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ASCENDING TO PROBLEMS:
ASTRONOMY AND
HARMONICS IN *REPUBLIC VII*

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In book VII of the *Republic* (521c ff.) Socrates describes five sciences (*mathemata*)—arithmetic, geometry, stereometry, astronomy, harmonics—which would play a fundamental role in the education of the rulers of the ideal state which he is constructing. From a modern point of view there is an important and obvious difference between the first three, which are purely deductive, and the last two, in which observation is of paramount importance. Socrates, however, appears to go out of his way to deny this difference. His students will “let be the things in the heavens” (530b7)¹ in studying astronomy and pay no attention to “heard concords” (531c1-2) in connection with harmonics. Most commentators have found Socrates’ descriptions of astronomy and harmonics puzzling, if not outrageous. I would like to try to minimize the seeming anomaly of these descriptions by invoking certain Greek scientific texts which, I believe, make clearer the kind of astronomy and harmonics Plato has in mind in the *Republic*. I shall argue that Plato assimilates astronomy to geometry (including stereometry) and harmonics to arithmetic, and that his doing so is not unreasonable in the light of these texts. I shall also attempt to explain what distinctions are needed in order to undermine the assimilations made by Plato.

I shall assume that underlying what Socrates says is a conception of scientific knowledge frequently ascribed to Plato. For my purposes there are two fundamental features of this conception; one is logical, the other ontological.² The former is relatively easy to state; for Plato, scientific know-

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ledge is deductive in structure. Although Plato does not describe the principles governing deduction, he does stress the role of assumptions and hypotheses and the development of a theory from them.³ From a linguistic point of view, the most important of these hypotheses are precise definitions. From a more ontological point of view, the hypotheses are assumptions that there exist objects satisfying precisely determined conditions. Plato develops this ontological point of view by arguing that ordinary physical objects do not satisfy the precisely determined conditions specified in scientific hypotheses. Hence, he concludes, the objects of science exist in an ideal realm outside the physical world. Socrates stresses this point in his discussion of the sciences, since their principal role in his educational scheme is to draw the minds of the students away from sensible becoming (and observation) to intelligible being (521d2-3). According to Aristotle, Plato held that the objects of the mathematical sciences exist in a realm between the forms and sensible things:

... Besides sensible things and forms [Plato] says there are objects of mathematics (*mathematika*), which occupy an intermediate position, differing from sensible things in being eternal and unchangeable, from forms in that there are many alike, while the form itself is in each case unique. (*Met.* A.6. 987b14-18)

In antiquity this account of Plato's view of science was taken for granted. The later neo-Platonists associated it with the view that science is a function of a faculty called *dianoia*, the objects of which are *dianoetika*, as opposed to *nous*, the objects of which include the forms.⁴ In the twentieth century there has been a great deal of controversy about the accuracy of this interpretation of Platonic theory. I hope to be able to by pass this controversy. For my purposes it is sufficient that Plato assumes every science to be the study of some kind of real but non-sensible object or objects.⁵

Most commentators have not hesitated to ascribe to Plato an ontology of non-sensible arithmetic, geometric, and stereometric objects. Many of them have balked at extending this ontology to astronomy and harmonics. If Plato did not

intend such an extension, he certainly made no attempt to apprise the readers of the *Republic* of the fact. In describing the position of stereometry between plane geometry and astronomy, the only relevant contrast Socrates makes is that astronomy studies "the solid in motion" (528a9) or the "movement of solids" (528e1) whereas stereometry studies the solid in itself (*auto kath' auto* 528b1). Socrates does not emphasize the introduction of motion into science, but treats it as a matter of course. Similarly, he does not say explicitly that the objects of astronomy are something other than the heavenly bodies. However it is natural to take him to mean this, given his remarks about arithmetic and geometry and his disparagement of astronomic observation. Thus if Socrates' description of arithmetic and geometry is based on a notion of intermediates satisfying the hypotheses of those sciences, then the description of astronomy implies the existence of moving intermediates, even if Plato never explicitly asserted the existence of such objects. On the other hand, if Socrates' description of arithmetic and geometry presupposes that these sciences are the study of forms, then his description of astronomy implies that it is likewise and that there is a form of motion or of the solid-in-motion or of some such thing. In other words, in whatever sense arithmetic is the study of "units which can only be conceived by thought" (526a6-7), geometry the study of the "eternally existent" (527b7), and stereometry the study of the "solid in itself" (528b1), astronomy would seem to be the study of the "solid in revolution" (528a9), and, to extend the discussion to the last part of Socrates' mathematical curriculum, harmonics the study of "concordant numbers" (531c3).

