

3388

FRIEDRICH, K.O.

1965

FRIEDRICH, K.O.

CHAPTER ONE

The Pythagorean Theorem

The theorem ascribed to Pythagoras is concerned with the sides of a right triangle. The three sides of such a triangle are the two legs adjacent to the right angle and the hypotenuse opposite to this angle. The Pythagorean theorem gives a relation between the lengths of the three sides; it enables one to compute the length of the hypotenuse if the lengths of the other sides are given.

To determine the length of a side, or of any segment of a straight line, one must adopt a unit, or rather a particular segment whose length is taken as unit. Then one may express the length of any segment as a multiple of such a unit segment. To say the length of a segment is three means that it is three times as long as the unit segment.

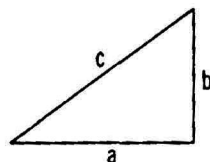


Figure 1. Right triangle with legs a , b and hypotenuse c

In this sense we denote the lengths of the two legs by a and b and the length of the hypotenuse by c (Figure 1). The Pythagorean theorem is embodied in the formula

$$c^2 = a^2 + b^2.$$

Clearly, when the lengths a and b are given, the length c can be computed since one can compute the square root of any positive number. Thus, by taking the square root of each member of the formula above, we find

$$c = \sqrt{a^2 + b^2}.$$

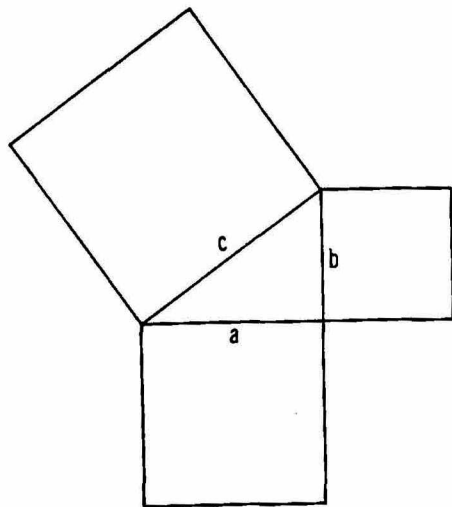


Figure 2. Right triangle with squares over legs and hypotenuse

The formula $c^2 = a^2 + b^2$ can be given a direct geometrical meaning. Let us erect the square over each of the two legs of the triangle and over its hypotenuse; see Figure 2. Clearly, the area of a square is the square* of the length of its side. The significance of the formula $c^2 = a^2 + b^2$ can therefore be expressed by saying that the area of the square over the hypotenuse is the sum of the areas of the squares erected over the sides. As a matter of fact, it is this geometrical assertion which is called Pythagoras' theorem by Euclid.

How can the Pythagorean theorem be proved? Many different proofs have been given and we shall present various quite different approaches leading to such proofs.

* Here the first word "square" refers to a figure, the second one to the product of a number by itself.

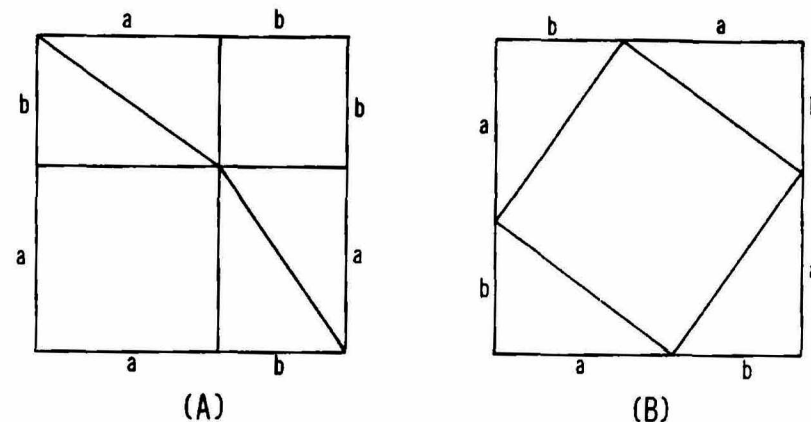


Figure 3. Proof of the Pythagorean theorem (A) by decomposing a square of area $(a+b)^2$ into two squares of areas a^2 and b^2 and four triangles of areas $\frac{1}{2}ab$; (B) by decomposing the same square into four triangles of areas $\frac{1}{2}ab$ and one square of area c^2

