

Inside the ancient triangle of the gods

Pythagorean triples and striking mathematical effects
arise in surprising ways in a 3:4:5 triangle

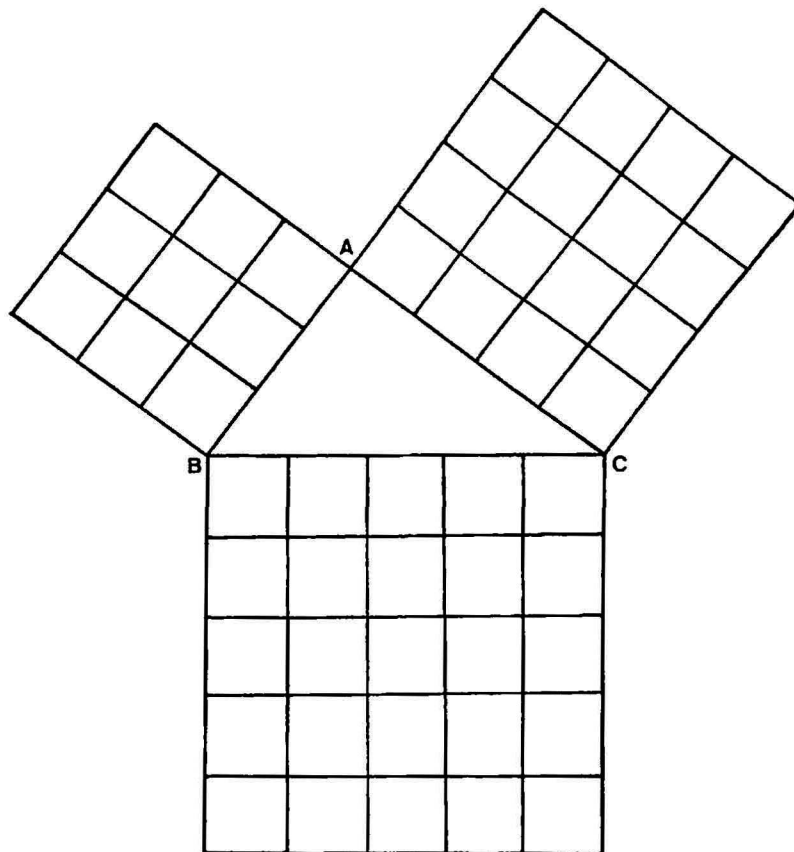
Leslie Greenhill

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The geometry that plays the central role in this essay is the well-known 3:4:5-proportion right triangle traditionally associated with the Pythagoreans. The triangle was known and utilised much earlier than the time of Pythagoras—at least eight Egyptian pyramids exhibit such proportions, according to Egyptologists Professor John Baines and Doctor Jaromir Malek, authors of the *Atlas of Ancient Egypt* (see “References”). One of the eight is the second largest of the structures at Giza, the pyramid of Khephren (Khafre), adjacent to the Great Pyramid. The sides of the 3:4:5-proportion pyramids slope at about $53^{\circ} 7' 48''$, which can be written as tangent $\frac{4}{3}$. Khephren's pyramid has a base about $215\frac{1}{3}$ metres (about 706 $\frac{1}{2}$ feet) square and is some 143 $\frac{1}{2}$ metres (about 471 feet) high.

The renowned Pythagorean theorem, which states that the square on the hypotenuse (the long side) of a right triangle is equal to the sum of the squares on the two shorter sides, is clearly visible in the diagram below— $9 + 16 = 25$.



The 3:4:5-proportion triangle with squares drawn on the sides

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The Greek Plutarch (ca. 46–120 AD), noted for his biographic works and essays on philosophy and ethics, discusses the divine status of the triangle in *On Isis and Osiris* in *Moralia V* (see “References”). He writes:

“One might conjecture that the Egyptians hold in high honour the most beautiful of the triangles, since they liken the nature of the Universe most closely to it, as Plato in the *Republic* seems to have made use of it in formulating his figure of marriage. This triangle has its upright of three units, its base of four, and its hypotenuse of five, whose power is equal to that of the other two sides. The upright, therefore, may be likened to the male, the base to the female, and the hypotenuse to the child of both, and so Osiris may be regarded as the origin, Isis as the recipient, and Horus as perfected result.”