The inclination of moderns to accept the ascription to Plato of an ontology which includes non-sensible arithmetic and geometric objects but does not include non-sensible astronomic and harmonic objects seems to derive in part from the modern versions of related sciences. Modern astronomy and harmonics obviously do study physical things, and modern pure mathematics, the outgrowth of Greek arithmetic and geometry, includes disciplines having no apparent connection with the physical world. It is important to realize that no such sharp dichotomy exists for the ancient versions of the

sciences in Socrates' curriculum. Some of the astronomic and harmonic texts which I will be considering shortly are not easily distinguished from their arithmetic and geometric analogues. These analogues themselves are very closely tied to the physical world. Greek geometry studies the spatial properties of idealized physical objects, Greek arithmetic the properties of finite sets of these objects, when the objects are conceived of as units, i.e., as single things. Aristotle's conception of mathematics as the study of physical objects thought of in abstraction from certain of their properties⁶ seems much more natural and appropriate for Greek arithmetic and geometry than Plato's postulation of separate mathematical entities. It seems reasonable to assume that Plato's mathematical ontology derives at least as much from his reflection on general philosophical problems as from a detailed study of the character of mathematics.⁷ Given the fairly loose connection between the Greek arithmetic and geometry of Euclid's *Elements* and this ontology, one should not expect any stronger connection between the astronomy and harmonics of analogous texts and what Socrates says or implies about these sciences in the *Republic*.

The astronomical texts to which I shall be referring are from the so-called *Lesser Astronomy*. They are Theodosius's *Spherica*, Autolycus's *On a Moving Sphere*, and Euclid's *Phenomena*. Although the *Spherica* was compiled at least two centuries after the death of Plato, there is no reason to doubt that it differs little in content or form from works extant in the late fourth century. For Autolycus and Euclid not only presuppose much of the content of the *Spherica*, they also quote propositions from it *verbatim*. The treatise of Autolycus is considered to be one of the earliest extant examples of Greek deductive mathematics, dating from the late fourth century. The *Phenomena* is assumed to be approximately contemporary.⁸

The *Spherica* is, in one sense, simply the theory of circles formed on the surface of a sphere by planes passing through it. But only excessively elaborate verbal paraphrase keeps the astronomical content out of sight. For example, Theodosius states and proves

Sph. III. 7. If in a sphere a greatest circle touches some circle of those in the sphere and some other greatest circle oblique to the circles parallel [to the one touched by the first greatest circle] but touching parallel circles larger than the one touched by the first, the points of contact being on the first greatest circle, and if equal arcs are cut off consecutively from the oblique circle on the same side of the greatest of the parallel circles and parallel circles drawn through the generated points, they will cut off between them unequal arcs of the first greatest circle and the arc nearer to the greatest of the parallel circles will always be greater than the further away one,

which means

Of arcs of the horizon determined by consecutive planes parallel to the equator and passing through beginning points of signs of the zodiac on one side of the equator, an arc closer to the equator is greater than one further away.

as is shown by Euclid's use of the proposition in

Ph. 8. The signs of the zodiac rise and set in unequal sections of the horizon, those toward the equator in the biggest sections, those which follow in less, those toward the tropics in the least.

A comparison of this proposition from the *Phenomena* with *Spherica III. 7* reveals two major and characteristic differences. Theodosius's terminology is geometric and his propositions more general, since he does not take as given the relative position of the horizon and such lines and points as the ecliptic and the poles. (These features of the *Spherica* account for the wordiness of *III.7*.) Euclid substitutes astronomical terms for geometric descriptions and sometimes takes for granted actual relative positions.⁹

A third difference is the presence of motion in the *Phenomena* and its absence from the *Spherica*. Mathematically this change is hardly noticeable, as can be seen by looking at Autolycus's *On a Moving Sphere*, which is a more fundamental treatise than the *Phenomena* in the sense that its content is presupposed in the *Phenomena*. Autolycus establishes certain properties of a sphere which rotates "uniformly." He assumes:

Points are said to move uniformly when they pass through equal and similar magnitudes in an equal time. If some point moving uniformly on some line passes through two lines [i.e., two segments of the line], the time in which the point passes

through one of the lines will have the same ratio to the other time as the line has to the line. (Aut. 195.3-8)