In the present chapter we shall give two geometrical proofs. In the first we consider the square with the side* $a+b$; see Figure 3. In one corner of this "large square" we place the square with the side* a ; in the opposite corner we place the square with the side b . Evidently two rectangles, each with sides a and b , remain. Thus the area of the large square is the sum of the areas a^2 and b^2 of the squares with sides a and b and twice the area ab of the rectangle with sides a and b ; see Figure 3A.

Each rectangle may be regarded as the sum of two right triangles, each with legs a and b . Now, one may place these four triangles differently in the large square, as indicated in Figure 3B, so that each is placed in a corner of the large square with its short leg to the left and long leg to the right if looked at from the outside. The figure bounded by the hypotenuses of these triangles is evidently a quadrilateral each of whose sides has length c . In fact this figure is a square; for, at a point of the side $a+b$ where two triangles meet, three angles combine to form a straight angle. Two of them, being opposite to legs a and b of two congruent right triangles, are complementary and therefore add up to a right angle; so the remaining angle is also a right angle.

* Here we use the term "side" where we should have used the expression "side with length". This is convenient and we shall continue to use this convenient simplification of language.

Thus we see that the large square, which can be decomposed into the squares a^2 and b^2 and four right triangles with legs a and b , can as well be decomposed into the square with area c^2 and again four right triangles with legs a and b . Imagining the four right triangles removed from both figures, we realize that the area $a^2 + b^2$ remaining in the one figure equals the area c^2 remaining in the other. In other words, we have derived the statement $a^2 + b^2 = c^2$.

This proof is very simple—perhaps it is grasped by intuition more easily than any other geometrical proof—but it is not quite direct. Instead of identifying the areas $a^2 + b^2$ and c^2 directly, the areas of two larger squares are identified. One may wonder whether or not a more direct proof of the Pythagorean theorem is possible. For example, can one cut the two squares a^2 and b^2 into pieces and then recombine these pieces to form the square c^2 ? Such a dissection and recombination is indeed possible; as a matter of fact, it is possible in infinitely many ways. We present a simple “recipe” for this purpose.

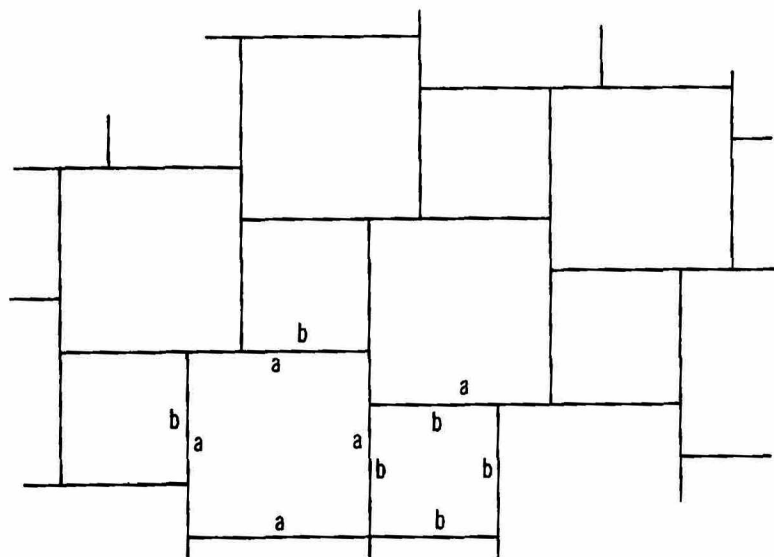


Figure 4A. Covering of plane by squares with sides a and b

We imagine the whole plane covered by squares with sides a and b in the manner shown in Figure 4A; we also imagine the plane covered by squares with side c in the manner shown in Figure 4B. We shall

refer to these coverings as the first and second “arrays”. The sides of the second array are to be parallel to the hypotenuses of the right triangles whose legs a and b are so placed that they are parallel or perpendicular to the sides of the squares of the first array.

