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2006

On Doubling the Cube: Mechanics and Conics¹

Michael J. White

It is indisputable that Plato was deeply impressed by mathematics and made it an important part of the tuition of his Academy. Much more disputable are the mathematical accomplishments with which Plato was credited in later antiquity. The most important of these is a solution of one of the famous geometrical problems of antiquity, the duplication of the cube: i.e., given a cube of a given side, to find the side of a cube having twice the volume of the given cube. According to tradition, Hippocrates of Chios had, in the fifth century BCE, reduced the problem of duplicating the cube to that of finding two line segments as mean proportionals in continued proportion between two straight line segments as extremes: that is, given line segments a and b such that $a > b$, to find line segments y and x such that $a:y = y:x = x:b$.²

1 I wish to express my thanks for valuable comments on an earlier draft of this essay to an anonymous referee for *Apeiron*.

2 With the aid of algebra, which the Greeks of course did not have, the relation is straightforward. The continued proportion yields the equations $x^2 = by$ and $x = aby$ and, thus, $x^3 = b^2a$. Hence, if we let $a = 2b$, we obtain $x^3 = 2b^3$. So, the cube on the mean proportional x is twice the volume of the cube on the given line b . Without claiming that it represents the method of Hippocrates, Knorr gives a reduction more in the spirit of ancient geometry: 'If, for any two given lines, A and B , we can insert the two mean proportionals, X and Y , then $A:X = X:Y = Y:B$. Thus, by compounding the ratios, one has $(A:X)^3 = (A:X)(X:Y)(Y:B)$, that is $A^3:X^3 = A^3:B^3$. Thus, X will be the side of a cube in the given ratio $(B:A)$ to the given cube (A^3) ' (Wilbur R. Knorr, *The Ancient Tradition of Geometric Problems* [Boston: Birkhäuser Publishing, Ltd. 1986] 23). For a hypothetical reconstruction of Hippocrates' method, see Ken Saito, 'Doubling the Cube, A New Interpretation of Its Significance for Early Greek Geometry', *Historia Mathematica* 22 (1995) 119-37).

On Doubling the Cube: Mechanics and Conics¹

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The attribution to Plato of a solution of the latter problem is late, occurring only in the commentary of Eutocius (first half of sixth century CE) on the second book of Archimedes' *On the Sphere and Cylinder*. The consensus of modern scholars is that the attribution is false, not only because of the lack of extant earlier references to such a solution, but also for several other reasons. The most important of these is that the proof involves a mechanical device, the use of which in geometry Plato is reported to have disapproved. The device in question is a sort of carpenter's or draftsman's square, together with a moveable straightedge that slides along one side of the square while remaining perpendicular to that side and parallel to the other side. (See Fig. 1³.) In view of Plato's general attitude toward geometry as a study that 'draws the soul from the realm of becoming to the realm of what is,'⁴ it is not surprising to find Plutarch reporting that Plato disapproved of the use of mechanical devices in

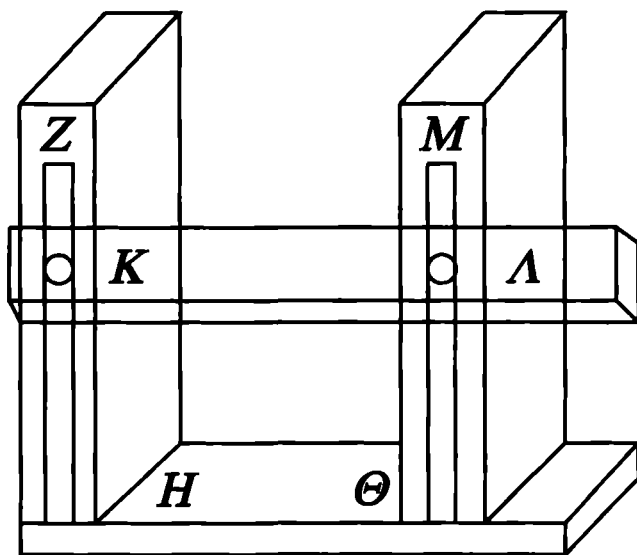


Figure 1

- 3 The device is pictured in the diagram (Figure 1) as having another stationary arm ΘM , parallel to arm HZ of the carpenter's square. But the only function of this arm appears to be to give mechanical stability to the device, and the arm ΘM is not actually used in the proof reported by Eutocius.