With these assumptions, questions about the time of movements are reduced to questions about lines which can be handled in an ordinary geometric way.¹⁰ Autolycus's terminology is almost as purely geometric as Theodosius's. However, he does use the phrase 'a great circle dividing (horizon) the visible and invisible hemispheres of the sphere', a phrase which he shortens to 'horizon'. He also speaks of points rising and setting.¹¹ Theodosius clearly strove to eliminate all astronomical terminology from the *Spherica*. Since Autolycus represents but does not mention the equator, the tropics, the ecliptic or zodiac, it seems likely that he strove to do the same, treating 'visible', 'invisible', 'rising', 'setting' as terms with an obvious enough geometric paraphrase not to need definition. Whatever terminology is used, a Platonist would have no trouble in assimilating both Autolycus's *On a Moving Sphere* and Theodosius's *Spherica* to Euclid's geometry. Both use geometrical reasoning to derive theorems from initial assumptions. In this sense they satisfy the Platonic methodological requirement that a science be deductive. In addition, since the deductions presuppose that the objects treated satisfy exact geometric conditions, it would be possible to argue along Platonic lines that the objects must exist outside the physical world. Aristotle provides us with such an argument:

But on the other hand astronomy cannot be dealing with perceptible magnitudes nor with this heaven. For neither are perceptible lines such lines as the geometer speaks of (for no perceptible thing is straight or round in the way in which he defines 'straight' and 'round'; for a hoop touches a straight edge not at a point, but as Protagoras used to say it did in his refutation of the geometers), nor are the movements and spiral orbits in the heavens like those of which astronomy treats, nor have points the same nature as the stars. *Met.* B.2. 997b34-998a6)

Aristotle here refers to the idealization involved in geometrical astronomy as a reason for saying that it does not deal with the perceptible heaven. He points out, and I would say correctly, that there is no difference in this regard between astronomy and geometry. If the character of geometry jus-

tifies the postulation of ideal objects, so does the character of astronomy.

Although the treatises under consideration can be applied to relatively few astronomic phenomena, there is no doubt that the techniques involved could be extended, at least in principle, to cover a much wider range of facts. Nor would there be any problem in substituting ordinary astronomical terms like 'horizon', 'ecliptic', 'star' for their more elaborate geometrical paraphrases. However, one is inclined to think that general statements like *Spherica* III.7 do not become astronomical statements simply by the substitution of astronomical terms for geometrical phrases. The statements also need to be given a specific content by fixing the positions and movements of various lines and points. What is missing from Socrates' account of astronomy is any attempt to explain how such a specific content would be justified. Euclid's *Phenomena* provides an example of how such justifications would ordinarily proceed. It includes a preamble in which Euclid argues for a certain geometric representation of astronomical phenomena. The beginning of the preamble is typical of the whole:

Since the fixed stars are always seen rising from the same place and setting in the same place, and those which rise simultaneously are always seen rising simultaneously, and those which set simultaneously are always seen setting simultaneously, and in their motions their distances from one another are always seen to be the same, and since this only occurs with things moving in a circle when the eye is equally distant from the entire circumference, as is proved in the *Optics*, let it be postulated (*theteon*) that the stars move in a circle and are attached in one body and that the eye is equally distant from their circumferences. (*Ph.* 2.1-10)

Here, of course, rising and setting are taken for granted, as is a long series of observations. The reference to the *Optics* is puzzling, since no corresponding proposition occurs in our texts of Euclid's *Optics*. Clearly the claim that the observed movements of the stars are only compatible with the hypothesis of a circular motion seen from a point equidistant from the whole circumference involves tacit assumptions about the forms which the movement could possibly take and our own stationariness. Similar points could be made about

the entire preamble. However the important point is that Euclid does attempt to justify certain astronomic-geometric correlations and explain the geometrical account of astronomical terms by reference to allegedly observable phenomena. Once these correlations have been established, the treatise proceeds in a strictly deductive way.¹²

It seems unlikely to me that Socrates would countenance such justifications in the astronomy taught to the future rulers of his ideal state. For in book VI of the *Republic* he says that within mathematics itself the justification of hypotheses plays no role:

. . . Students of geometry and reckoning and such subjects first postulate the odd and the even and the figures and three kinds of angles and other things akin to these in each branch of science, regard them as known, and, treating them as hypotheses, do not deign to render any further account of them to themselves or others. . . (510c2-7)