The treatment of the 3:4:5 triangle in this essay produces an intriguing range of features. And the way certain Pythagorean triples arise is memorable, indeed. (Simply put, a Pythagorean triple is a right angle triangle with whole number sides—like 3, 4 and 5.) Triples have long interested many mathematicians. A search of the Internet yields much material in this regard.

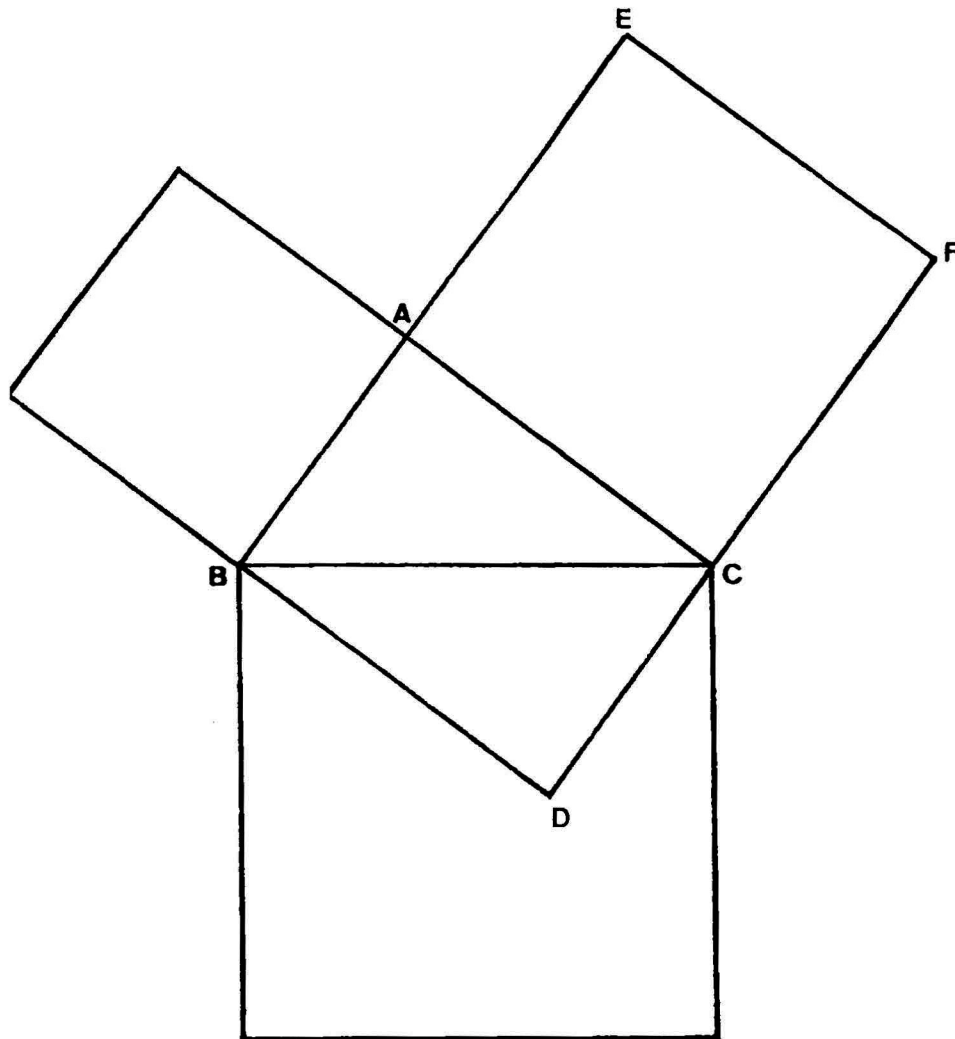
The branch of mathematics that treats of the relations between the sides and angles of triangles is known as trigonometry. It has a long history, being used in antiquity, in one form or another, in such fields as navigation, surveying and astronomy, principally to measure distances between objects, especially ones that might be far apart.

The French mathematician François Viète (1540–1603) developed the subject of trigonometry, perceiving that a trigonometric ratio could be used to solve an algebraic equation. In this exposition, though, only basic arithmetic is utilised: multiplication, division, addition and subtraction. And mathematical terminology is employed as little as possible.

Vulgar, mixed and improper fractions are used to describe measures found in the geometry for reasons that become obvious.