- 4 *Republic* 521d3-4, trans. G.M.A. Grube and C.D.C. Reeve

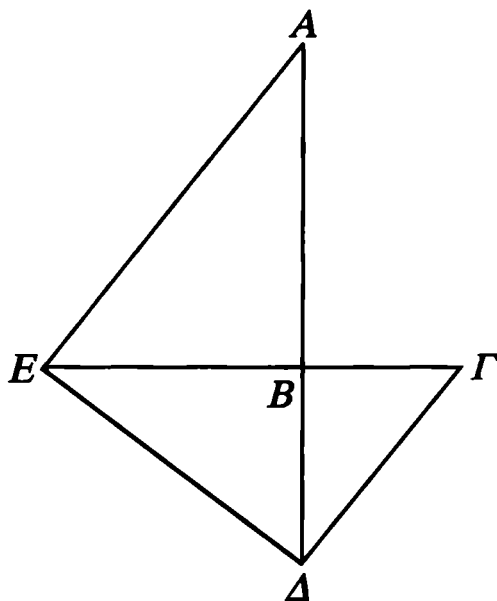


Figure 2

geometry. According to Plutarch's account, Plato believed that when such mechanical instruments are used, 'the good of geometry is thereby lost and destroyed, as it is brought back to sensible things instead of being carried upward and laying hold of eternal and incorporeal images.'⁵ This remark immediately follows Plutarch's claim that 'Plato himself reproached those in the circle of Eudoxus and Archytas and Menaechmus for attempting to reduce the duplication of the cube to the use of tools and mechanical constructions, as though trying to find two mean proportionals not by the use of reason but in whatever way would be practicable.'⁶ The main thesis that I advance in what follows is that, although contemporary authors have thought that the mechanical proof (mis)attributed to Plato is derivative from a 'theoretical' proof, which employs the intersection of two parabolas, attributed by Eutocius to

5 Plutarch, *Quaestiones conviviales* VIII 2, 718f.

6 Ibid. 718e Note the pun: 'ὥσπερ πειρομένους δίχα λόγου δύο μέσας ἀνὰ λόγον, ἢ παρείκοι, λαβεῖν.'

Menaechmus, the opposite is much more likely the case: the configuration of line segments in the proof of Menaechmus employing two parabolas is derivative from the configuration of line segments of the mechanical proof.

Eutocius, in his account of solutions to the problem of duplication of the cube, credits Menaechmus (fl. mid-fourth century BCE) with two solutions employing conic sections, the investigation of which he was one of the first geometers to undertake. Little more is known about Menaechmus; but Proclus does report that he was a pupil of Eudoxus and 'associate' of Plato, who, with his brother Dinostratus, 'made the whole of geometry more perfect.'⁷ The first of the solutions attributed to Menaechmus takes two straight line segments, A and E, given as extremes, and proceeds to determine the straight line segments that are the two mean proportionals in continued proportion between them, B ($=\Theta Z = \Delta K$) and $\Gamma (= \Theta K = \Delta Z)$. (See Figure 3.⁸) In contemporary (but anachronistic) mathematical terms, B and Γ are the ordinate and abscissa of the point of intersection of two conic sections in the same plane. The two sections in question are (i) a right-opening parabola with *latus rectum*⁹ A and with origin at Δ and (ii) one branch of a rectangular hyperbola constructed (in the first quadrant, in contemporary Cartesian-coordinate terminology) with reference to its asymptotes (as reference axes, in Cartesian-coordinate terminology intersecting at Δ as origin) and such that the product of 'Cartesian ordinate' and 'Cartesian abscissa' is equal to the 'constant rectangle' ($A \times E$) — which, of course, is defined by the two given linear extremes A and E.

7 Proclus, *In primum Euclidis elementorum librum commentarius*, ed. G. Friedlein (Hildesheim: Georg Olms Verlagsbuchhandlung 1967) [repr. of Leipzig: B.B. Teubner 1873], 67. 9-12

8 Let $a = A$, $y = \Theta Z = \Delta K$, $x = \Theta K = \Delta Z$, $b = E$. Then, an algebraic deduction from the equations $ax = y^2$ and $xy = ab$ for parabola and hyperbola, respectively, yields the sought solution for the identity $a/y = y/x = y/b$.

9 In terms of the modern focus-directrix definition of a parabola, the *latus rectum* is the line segment through the focus of parabola, parallel to the directrix (and perpendicular to the major axis), which has both endpoints on the curve. Thus, the length of a parabola's *latus rectum* is $4p$, where p is the 'coefficient' of the parabola — the axial distance from the focus to the vertex. As noted later, a common ancient conception of the *latus rectum* (ὀρθία πλευρά) is that of an equivalent but differently conceived constant parameter of the parabola.

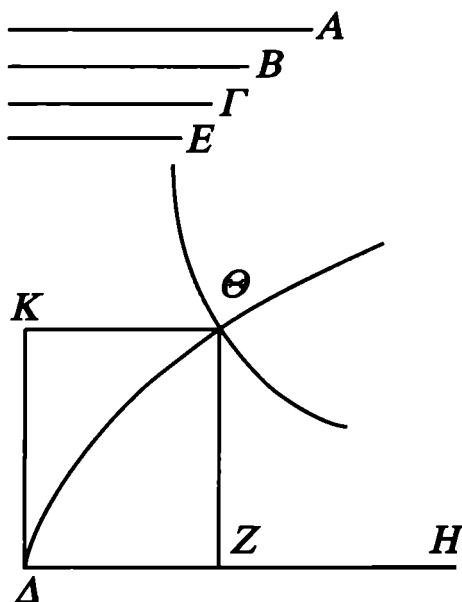


Figure 3

The second solution attributed by Eutocius to Menaechmus is similar, finding the two mean proportionals as the ordinate and abscissa (in terms of Cartesian coordinates)¹⁰ of the point of intersection of two parabolas the vertices of which coincide and the axes of which are at right angles: (i) an upwards-opening parabola with *latus rectum* BΓ (= the