If this account were applied to astronomy, then the purported justifications of hypotheses in the preamble to the *Phenomena* would have no place in Platonic astronomy. Rather the astronomer would simply lay down some hypotheses and derive consequences from them. Socrates may have such derivations in mind when he says that astronomy should be studied "by means of problems. . . as in geometry." (530b6-7)¹³

Socrates' answer to the question 'How are astronomical hypotheses justified?' would seem, then, to be that hypotheses are not justified within astronomy itself. Such an answer is, of course, appropriate within Socrates' educational scheme, in which the justification of hypotheses is reserved for dialectic. However it would seem that Plato must have realized that in ordinary astronomy hypotheses are thought to be justified by reference to the things in the heavens, as they are justified in the *Phenomena*. How is it possible to explain Socrates' explicit denial that the phenomena are relevant to astronomy as he conceives it? I am inclined to think that the answer to this question lies in a Platonic assimilation of astronomy to geometry. In other words, I believe that Plato thought the relationship of the hypotheses of geometrical astronomy to the heavens to be the same as the relation of geometric hypotheses to ordinary physical ob-

jects. It is now a philosophical commonplace that reference to physical objects or other phenomena cannot provide a justification for geometrical hypotheses, whatever psychological effect such references might have on an individual. The explanation for this commonplace is that ordinary objects do not satisfy geometric hypotheses because of the idealization involved in geometry. In the quotation given above, Aristotle points out that geometrical astronomy also involves idealization. My suggestion is that this idealization may have been Plato's only ground for ignoring the role of phenomena in the justification of astronomical hypotheses.

The last question I would like to consider in connection with Platonic astronomy concerns the tenability of Plato's assimilation of astronomy to geometry. There are, of course, many ways in which a person might try to argue against this assimilation. One might be to invoke falsification. Astronomical hypotheses would seem to be falsifiable in a way in which geometric ones are not. I doubt that such an argument would work against Plato. For, if I have interpreted him correctly, he believes that the phenomena do not satisfy either geometric or astronomic hypotheses (although, of course, they do not falsify them either, since the hypotheses don't concern phenomena at all). To distinguish geometry and astronomy in a way satisfactory to Plato, one would have to distinguish between the ways in which phenomena fail to satisfy the hypotheses of the two sciences. Geometrical hypotheses involve idealization. So do astronomical hypotheses, but these also involve what might be called approximation. For example, Euclid treats the earth as an eye, the eye as a point. His doing so is justified by the assumption that the stars are extremely far away relative to the earth. Can one distinguish conceptually between the idealization involved in Euclid's assumption in the *Elements* that two points determine a unique straight line and his treatment of the earth in the *Phenomena*, which, I suggest, involves both idealization and approximation? I do not know the answer to this question, but it seems to me that without such a distinction Socrates' account of astronomy is as tenable as his account of geometry.

When Socrates turns from astronomy to harmonics, he

describes them as "kindred sciences" which study two forms of movement. "We may venture to suppose," he says, "that as the eyes are framed for astronomy so the ears are framed for the movements of harmony." (530d6-7) After he has disparaged both the attempt to determine minimal musical intervals by experimentation and the search for numbers in heard concords, he calls for a harmonics which ascends to problems and considers "which numbers are concordant and which not and why in each case." (530e1-531c4) The subject matter of Platonic harmonics would appear, then, to be numbers or, more exactly, relations between numbers, ratios. The principal distinction between harmonics and arithmetic will be the numerical properties with which each is concerned. Whereas arithmetic will deal with such properties as being composite or relatively prime, harmonics will be concerned with numerical concordance and discordance in general and such particular examples of them as being an octave or a tone.

In modern times the idea of concordant numbers has found few defenders; yet in antiquity the notion of harmonics as a branch of arithmetic seems to have flourished. The text which best embodies this notion is the *Sectio Canonis*, usually ascribed to Euclid. In it Euclid attempts to establish numerical expressions for the concordant intervals using only arithmetic and some further assumptions. Like the *Phenomena*, the *Sectio* begins with an explanatory preamble.