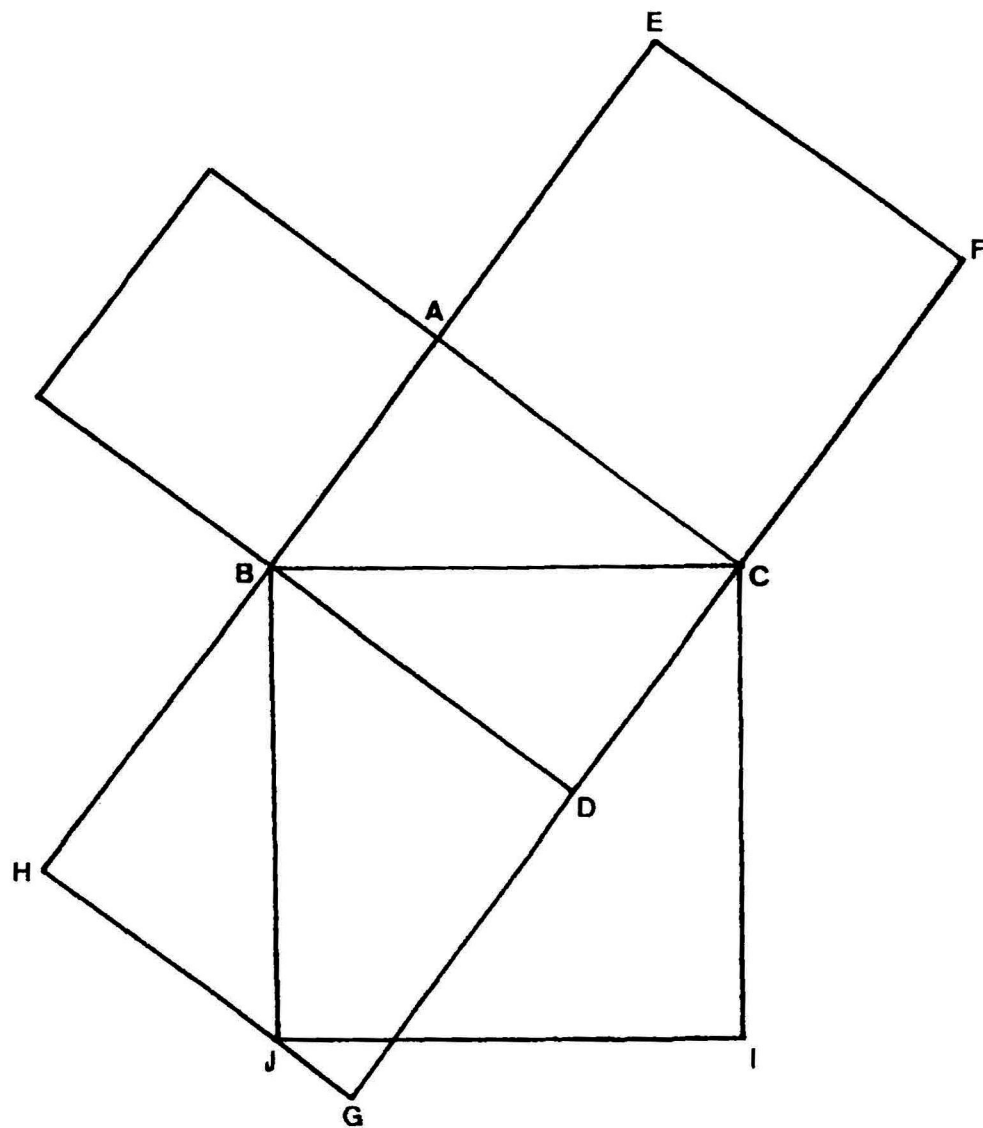
All measures are developed by means of naturally occurring phenomena.

The geometry is developed in four easy-to-follow steps.

Step 1.***Developments***

- 1) ABC is the initial 3:4:5-proportion triangle with squares drawn on its sides.
- 2) A symmetrical 3:4:5 triangle BDC is formed by extending the lower line on the "three" square and the right hand side of the "four" square.
- 3) The letters E and F are placed in locations of relevance.