10 The point Z of intersection of the two parabolas defines the two linear magnitudes BΔ and BE. In the terminology employed by Apollonius of Perga (fl. latter half of third century BCE) — although it apparently was not used by Menaechmus himself — the longer of the two mean proportionals BE is an abscissa (ἡ ἀπολαμβανομένη) of the upwards-opening parabola and is equal to an ordinate (ἡ τεταγμένη) ΔZ of the right-opening parabola. The shorter of the two proportionals BΔ is an abscissa of the right-opening parabola and is equal to an ordinate EZ of the upwards opening parabola. According to the terminology of Apollonius, each of the lines parallel to one another that are bisected by the axis or principal diameter of a parabola is ‘drawn ordinatewise to the diameter (τεταγμένως δὲ ἐπὶ τὴν διάμετρον κατῆχθαι)’ (*Conica* I Def. 4). And the segment of the parabola’s axis/principal diameter extending from the vertex to where it is ‘cut off’ by an ordinate (τῆς ἀπολαμβανομένης ὑπ’ αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῇ κορυφῇ τῆς τομῆς) is an abscissa (*Con* I 11).

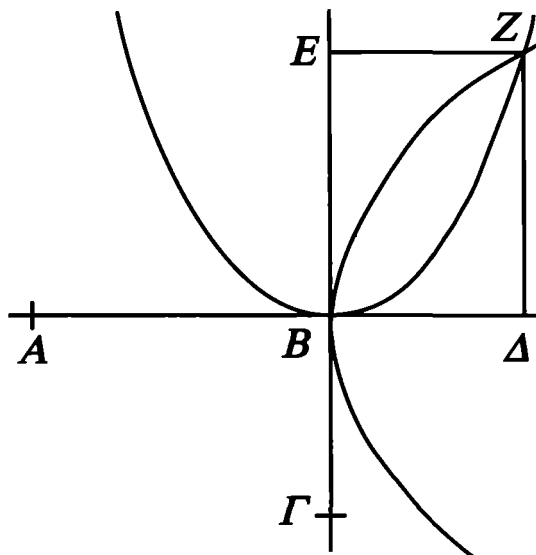


Figure 4

shorter of the two given linear extremes = E in the preceding solution) and with vertex B ; (ii) a right-opening parabola with *latus rectum* AB (= the longer of the two given linear extremes = A in the preceding solution) and also with vertex B . (See Figure 4.¹¹)

To return to the mechanical solution attributed to Plato by Eutocius, it arranges the two given linear extremes AB and $B\Gamma$ at right angles and as meeting at common end-point B . Line segment AB is thought of as being extended indefinitely vertically (downward, 'in the direction of Δ ' of Figure 2), and line segment $B\Gamma$ is thought of as being extended indefinitely horizontally (leftward, 'in the direction of E ' of Figure 2).¹²

11 Let $a = AB$, $y = BE = \Delta Z$, $x = EZ = B\Delta$, $b = B\Gamma$. Then an algebraic deduction from the equations $ax = y^2$ and $by = x^2$ for the right-opening and upwards-opening parabolas, respectively, yields a solution for the identity $a/y = y/x = x/b$.

12 As Netz notes in his commentary, E and Δ initially represent the direction of the indefinite extension of line ΓBE 'to the left' and line ABA 'downward', respectively. They become 'determinate points' only as a result of the application of the mechanical

The mechanical device of Figure 1 is manipulated onto the diagram of Figure 2 in such a way that the following conditions all hold: (i) point Γ of the figure's diagram lies on fixed straight-arm ΘH of the device; (ii) point A of the diagram falls on moveable straight-arm KA of the device; (iii) vertex H of right angle ΘHZ of the device lies on vertical line BA of the diagram; (iv) vertex K of right angle ΛKZ (or, equivalently, vertex K of right angle ΛKH) of the device lies on horizontal line BE of the diagram. Point H of the device then coincides with a determinate point, call it Δ , on the lower half of the vertical line of the diagram and defines a line segment BA ; similarly, point K of the device coincides with a determinate point, call it E , of the left half of the horizontal line of the diagram and defines a line segment BE . Since the device guarantees that the angle ΛKH ($= \Lambda E\Delta$) is a right angle, it follows from the porism (corollary) to Euclid VI 8 that BE is the mean proportional between AB and BA : 'if in a right-angled triangle a perpendicular be drawn from the right angle to the base, the straight line so drawn is a mean proportional between the segments of the base.' So, $AB:BE = BE:BA$. But, since the device also guarantees that the angle ΘHZ ($= \Gamma\Delta E$) is a right angle, it follows by the same theorem of Euclid that BA is a mean proportional between BE and $B\Gamma$, i.e., $BE:BA = BA:B\Gamma$. Hence it follows that $AB:BE = BE:BA = BA:B\Gamma$; and, if AB is taken to be the longer and $B\Gamma$ the shorter of the two given linear extremes,¹³ BE will be the longer and BA the shorter of the two mean proportionals in continuing proportion between them. That is, if two linear extremes a and b are given such that $a > b$, then BE is the y and BA the x such that $a:y = y:x = x:b$.