If there were rest and motionlessness there would be silence; if there were silence and nothing moved, nothing would be heard. Therefore if something is to be heard there must be beforehand a blow and motion. Thus since all notes occur when a blow occurs but it is impossible for a blow to occur unless a movement occurs beforehand, and [since] of motions some are denser, some rarer, and the denser make higher notes, the rarer deeper, necessarily there are some higher notes since they are composed of more compact and more numerous movements, and some deeper since they are composed of rarer and fewer movements. Thus those notes which are higher than fitting attain the fitting when relaxed by a subtraction or reduction of motion, the deeper attain the fitting when tensed by an addition or increase of motion. Therefore notes are said to be composed of parts since by addition and subtraction they attain the fitting. All things

which are composed of parts are said to have the ratio of a number to one another so that notes are also necessarily said to be in the ratio of a number to one another. Of numbers, some are said to be in a multiple ratio, some in an *epimorios* (expressible in the form $n+2/n+1$), some in an *epimeres* (expressed in least numbers by $m+1+n+2/n+2$)¹⁴ so that notes are said necessarily to be in such ratios to one another. We also know that of notes some are concordant, some discordant, and the concordant make one blend from two things, the discordant do not. It is therefore reasonable (*eikos*) that the concordant notes be among the numbers which are called by one name in relation to one another, i.e., the multiples and the *epimoria*, since they make one blend of voice out of two things. (Sectio 158.1-160.4)

In this preamble Euclid makes a general reference to empirical phenomena in order to "justify" a frequency theory of pitch and thereby the numerical representation of pitch relations.¹⁵ This seems to be the only purpose of the preamble. For once the representation is given, the *Sectio* proceeds in a purely arithmetic way except for a curious blend of musical and arithmetic terminology.¹⁶ The main difference between the preambles of the *Sectio* and the *Phenomena* is that the preamble of the former does not even attempt to provide any genuine observational basis for the mathematical model adopted in the way that the preamble of the latter does. In particular, the frequency theory is not used and, given Greek experimental capacity, could not have been used to justify the assignment of particular kinds of ratios to consonant intervals. From a modern point of view the justification of this assignment in the preamble is extremely tenuous. In Greek, *epimoria* and multiples can be expressed by one word, the *n*th multiple by adding the suffix '-plasios' to a form of the word for *n*, the ratio $n+1/n$ by adding the prefix 'epi-' to a stem meaning *n*th. (The latter rule is true for *n* greater than 2; $3/2$ is called *hemiholios*.) On the other hand, ratios of the general form $m+n/n$ are expressed by '*m+n* nths' and hence not called by one name. Euclid "argues" that concordant intervals should be associated with multiples and *epimorios* ratios, since notes separated by such intervals combine to produce a unified sound. Although no trace of this particular piece of reasoning is found elsewhere, there is no reason to doubt that it is seriously meant. Others

equally insubstantial are ascribed by ancient authors to the Pythagoreans. For example, Ptolemy says that they chose to assign multiples and *epimoria* to the concordant intervals "because of the simplicity of the comparison: in *epimoria* the excess [of the greater term over the less] is a part, in multiples the less is a part of the greater." (*Har.* 11.15-17) Ptolemy chastises the Pythagoreans for their arbitrary procedure, particularly on the grounds that it leads to the exclusion from the concordant intervals of the octave-plus-fourth, which turns out to have the ratio 8/3. There is no reason to doubt that Ptolemy and others who repeat his criticism are correct from an empiricist point of view. For the Greeks the octave-plus-fourth is a concordant interval so that any theory, like that of the *Sectio*, which excludes it is simply contradicting the phenomena.

The preamble of the *Sectio* purports to provide a justification for the fundamental hypothesis of harmonics: the concordant notes are among the numbers which are called by one name in relation to one another. This assumption can be expressed as

SCA: If two numbers are concordant, the greater is either a multiple or an *epimorion* of the lesser.

The basic program of the *Sectio* is, first, the derivation from SCA of specific numerical assignments for the concordant intervals and the tone, and, second, the "refutation" of certain claims made by more empirical music theorists.¹⁷ In making his derivations Euclid treats addition and subtraction of musical intervals as the multiplication and division of ratios respectively. He also takes for granted the following:

SCB. The concordant intervals are, in order of increasing size, fourth, fifth, octave, octave-plus-fifth, double octave.

SCC. The octave is composed of a fourth plus a fifth.

SCD. A tone is the difference between a fifth and a fourth.

As has already been mentioned, Euclid carries out his program in a strictly arithmetical way. Thus, for example, to prove

SC10. The octave interval is a multiple,

Euclid argues that the double octave must be either a multiple or an *epimorion* (SCA and SCB) and invokes two previously proved arithmetic facts; according to SC3 an *epi-*

morion is never the square of a ratio, so that the double octave is a multiple, but then by SC5, according to which a ratio with a multiple as square is itself a multiple, the octave is a multiple. Although Euclid does not actually succeed in carrying out his program due to a fallacy in the argument for SC11,¹⁸ there seems to me no doubt that the body of the *Sectio* can be adequately characterized as the investigation, by means of problems, of which numbers are concordant and which are not. If the preamble were simply replaced with SCA-SCD, the result would be a Platonic harmonics.