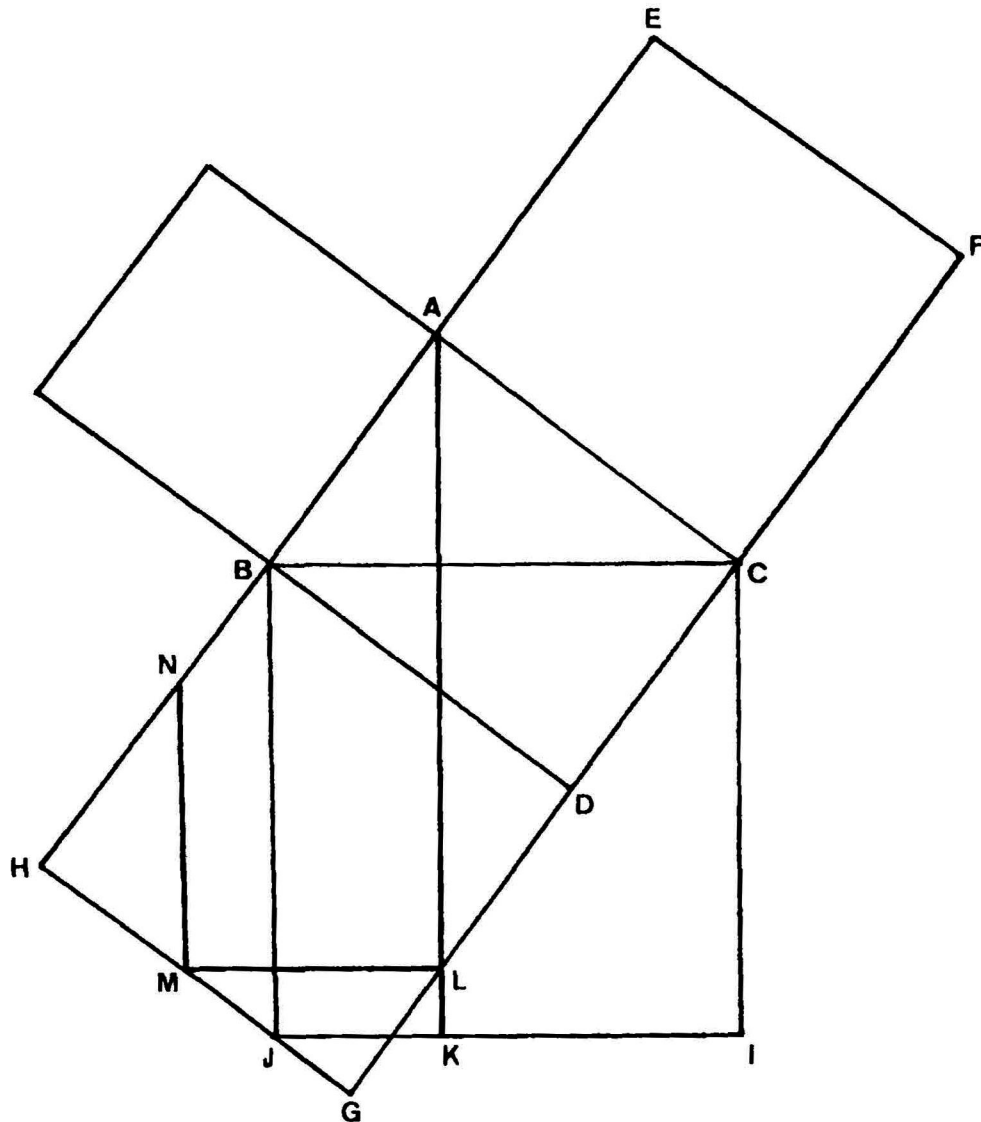
Step 2.