As Heath noticed, the manuscript diagrams printed in Heiberg's text with the mechanical solution and Menaechmus' second solution (the one

cal device, as explained in the following text. See Reviel Netz, *The Works of Archimedes: Translated into English, together with Eutocius' commentaries, with commentary, and critical edition of the diagrams*, Volume I, *The Two Books On the Sphere and the Cylinder* (Cambridge: Cambridge University Press 2004), 273 fn. 18 and 274 fn. 26.

13 Netz reports that codex E represents the length of the line segments in the opposite manner: 'codex E has the line-segments in "correct" proportions ($B\Gamma > BA > BE > BA$) and, since codex E is on the whole the most conservative visually, it may perhaps be preferable' (Netz, *Archimedes*, 275). However, one probably should here bear in mind Netz' own cautions (ibid., 8-10) about assuming that diagrams accompanying ancient proofs convey metrical information.

using the intersection of two parabolas) are identical in terms of their relative placement of the four line segments — the two given linear extremes AB and BG, and the two mean proportionals BE and BD in continuing proportion between them. In fact, if one thinks of the diagram accompanying Menaechmus' second solution as being first rotated 180 degrees on its horizontal axis ABA and then rotated 90 degrees clockwise, the placement of the four lines coincides exactly with that of the diagram accompanying the mechanical solution. Heath concludes: 'hence it seems probable that some one who had Menaechmus's second solution before him wished to show how the same representation of the four straight lines could be got by a mechanical construction as an alternative to the use of conics.'¹⁴ Particularly in view of the attribution of a mechanical solution of the problem to 'those in the circle of Eudoxus and Archytas and Menaechmus,' Bulmer-Thomas, in his entry on Menaechmus in the *Dictionary of Scientific Biography*, takes Heath's speculation a step further: 'But why not Menaechmus himself [as Heath's "some one"]? If he was the author [of the mechanical solution], it would be easy for the tradition to refer it to his master, Plato. This cannot be proved or disproved, but it would be the simplest explanation of the facts.'¹⁵

I believe, however, that Heath got it backwards. In other words, I suggest that the relative placement of the four line segments common to the two diagrams belonged *first* to 'Plato's mechanical solution' (that is, *Menaechmus'* mechanical solution, according to the hypothesis of Bulmer-Thomas). Then, essentially the same diagram was adopted for the representation of Menaechmus' solution in terms of the intersection of the two parabolas. The summary reason for this claim is that the relative placement of the four lines is essential to the mechanical solution but not to Menaechmus' parabolic solution. In what follows, I shall attempt to elaborate the evidence for this claim.

As Netz notes, the two solutions using conic sections attributed to Menaechmus are set forth in 'an analysis/synthesis structure.'¹⁶ That is,

14 Sir Thomas Heath, *A History of Greek Mathematics*, Volume I, From Thales to Euclid (New York: Dover Publications, Inc. 1981) [repr. of Oxford: Clarendon Press 1921], 256

15 Ivor Bulmer-Thomas, 'Menaechmus,' in *Dictionary of Scientific Biography*, ed. Charles Coulston Gillispie, Volume 9 & 10 (New York: Charles Scribner's Sons 1980), 275

16 Netz, *Archimedes*, 286 fn. 99

both begin 'analytically', by assuming that the problem of finding the two mean proportionals, given the extremes, has been solved and then proceed to construct the relevant conic sections that provide the solution using this information. The 'synthetic' part of the arguments begins with the linear extremes as given and uses them to construct the relevant conic sections, which yields the two mean proportionals (as 'ordinate' and 'abscissa' of the point of intersection of the two sections).

One of the few bits of information provided by Proclus about Menaechmus was that he (or, rather, 'the mathematicians associated with Menaechmus and Amphinomus'¹⁷) were concerned with the issue of conversion or 'reversibility' (ἀντιστροφή), particularly in the sense of this term in which the 'hypothesis' used to deduce a 'theorem' is also a logical consequence of that theorem (taken together, perhaps, with the other hypotheses necessary to deduce the theorem).¹⁸ This sort of ἀντιστροφή is crucial to the application of the method of analysis and synthesis. For example, in the application of the conic-section constructions above, it is necessary for the analytic component of the construction that, given the extremes, one can construct the conic sections with their particular manner of intersection if one is given the two mean proportionals. For the synthetic component of the construction, it is necessary that, given the extremes, one can construct the two linear mean proportionals if one is given the conic sections with their particular manner of intersection.

My suggestion is the following. Anyone familiar with the fourth-century geometry that was eventually incorporated into the sixth book of Euclid would have been familiar with the representation of a *single* mean proportional y between extremes a and x as follows. The extremes a and x are represented as line segments placed end-to-end on a given line, and their mean proportional y is represented as a line segment perpendicular to this line, originating at the common end-points of a and x . Material from Euclid VI pertaining to similar triangles yields the following result. Suppose that we begin with the 'sum' or linear concatenation of a and x (henceforth represented as ' $a + x$ ') taken to be one side of a triangle. Let the remaining two sides be produced by joining the two end-points of the line segment $a + x$ with the end-point of y that does not fall on $a + x$.

17 Proclus, in *Euc.*, 254.4-5

18 *Ibid.*, 252ff.