An interesting feature of the *Sectio* is its incompatibility with empirical facts. I have already mentioned that the failure to include the octave-plus-fourth among the concordant intervals is simply a case of ignoring musical facts. Other interesting cases are provided by Euclid's mathematical refutations of the claims of more empirical music theorists. For example, in SC16 Euclid proves that there is no division of the tone into equal intervals. In mathematical terms this proposition is simply the true assertion that the equation ' $9x^{m+1} = 8y^{m+1}$ ' has no integral solutions. However, because of the limits of human auditory discrimination, any careful experiment with a stringed instrument would falsify this claim, interpreted musically. In this respect one might compare proposition 16 with the assertion of the incommensurability of the side and the diagonal of a square, which likewise is always disconfirmed by careful measurement.

The separation of harmonics from observation seems to have been a tradition in the subject.¹⁹ When Aristoxenus opposes his own procedures and views to those of people who "reason in another way and turn away from perception as not being accurate, constructing intelligible causes and saying that high and low pitch come to be in certain numerical ratios and relative speed" (*Har.* 32.21-26), he is undoubtedly referring to the kind of theorizing found in the *Sectio Canonis*. Even Ptolemy, who insists upon the importance of observation and experiment in music,²⁰ allows *logos* to be the ultimate judge, since "sensation finds the approximate and receives from elsewhere what is accurate, *logos* receives the approximate from elsewhere and finds what is accurate" (*Har.* 3.6-8).

It seems reasonably clear, then, that Plato would have had no trouble in interpreting the *Sectio*, purged of its preamble, as a piece of deductive arithmetic based on SCA-SCD and ordinary arithmetic hypotheses. Presumably he would not have felt it incumbent upon the music theorist to justify either his hypotheses or such terms as 'concordant' and 'octave'. These terms and hypotheses are simply ways of referring to and describing perfectly clear characteristics of numbers. Perhaps the first thing that strikes us as wrong with the Platonic conception, as I have explained it, is that it does not treat the numbers of harmonic theory as numbers of anything. We understand ratios in harmonics as ratios between physical magnitudes: string lengths or frequencies, for example. Plato may have understood the ratios of harmonics in this way, but there is no evidence of his doing so in the *Republic*. Socrates speaks of harmonics as the study of numbers and not as the study of measurable characteristics.²¹ A similar conception of harmonics is found in a Pythagorean division of the sciences, which has often been compared with the scientific curriculum of the *Republic*:

The Pythagoreans considered all mathematical science to be divided into four parts: one half they marked off as concerned with quantity, the other half with magnitude; and each of these they posited as twofold. A quantity can be considered in regard to its character by itself or in its relation to another quantity, magnitude as either stationary or in motion. Arithmetic then studies quantity as such, music the relation between quantities, geometry magnitude at rest, spherics magnitude inherently moving. (Proclus, *In Pr. Eucl.* 35.21-36.3)

If harmonics is simply the study of certain properties of numbers, then the relation between Platonic harmonics and the physical world will probably not be the same as the relation between Platonic astronomy and the physical world. For, whereas points on a rotating sphere are a direct representation or idealized picture of the sphere of the fixed stars, the ratio of two to one is not in the same sense a picture of the octave. This difference between Platonic harmonics and astronomy has a parallel in the case of geometry and arithmetic, which might be brought out in the following way. When we measure the angles of a triangle having sides which appear to be equal, we expect the angles to

appear more or less equal too, but we do not insist that they be exactly equal. But when we arrange objects in m rows of n objects each, we expect there to be exactly $n \times m$ objects in the array. In other words, although we may concede that there are no perfect physical realizations of geometric truths, there is no reason for us to concede the same thing in the case of arithmetic truths.