Developments

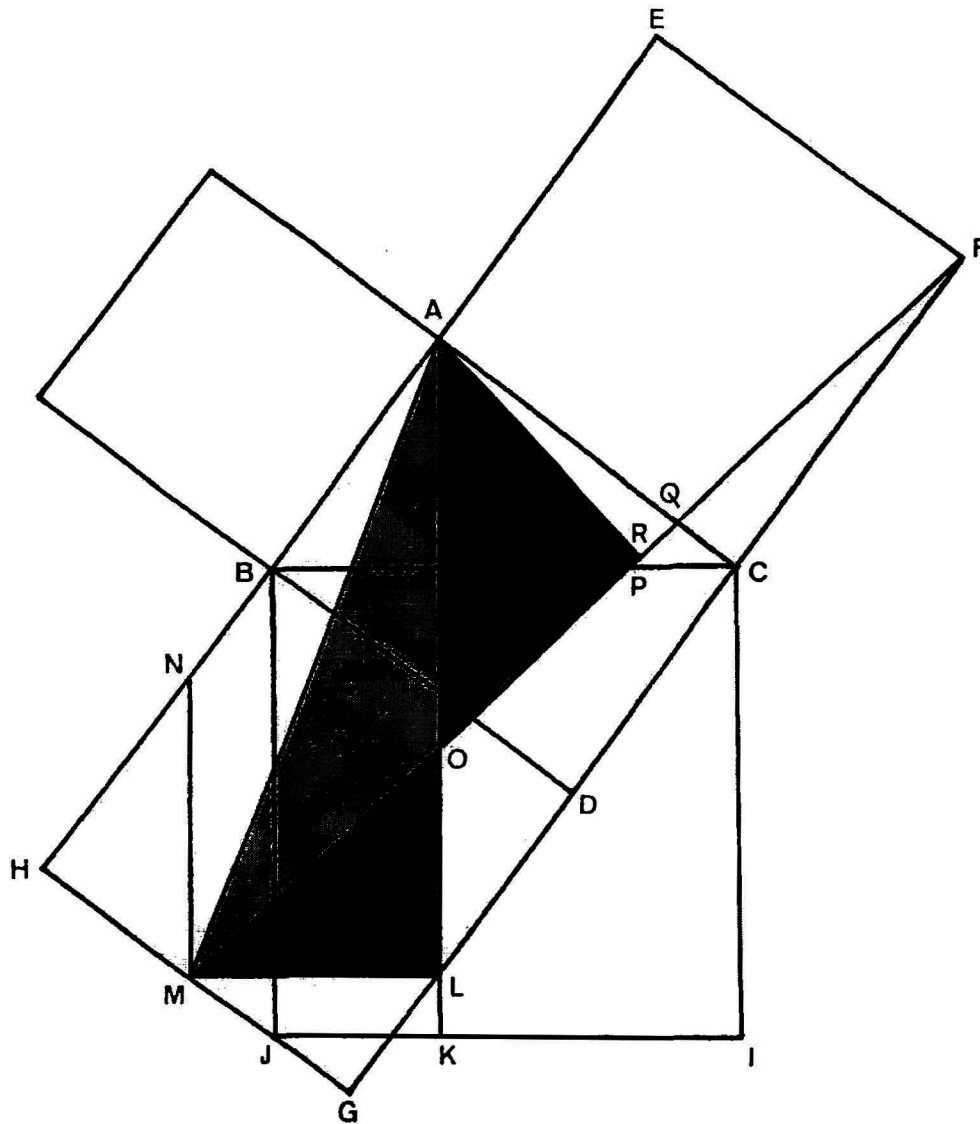
- 1) A “four” square (like AEFC) is drawn on BD.
- 2) The letters G, H, I and J are placed in locations of relevance.

Step 3.



Developments

- 1) From A, a line perpendicular to JI is drawn to K. AK crosses FG at L.
- 2) From L, a line parallel to JI is drawn to M.
- 3) From M, a line parallel to BJ and AK is drawn to N on BH.

Step 4.***Developments***

- 1) From M, lines are drawn to both A and to F.
MF crosses—
 - AK at O • BC at P • AC at Q
- 2) From A, a line perpendicular to FM is drawn to R to form right triangle **ARQ**.
- 3) AR also creates right triangle **ARM**.

Commentary on features in Step 4

ABC is the initial 3:4:5 triangle and it establishes a basic unit measure. This unit is from now on referred to as “inch” to avoid confusion with the term “unit” in individual triples. For example, in triple NHM (3:4:5 proportions) a unit measures $\frac{16}{27}$ of the initial unit/inch established by ABC.

New Pythagorean triples

1) Triangle **ALM** has the proportions **5:12:13**. Its sides measure:

- ML $\frac{25}{9}$ inches ($\frac{25}{9}$ is $\frac{5}{3}$ times $\frac{5}{3}$, that is, $\frac{5}{3}$ squared)
- AL $6\frac{2}{3}$ inches
- AM $7\frac{2}{9}$ ($\frac{65}{9}$) inches

A unit in ALM measures $\frac{5}{9}$ of an inch.

2) Triangle **AHM** has the proportions **16:63:65**. Its sides measure:

- HM $\frac{16}{9}$ inches ($\frac{16}{9}$ is $\frac{4}{3}$ times $\frac{4}{3}$, that is, $\frac{4}{3}$ squared)
- AH 7 inches
- AM $7\frac{2}{9}$ ($\frac{65}{9}$) inches

Note that AM is the hypotenuse for the triple ALM as well.