The result will be a right triangle of which $a + x$ is the hypotenuse and the opposing right angle has its vertex at the end-point of y . It would be a natural (but clever) extrapolation to represent *two* mean proportionals in continued proportion between extremes a and b in the manner of Figure 2. Let a be AB , y be EB , x be BA , and b be $B\Gamma$. Then the mean proportional y between extremes a and x is represented as perpendicular to the linear junction of $a + x$ (originating at point B). A similar construction is produced at a ninety-degree rotation: x is now the mean proportional between extremes y and b and is represented as perpendicular to the linear junction of $y + b$ (also originating at point B). Thus y and x are the two mean proportionals in continued proportion between a and b .

So, the analytic component of the proof accompanying Figure 2 yields the following claim: If four line segments, the two given extremes and two mean proportionals in continued proportion between them, are placed as in Figure 2, then the angles $AE\Delta$ and $E\Delta\Gamma$ produced when the end-points of the line segments are joined must be right angles. A theorem such as Euclid VI 8 is crucial to the synthetic component of the proof. If a ($= AB$) and b ($= B\Gamma$) are joined at right angles, and if each is extended as in the diagram, and if it can be contrived that triangles $AE\Delta$ and $\Gamma\Delta E$ are produced with right angles at vertices E and Δ , respectively, then the two 'heights' y ($= BE$) and x ($= BA$) will be the two mean proportionals in continued proportion between a and b . The mechanical device of Figure 1 is designed to yield such right triangles with the vertices of the right angles on rays BE and BA . For this construction to succeed, it is obviously crucial that relative placement of lines of Figure 2 or Figure 4 (or some other rotational variant of that placement) be employed. That is, the two linear extremes must be perpendicularly joined at their end-points (B , in both diagrams); and the two mean proportionals must fall on their linear extensions, at right angles, on the other side of the common end-point B .

As we saw, there is the same relative placement of the four line segments in the diagram accompanying Menaechmus' second solution (Figure 4) using the intersection of two parabolas the axes of which are at right angles and the vertices of which coincide at point B . However, in the constructive component of the proof, the diagram's placement of the two given linear extremes, AB and $B\Gamma$, is not at all necessary. In the synthetic construction, these two lengths are the *latera recta* of the parabola constructed on axis BA and of the parabola constructed on axis BE , respectively. In the common contemporary formulation, the *latus rectum* of a parabola is a chord passing through the focus and perpendicular to the major axis (or parallel to the directrix line of the parabola). Although

we have little evidence about Menaechmus' construction of parabolas, it is more likely that his construction was more similar to that of the first book of Apollonius' *Conics* than to the modern account, which appeals to the focus-directrix definition of a parabola. For Apollonius, the *latus rectum* (= ὀρθία or ὀρθία πλευρά¹⁹) is defined at *Con* I 11 to be the 'constant' line segment (for a given parabola) such that the rectangle formed by it and any variable abscissa of the parabola (line segment along the parabola's major axis or principal diameter considered as originating at the parabola's vertex) is equal to the square on the 'ordinate' chord (from axis to the parabola) defined by that abscissa.²⁰

There are two sorts of diagrams that are printed with Heiberg's text of the first book of Apollonius' *Conics* that show line segments representing *latera recta* of parabolas.²¹ One sort, that accompanying *Con* I 52, shows the *latus rectum* simply as a line segment detached from the figure of the parabola and the cone used to generate it. The other sort, that accompanying *Con* I 11, shows the *latus rectum* as a line segment one end of which is tangent to the vertex of the parabola and which is *perpendicular* to the parabola's axis or principal diameter (although usually not, it appears, in the plane of the parabola).²² This second sort of representation of the *latus rectum*, as perpendicular to the major axis of the

19 This term is actually introduced at the conclusion of the proof of *Con* I 11 as a shorter equivalent for the periphrastic phrase, 'ἡ δὲ [εὐθεία γραμμὴ] ... παρ' ἣν δύνανται αἱ καταφόμεναι τεταγμένως ἐπὶ τὴν ... διάμετρον,' which Taliaferro renders as 'the straight line to which the straight lines drawn ordinatewise to the diameter ... are applied in square' (Apollonius of Perga, *Conics, Books I-III*, revised edition, Dana Densmore [Santa Fe, New Mexico: Green Lion Press 2000], 21).

20 Strictly speaking, this is a property of the *latus rectum* established by Proposition I 11. In the statement of the proposition, the *latus rectum* is, in effect, defined in terms of a proportion: it is the straight line x such that the following proportion holds: x :(straight line from vertex of the parabola along the side of the defining cone to its apex) = (square of the base of the axial triangle):(product of the remaining two sides of the axial triangle). The 'axial triangle' is a vertical cross-section of the cone used to generate the parabola, which contains the apex of the cone, a diameter of the circular base of the cone, and the major axis of the parabola generated by the cone.

