical devices used to defend his beloved Syracuse enhanced his reputation in the ancient world. In mathematics, Archimedes extended the ideas of Eudoxus and made use of potential infinity in finding areas and volumes using infinitesimal quantities. By these methods, he derived the rule stating that the volume of a cone inscribed in a sphere with maximal base equals a third of the volume of the sphere. Archimedes thus showed how a potential infinity could be used to find the volume of a sphere and a cone, leading to actual results. After Archimedes' death at the hands of a Roman soldier,

a stone mason chiseled the cone inscribed in a sphere on his gravestone to commemorate what Archimedes considered his most beautiful discovery.

Greek philosophers and mathematicians of the Golden Age, from Pythagoras to Zeno to Eudoxus and Archimedes, discovered much about the concept of infinity. Surprisingly, for the next two millennia, very little was learned about the mathematical properties of infinity. The concept of infinity, however, was reborn during medieval times in a new context: religion.

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