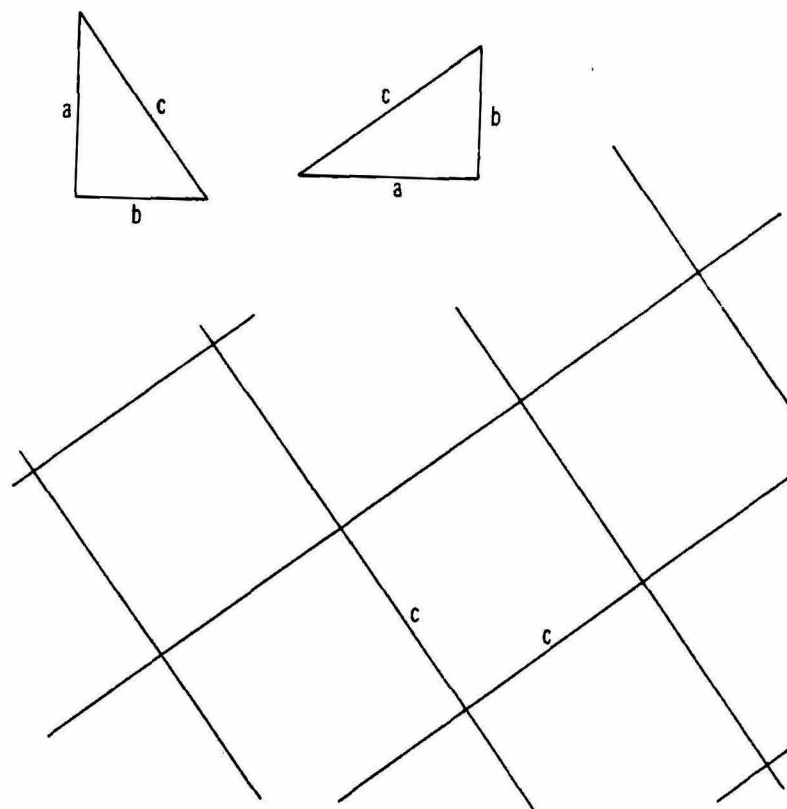


Figure 4B. Covering of plane by squares with side c

There are many possibilities for the location of the second array in relation to the first one. We certainly can choose the location of one of its vertices at pleasure; of course, the location of all other vertices is then fixed (see Figure 5). If the selected vertex of the second array lies in a particular square of the first array, all other vertices of the second array lie in corresponding positions in congruent squares.