21 *Apollonii Pergaei, Quae Graece exstant cum commentariis antiquis*, ed. and Latin trans., I.L. Heiberg (Stuttgart: B.G. Teubner 1974) [repr. of Leipzig: B.G. Teubner 1891]

22 Diagrams printed with *Con* I 12 and I 13 similarly represent *latera recta*: as line segments perpendicular to the end points of major axes — for a branch of a hyperbola and for an ellipse, respectively.

parabola and having one end-point at the parabola's vertex, is particularly appropriate in view of the function, so to speak, of the *latus rectum*. It is that constant linear magnitude (relative to a particular parabola), which when multiplied by any 'abscissa' or line segment along the axis of the parabola beginning at its vertex, produces a *rectangle* equal (in area) to the *square* constructed on the line segment 'drawn ordinatewise' from the end of that abscissa to the conic section. In other words, since the *latus rectum* is to form a rectangle with a length of the axis, it is quite natural to represent it as perpendicular to the axis.²³

However, the representation of the two *latera recta* in Figure 4 is an anomaly. Line segment AB, considered as the *latus rectum* of the right-opening parabola, is represented as a linear continuation of the axis of that parabola leftward, beyond its vertex B. Similarly, BF, considered as the *latus rectum* of the upwards-opening parabola, is a continuation of its axis downward, beyond its vertex B. Insofar as I have been able to determine, such a representation of *latera recta* relative to their parabolas is unique. In terms of the constructive component of the proof, it also appears to be unnecessary.

The following seems to me to be a quite plausible hypothesis. Somewhere along the way the quite natural '*analytic* relative placement' of the four lines of Figure 2 has been borrowed. That same placement worked well — and indeed was necessary — for the *synthetic* component of the mechanical proof attributed to Plato by Eutocius. For the synthetic component of the proof using the two parabolas, the two mean proportionals being constructed (BE and BA) will end up being joined at a right angle at their end-points, just as in Figure 2. But the two given extremes, AB and BF, when employed as *latera recta*, do not need to be placed relative to the extremes as they were in Figure 2. It is true that this placement of the given extremes does no 'harm' in the proof accompanying Figure 4; but, in terms of the structure of the synthetic component of this proof, it appears to be unnecessary and unmotivated. The 'motivation', I am suggesting, comes from the fact that the relative placement of the four lines of Figure 2 (which placement is necessary for both the analytic and synthetic components of the mechanical proof) has been

23 I should not want to place too much weight on the diagrams printed with Heiberg's text, since it is known that he sometimes constructed his own diagrams and I have not done a study of the history of the particular diagrams that he uses.

transposed to the 'second' proof using conic sections attributed to Menaechmus by Eutocius.

Additional evidence that the placement of the given extremes in Figure 4 is not at all crucial comes from the diagram (Figure 3) accompanying the first proof using conic sections attributed to Menaechmus — that employing the intersection of a parabola and (one branch of) a rectangular hyperbola. Note that in Figure 3, the given extremes A and E, are *not explicitly represented at all* in the diagram accompanying the proof. Of course, as the text of the proof makes clear, the two mean proportionals (B and Γ , among the detached line segments) being constructed in the synthetic component of the proof *do* end up being represented in the diagram as $\Theta Z (= \Delta K)$ and $\Delta Z (= \Theta K)$, respectively. But the given 'parameters', A and E, of the construction of the two conic sections in the synthetic component of the proof appear *only* as detached line segments, similar to the style of the diagram accompanying Apollonius' *Con* I 52 that is printed by Heiberg.

My principal conclusion, then, is that there is reason to think that the relative placement of the four line segments common to Figures 2 and 4 originally belonged with the mechanical proof (Figure 2) and then was transposed to Menaechmus' proof using two parabolas (Figure 4). Of course, this conclusion leaves open the question of when the transposition occurred, and by whose hand. It also assumes a close relation between the two proofs, on the one hand, and the two sorts of diagram accompanying them, on the other. Unfortunately, we do not have ancient manuscripts, with accompanying diagrams, of Archimedes' *On the Sphere and Cylinder* or of Eutocius' commentary on its two books. However, in his edition of these works, R. Netz comments that 'it should be stressed that the diagrams across the manuscript tradition are strikingly similar to each other, often in quite trivial detail, so that is hardly a question that they derive from a common archetype.'²⁴ Netz argues that diagrams of ancient geometrical proofs are typically as much an 'essential part' of the proof as is the text:

diagrams, largely speaking, provide a schematic representation of the pattern of configuration holding in the geometrical case studied. This pattern is what can be reliably "read" off the diagram and used as part

24 Netz, 'Introduction', *Archimedes*, 8

of the logic of the argument, since it is independent of metrical values. Ancient diagrams are taken to represent precisely that which can be exactly represented and are therefore, unlike their modern counterparts, taken as tools for the logic of the argument itself.²⁵

If Netz' views are correct, there is some reason to think that the diagrams of Figures 2 and 4 (or minor variants of them) accompanied the two proofs — that using the mechanical method and that using the intersection of two parabolas — from the ancient origins of those proofs. In which case, the following conjectural story can be devised. Someone in the circle of Menaechmus (and, in the words of Bulmer-Thomas, 'why not Menaechmus himself?') *first* devised the mechanical proof, which developed very straightforwardly from the fourth-century geometry that eventually found its way into the sixth book of Euclid. Then Menaechmus, either stung by Plato's reproach about the use of mechanical tools in geometry or troubled in this respect by his own 'Platonic' methodological convictions, devised the proof using the intersection of two parabolas borrowing his own relative placement of the four line segments from his earlier mechanical proof. The probability of this hypothesis is perhaps increased a bit if, as Charles Taylor²⁶ and Bulmer-Thomas believe, the 'evidence ... [is] inescapable that Menaechmus attempted to find some mechanical device for tracing the curves [i.e., the conic sections he theoretically investigated].'²⁷ That is, there would then be a certain continuity between Menaechmus' 'mechanical' solution, which evidently did not prove to be amenable to a purely theoretical formulation, and his solution using the two parabolas, which perhaps was capable of both a theoretical formulation and a 'practical' application.²⁸