When Socrates contrasts pure and applied arithmetic in the *Republic* (525c8-526a7), he stresses the difference between the abstract units of pure numbers, which are "equal to every other without the slightest difference and admitting no division into parts," and the unequal divisible things which we count in everyday life. The distinction which Socrates makes here is a conceptual one and does not affect the question whether the truths of arithmetic are perfectly realized in the physical world, whether, for example, there are always twenty cows in five rows of four cows each. I am unable to find any evidence that Plato was conscious of the difference between the two contrasts: the geometric one between idealized picture and physical realization and the arithmetic one between abstract object and physical one. Nevertheless I would like to make use of the difference in discussing briefly the question of how we should understand the relation between arithmetic harmonics and the physical world. Clearly the answer to this question depends upon what we take the physical embodiment of sound to be: in other words, what we take the physical interpretation of harmonics to be. If the numerical ratios are understood as ratios of string lengths, the relation of pure harmonics to the physical world would presumably be like the relation of geometry to it; just as two strings cannot be exactly equal, neither can they be exactly in the ratio of two to one. To argue against the assimilation of harmonics to pure mathematics, one would have to construe harmonics as involving approximation like astronomy.

On the other hand, if Plato believed that the ratios of harmonics are properly interpreted as ratios of frequencies,²² he might claim that, like the truths of arithmetic, the truths of harmonics have perfect realizations. For clearly, if strings a and b are vibrating at frequencies in the ratio of three to

two and b and c are vibrating in the ratio of four to three, then a and c are vibrating in the ratio of two to one. The difficulty arises, however, when we try to translate this truth into a statement about sound. Presumably Plato believed that our perceptions of fourths, fifths, and octaves are too imprecise to enable us to determine ratios between frequencies, and hence believed that we are never in a position to confirm or disconfirm the claim that an octave is composed of a fourth and a fifth. It is perhaps from this point of view that he belittles the Pythagorean search for numbers in heard concords. If the position I have just been describing is Plato's, then perhaps the relevant way to distinguish harmonics and arithmetic is, in terms of relative accuracy of perception. Although my senses never enable me to determine the exact ratio of the frequencies of two vibrating strings, they do allow me to determine the number of people in a room. The differences between harmonics and arithmetic would seem to involve a notion of approximation, but a slightly different one from that involved in the difference between astronomy and geometry. Both arithmetic and harmonics can be treated as pure sciences of number, but, whereas the truths of arithmetic can sometimes be determined to hold exactly in the physical world, the laws of harmonics are never determined to be more than approximately true.

NOTES

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1. In this paper I use, with occasional changes, the following translations: Shorey's *Republic* of Plato, Ross's *Metaphysics* of Aristotle, and Morrow's *Commentary on Book I of Euclid's Elements* of Proclus. Other translations are mine. References are made to the standard editions listed in the bibliography.

2. What I take as the two fundamental features correspond to points II and VI of A. Wedberg's account of Plato's philosophy of arithmetic and geometry, although Wedberg ascribes to Plato a theory of intermediate objects. See A. Wedberg, *Plato's Philosophy of Mathematics* (Stockholm, 1955), pp. 61, 62, 65, 67.

3. See especially *Rep.* VI 510b2 ff.

4. See, e.g., Proclus, *In Pr. Eucl.*, 10.16-11.25. The view is, of course, a plausible interpretation of the middle books of the *Republic*.

5. I must admit that some views I ascribe to Plato in this paper are more simply interpreted with intermediates than without them, particularly if forms are construed as abstract concepts rather than perfect exemplars. It seems to me, however, that the same could be said of many of Plato's remarks about arithmetic and geometry.

6. For substantiation of these claims about Greek mathematics and Aristotle's interpretation of it, see my papers "Euclid's *Elements* and the axiomatic method," *British Journal for the Philosophy of Science* 20(1969): 289-309, and "Aristotle on geometric objects," *Archiv für Geschichte der Philosophie* 52 (1970): 156-71.

7. Thus it is necessary to distinguish between Plato's mathematical Platonism and the modern mathematical philosophy called Platonism. The latter is based on particular features of modern mathematics which are foreign to Greek mathematics. For a description of modern mathematical Platonism, see P. Bernays, "On Platonism in mathematics," in P. Benacerraf and H. Putnam (eds.), *Readings in the Philosophy of Mathematics* (Englewood-Cliffs, N.J., 1964), pp. 274-86.

8. See T. Heath, *A History of Greek Mathematics* (Oxford, 1921), Vol. I, pp. 348-53 or, for more detail, F. Hultsch, "Autolykos und Euklid," *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Philologisch-Historische Klasse*, 38(1886): 128-55. I have chosen to treat the mathematical texts with which I deal in this paper as entirely genuine rather than enter into the thorny questions which have been raised about the authenticity of various parts of them.

9. Euclid does not always do so, however. He adds to the antecedent of proposition 2 the condition that the pole of the horizon be between the summer tropic and the visible pole and imposes a similar condition in

proposition 7. In his treatise *On Risings and Settings* Autolycus always takes such things for granted.