A unit in AHM measures $\frac{1}{9}$ of an inch. Accordingly, the perimeter of AHM measures 16 inches, the same measure as the perimeter of each of the “four” squares.

3) Triangle **FGM** has the proportions **20:99:101**. Its sides measure:

- MG $2\frac{2}{9}$ ($\frac{20}{9}$) inches
- FG 11 inches
- FM $11\frac{2}{9}$ ($\frac{101}{9}$) inches.

A unit in this triple also measures $\frac{1}{9}$ of an inch, the same as the unit in AHM, which has 16:63:65 proportions.

4) Triangle **OLM** has the proportions **336:377:505**. Its sides measure:

- OL $\frac{8400}{3393}$ ($2\frac{1614}{3393}$) inches
- ML $\frac{5}{3} \times \frac{5}{3}$ ($2\frac{7}{9}$) inches
- MO $\frac{12,625}{3393}$ ($3\frac{2446}{3393}$) inches

A unit in this triangle measures $\frac{25}{3393}$ of an inch. The number 3393 in the denominator of the fractions is the product of 377 times 9.

The number 377, which appears in the proportions of OLM, is the product of 13 times 29.

The number 13 manifests in triple ALM 5:12:13 and triple AHM 16:63:65 (65 is 13 times 5).

Observe that MO, the hypotenuse of OLM, contains 505 units: 505 is 101 x 5. FM in the triple FGM discussed in point (3) above contains 101 units.

5) Triangle **ARM** has the proportions **2844:5917:6565**.

ARM shares the same hypotenuse AM as triangle AHM, which has the proportions 16:63:65, and triangle ALM, which has the proportions 5:12:13. The number 65, as noted above, is 13 times 5. And the number 6565 is 13 x 5 x 101.

The measures of the sides of triple ARM are (the numbers in the numerators reflect the triangle's proportions):

- AR $\frac{2844}{909}$ ($3\frac{117}{909}$) inches
- RM $\frac{5917}{909}$ ($6\frac{463}{909}$) inches
- AM $\frac{6565}{909}$ ($\frac{65}{9}$ or $7\frac{2}{9}$) inches

A unit in this triangle measures $\frac{1}{909}$ of an inch.

(The measure of AR is readily deduced because it is part of triple ARQ, which has the same proportions as FCQ and **FGM**, that is, 20:99:101.)

Special features of interest

- In the 3:4:5-proportion triangle MGL, GL measures $\frac{5}{3}$ inches, MG is $2\frac{2}{9}$ inches and ML measures $\frac{5}{3}$ **squared** ($2\frac{7}{9}$) inches. ML is also one leg of the triples ALM and OLM.
- In the 3:4:5-proportion triangle NHM, HM measures $\frac{4}{3}$ **squared** ($\frac{16}{9}$) inches and HN measures $\frac{4}{3}$ **cubed** ($\frac{64}{27}$) inches.

- c) The following is especially interesting. If $HN \frac{4}{3}$ **cubed** inches is subtracted from AH 7 inches, the remainder AN measures $\frac{5}{3}$ **cubed** ($\frac{125}{27}$) inches.

In other words, AH 7 inches has two notable components—

- $HN (\frac{4}{3} \times \frac{4}{3} \times \frac{4}{3})$ inches
- $AN (\frac{5}{3} \times \frac{5}{3} \times \frac{5}{3})$ inches.

Furthermore, triple ALM, which has 5:12:13 proportions, has these $\frac{5}{3}$ elements in its makeup:

- (i) ML, as mentioned above, measures $\frac{5}{3}$ squared inches.
- (ii) Appended to L is GL, which measures $\frac{5}{3}$ inches.
- (iii) Appended to A is AN, which measures $\frac{5}{3}$ cubed inches.
- (iv) The perimeter of triple ALM measures $16 \frac{2}{3}$ inches. The number $16 \frac{2}{3}$ is $\frac{5}{3}$ multiplied by ten, that is, $\frac{50}{3}$.

These outcomes arise as a result of the development of location L: see step 3.

All this is complemented, and perhaps surpassed, by material unveiled in another work by the present writer entitled *The strange nature of certain Pythagorean triples*.

References

Babbitt, F. C. *Plutarch: Moralia, Volume V*. Harvard University Press, 1999 edition, page 135 (Stephanus index 373–4)

Baines, J. & Malek, J. *Atlas of Ancient Egypt*. Time-Life Books, Amsterdam, 1996 edition, pp. 140–1.

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