25 Ibid., 9

26 Charles Taylor, *Introduction to the Ancient and Modern Geometry of Conics* (Cambridge: Cambridge University Press 1881), xxxi-xxxiii

27 Bulmer-Thomas, 'Menaechmus,' 274

28 Such a hypothesis helps to explain the comment in a letter purported to be by Eratosthenes quoted by Eutocius to the effect that, among the geometers associated with Plato (such as Archytas and Eudoxus) who attempted to solve the problem, they did so theoretically — that is, they 'wrote demonstratively (ἀποδεικτικῶς γεγραμέναι)' — but none of them was able to solve it practically 'by hand' except

However, an alternative account of the origin of the solution to the problem, as reported by Eutocius, that employs the intersection of two parabolas must be considered. This is the hypothesis of G.J. Toomer that the solution employing two parabolas was due to Diocles rather than Menaechmus. Although there has been some uncertainty concerning Diocles' dates, Toomer convincingly argues that he lived from the late third into the early second century BCE. As Toomer points out, this would make him an exact contemporary of Apollonius; and he thus would have postdated Menaechmus by roughly a century and a half. Toomer's hypothesis is based upon Proposition 10 of an Arabic manuscript of Diocles' *Περὶ πυρρείων* (*On Burning Mirrors*), in which Diocles 'shows how to find a cube which is twice a (given) cube by means of a geometrical proof.'²⁹ This proposition does indeed employ the intersection of two parabolas the axes of which are perpendicular to one another. Toomer argues that 'Eutocius does not mention the author [of the solution employing two parabolas that he reports]. It has been commonly assumed that it is due to Menaechmus, on the not very cogent grounds that in the text of Eutocius it follows a solution by Menaechmus, being introduced by the remark ἄλλως ("another way"). ... Since we now find the solution in Diocles' work, and since Eutocius provides two other excerpts from Diocles, it is certain that Diocles is his source for this too.'³⁰

Menaechmus, and he only 'barely and this with difficulty' (πλὴν ἐπὶ βραχύ τι τὸν Μέναιχμον καὶ ταῦτα δυσχερῶς) (*Eutocii comm. in lib. ii*, in Archimedes, *Opera omnia*, Vol. III, ed. I.L. Heiberg [Stuttgart: B.G. Teubner 1972] [repr. of Leipzig: B.G. Teubner 1915], 90.8-11). Netz keeps the 'manuscripts' reading [βραχυτητα] against Heiberg's (possible) emendation [βραχύ τι]. The resulting phrase would then read as follows: 'except Menaechmus, by the shortness — and this with difficulty.' Netz' suggestion seems to be that the phrase 'by the shortness' could be a 'tantalizing, vague' allusion to the 'practical' method used by Menaechmus to construct the conic sections used in the proofs attributed to him by Eutocius. See Netz, *Archimedes*, 294-5 and 295 fn. 157. Another possibility is that this text is an allusion to Menaechmus' 'first' mechanical solution mis-attributed to Plato by Eutocius — not to Menaechmus' solution(s) using conic sections. As anyone who has tried the mechanical solution can attest, it is doable 'with difficulty'. For more on the 'practical' application of the method using the intersection of two parabolas, see the discussion of Diocles below.

29 Diocles, *On Burning Mirrors: The Arabic Translation of the Lost Greek Original*, ed. with English Translation and Commentary, G.J. Toomer (Berlin, Heidelberg, and New York: Springer-Verlag, 1976), 90.186

30 Toomer, commentary on Diocles, *On Burning Mirrors*, 169-70

Although I do not think that Toomer's hypothesis can be completely discounted, I share the skepticism concerning it expressed by Bulmer-Thomas.³¹ In the first place, the fact that Eutocius elsewhere cites Diocles would lead one to expect that he would cite him here as well if he were believed by Eutocius to be the author of the solution, especially since Eutocius does explicitly cite the authors of the other solutions to the problem. It seems that the traditional understanding of the 'ἄλλως' — that it introduces a different kind of solution due to Menaechmus — is at least as likely as Toomer's conjecture.