10. This is a slight exaggeration, since the transformation from spatio-temporal to purely spatial considerations presupposes the description of the path of a point on the surface of a rotating sphere and an extension of Autolycus's assumptions to such points moving on different paths. Autolycus establishes these presuppositions in propositions 1 and 2 with arguments which are in fact question-begging.

11. 'Earth' and 'stars' each occur once in proposition 9. Hultsch ("Autolykos und Euklid," 144, fn.) ascribes their occurrence to "later reworkings of the original text."

12. There is one puzzling exception to this generalization, namely, the first proposition of the *Phenomena*. It says that the earth is the center of the cosmos. To prove this assertion Euclid invokes a measuring instrument, the diopter, and describes "observations" which establish that the zodiac is bisected by the horizon, a claim already invoked in the preamble. Since actual observations with a diopter would not yield the results envisaged by Euclid and since the preamble includes material for proving proposition 1 in a strictly geometric way, it is difficult to determine why Euclid proceeds as he does.

13. I am inclined to think that the word 'problem' here does not have the technical sense of 'construction' but the more general sense of 'something set out for proof or refutation'. See, e.g., Aristotle, *Topics* A.4.101b28-37, or the scholium, perhaps by Proclus, on the similar use of the word at Th. 180c5. (W. C. Greene (ed.), *Scholion Platonica* [Haverford, 1938], p. 33) There does not seem to be any particular reason for Plato to focus on constructions in particular in the *Republic* passage.

14. Theon (*Expos.* 78.6-22) and Nicomachus (*Int.* I.20) both restrict the term *epimeres* to ratios expressed in least numbers in the form $m+1+n+2/n+2$ with $n+2$ greater than $m+1$. If Euclid does the same, he has not, of course, accounted for all ratios of a greater to a lesser number in his threefold classification.

15. See B. L. van der Waerden, "Die Harmonielehre der Pythagoreer," *Hermes* 78(1943): 192-97. The justification of the numerical representation is, of course, illegitimate, since frequencies can be irrational.

16. The claim that the *Sectio* proceeds purely arithmetically is true of all but the last four propositions, which involve the determination of certain musical "systems" (tunings or scales). However these propositions too are totally independent of the frequency theory of the preamble. An example of the curious terminology is the use of the word *diastema* to mean both 'ratio' and 'interval'.

17. The doctrines of these theorists are most easily found in the *Harmonics* of Aristoxenus.

18. The fallacy was first pointed out by P. Tannery, "Inauthenticité de la 'Division du Canon' attribuée à Euclide," *Mémoires Scientifiques* (Toulouse and Paris, 1911 ff.), vol III, p. 215. It amounts to taking SCA as asserting that all multiples are concordant. In fact SCA-SCD are satisfied

by any assignment of ratios of the following form: tone: $m+n+1/n+1$; fourth: $n+1/1$; fifth: $m+n+1/1$; octave: $(m+n+1)(n+1)/1$; octave-plus-fifth: $(m+n+1)^2(n+1)/1$; double octave: $(m+n+1)^2(n+1)^2/1$.

19. See W. Burkert, *Lore and Science in Ancient Pythagoreanism*, translated by E. L. Minar Jr. (Cambridge, Mass., 1972), pp. 383-86.

20. Ptolemy's notion of an experiment is, however, rather unsatisfactory from a modern point of view. For example, he describes (*Har.* 26.3-14) an experiment with an eight-stringed canon to show (against Aristoxenus) that the octave is less than six tones; but the experiment presupposes that the lengths of strings producing notes a tone apart must be in the ratio of 9/8. He also describes (*Ibid.*, 25.5-11) a more neutral experiment in which a "very musical" person produces seven consecutive pitches, each a tone above its predecessor, and checks to see whether the last is an octave above the first. However, Ptolemy concludes that, if the experiment doesn't refute Aristoxenus, the tones must have been inexact.

21. Van der Waerden makes this point in "Platon et les sciences exactes des Pythagoriciens," *Bulletin de la Société Mathématique de Belgique* 21 (1969): 121. I think, however, he exaggerates when he refers to a "very clear distinction between sounds and musical intervals on the one hand and numbers and their ratios on the other" in Ptolemy's *Harmonics*.

22. I. M. Crombie ascribes something like a frequency theory of pitch to Plato in *An Examination of Plato's Doctrines* (London, 1963), vol. II, pp. 182-83.

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