Second, although Diocles' solution certainly is, in some sense, the same solution as that reported by Eutocius, it is set forth in a quite different way. Diocles' argument is entirely synthetic: that is, he simply specifies the construction that will yield the two mean proportional line segments, given two extreme line segments one of which is half the length of the other. The proof is also very constructive and makes crucial use of the focus-directrix property of parabolas. Two lines, which are to be the axes of the parabolas, are drawn as intersecting perpendicularly. It is clear from the text (as well as the diagram supplied by Toomer) that one axis is thought of as extending beyond the point of intersection by one-fourth of the longer 'given' extreme line segment and the other axis as extending beyond the intersection by one-fourth of the shorter given line segment. So there is a major difference from the arrangement of line segments of Figures 2 and 4. What is needed for Diocles' construction is not the continuation of each parabola's axis beyond the point of their intersection (which will mark their common vertex) by the parabola's *latus rectum*, but the continuation of each parabola's axis by one-fourth of its *latus rectum*. In contemporary terminology, each such extension will mark the distance of the directrix line from the vertex of its parabola. Diocles proceeds to define the focus of each parabola as lying on its axis the same distance from its vertex but in the opposite direction. He then, in effect, gives directions for 'graphing' points of each parabola. The procedure is the following. (1) Select a point P some distance along the axis from the focus F. (2) Extend a line *l* perpendicular to point P (toward the quadrant where the two parabolas will intersect). (3) Inscribe a circle *c* with radius equal to the line segment beginning at the end of the extension of the parabola (one-fourth of the parabola's *latus rectum*

31 Bulmer-Thomas, 'Menaechmus', 272.

beyond its vertex) and ending at P. (4) Mark the point of intersection R of circle *c* and line *l*. In the words of Diocles, we then repeat this procedure 'for many points close to'³² P.

From the contemporary perspective, it is easy to see that for each such 'graph-point' R, the linear distance from P to R (= radius of the circle *c*) is equal to the distance of a perpendicular line segment from the directrix of the parabola to R. The directrix will be the line perpendicular to the axis of parabola and lying a distance of one-fourth of the parabola's *latus rectum* when the axis is extended beyond the vertex. So the construction appeals quite directly to a characteristic of parabolas that is now often taken to be definitory: a parabola is a set of points that are each equidistant from a fixed line (the directrix) in a plane and a point (the focus) in that plane not lying on the directrix. After having graphed a sufficient number of points, Diocles directs us to 'draw with a curved ruler a line passing through' these points. His assumption is that this will give us a (segment of a) smooth curve, but he says nothing more about this curve at this point in the proof. He then directs us to repeat the same process on the axis of the other parabola, perpendicular to the one to which we have been attending. And here he assumes that the two curves so constructed intersect and proceeds to a proof that the perpendiculars dropped from the point of intersection to the two axes are the two mean proportions being sought. This proof is clever but elementary, using little more geometry than the Pythagorean Theorem and Euclid II 8. It is only at the very conclusion of the proof that Diocles makes the following claim: 'and, furthermore, it is obvious that the [two curves] are parabolas, and that each of them passes through point N; hence the (conic) sections necessarily intersect each other.'³³ The fact that the two curves that he has graphically defined are parabolas appears to be invoked only to guarantee the existence of a point of intersection.

My conclusion is that close examination of Diocles' construction shows that it has relatively little to do with the argument employing two parabolas reported by Eutocius. Eutocius' construction makes no use of the focus-directrix property, which is essential to Diocles' construction. Diocles' construction *requires* the extension of each axis by one-fourth of its *latus rectum* (= *f*, the parabola's 'focal length') beyond its vertex — not

32 Diocles, *On Burning Mirrors*, 92.191

33 Ibid., 96.207

by the whole *latus rectum* (= $4f$), as in Figures 2 and 4 (a placement of *latera recta* which, as we saw, was crucial to the mechanical method attributed to Plato by Eutocius, as in Figure 2, but not at all crucial to the solution using the intersection of two parabolas, as in Figure 4). Diocles' concern with burning mirrors makes it natural that he should have developed his focus-directrix method of constructing the two parabolas because of his preoccupation with the foci of parabolic reflectors. This 'naturalness', of course, in no way detracts from the ingenuity of his construction.

Consequently, despite Toomer's claim, I believe that the two following claims are quite plausible: (1) Eutocius *does* mean to attribute to Menaechmus the solution using two parabolas; (2) this solution, as reported by Eutocius, is in some way derivative from the solution (mis)attributed to Plato — not the other way around, as contemporary scholars beginning with Heath have hypothesized. It is also certainly possible, particularly in view of Plato's reproach concerning the use of mechanical methods, that Menaechmus was responsible for *both* proofs, the mechanical one associated with Figure 2 and the theoretical one employing the intersection of two parabolas, associated with the essentially identical Figure 4. In the words of Bulmer-Thomas, 'this cannot be proved or disproved, but it would be the simplest explanation of all the facts.'³⁴

But where does this leave Diocles' proof, as reported in the Arabic manuscript of the *Περὶ πυρρείων*? My conjecture — which I think also, at this junction, 'cannot be proved or disproved' — is that Diocles was familiar with an earlier 'theoretical' solution of the problem using the intersection of two parabolas, of the sort reported by Eutocius and due to Menaechmus. The contribution of Diocles, who need not have been much concerned with Plato's contempt for the mechanical, was to devise a practical (and 'mechanical') means for constructing the two intersecting parabolic curves, given 'parameters' (*latera recta*) for each. According to this conjecture, he already knew, from Menaechmus' theoretical proof, that the intersection of two such parabolas solves the problem. He sets out to find a (relatively) practical mechanical way to construct the curves 'in real space', so to speak. And, if this was indeed his goal, I think that he succeeded in achieving it by means of his graphing technique,

34 Bulmer-Thomas, 'Menaechmus,' 275.

which requires only a compass, straightedge, and whatever he used as a 'curved ruler' to fit his graph-points onto a smooth curve.

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