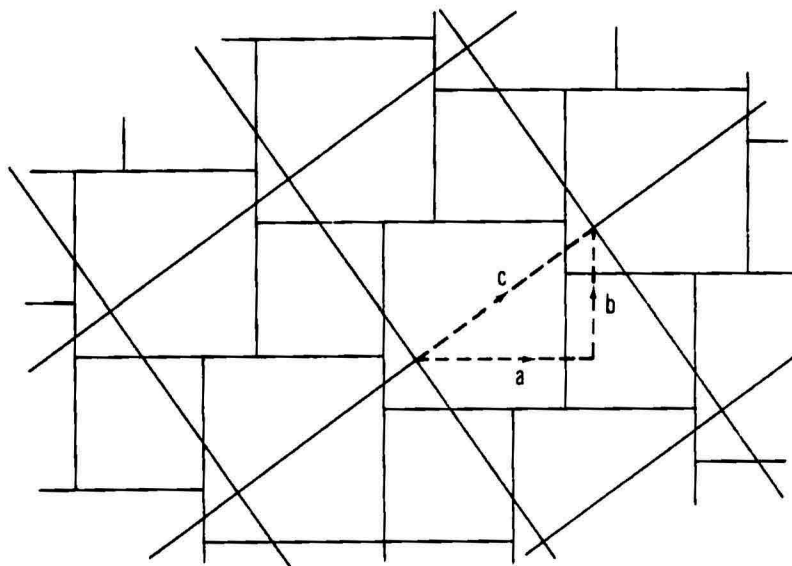


Figure 5. Second array superimposed on first array

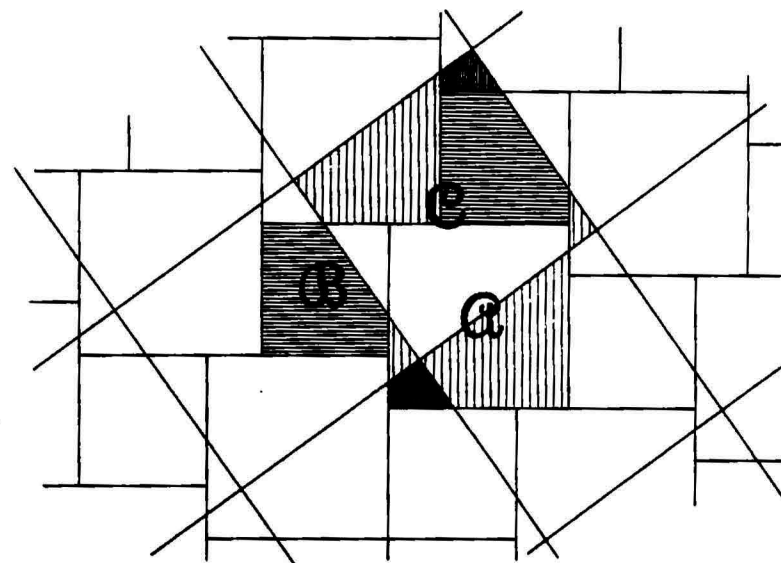
In order to explain what we mean by "corresponding position" we observe that any square of the first array can be displaced into a congruent one simply by first moving it a distance a horizontally to the right and then a distance b vertically up, as is evident from Figure 5. In this displacement every point of the square goes over into a point in a corresponding position in the new square.

The displacement of each point may also be described by saying that the point moves along the legs a and b of a right triangle; clearly, this displacement can be effected more directly just by moving the point along the hypotenuse c of the triangle. If a vertex of the second array is moved along such a hypotenuse, it will evidently end up at a vertex of a congruent square.

Applying the same kind of argument to all four directions leading from a vertex of the second array to neighboring vertices, and repeating this any number of times, we realize that all vertices of the second array have corresponding positions in congruent squares of the first array.

Now, consider a pair of adjacent squares of the first array, say α , β with sides a and b , and a square ϵ of the second array intersecting the squares α and β ; see Figure 6. Clearly, the sides of the

squares of the second array cut the squares α and β into pieces. Each such piece either belongs to ϵ or it belongs to one of the squares congruent to ϵ . In the latter case this piece is congruent to a piece lying within ϵ cut out by the lines of the first array. If now all the pieces of α and β are cut out and reassembled within ϵ , each piece being placed in its corresponding position, then they completely fill out the square ϵ . Thus we recognize that the area of ϵ is precisely equal to the sum of the areas of α and β and are led to the statement $c^2 = a^2 + b^2$, that is to say, to the statement of the Pythagorean theorem.

Figure 6. Subdivisions of squares α and β reassembled in square ϵ

To be sure, our present argument is not complete inasmuch as we have taken for granted a number of simple geometrical facts without deriving them rigorously. It would not be difficult to supply these missing links for each way of cutting up and reassembling our squares. If this is done, a proof of the Pythagorean theorem results; however, we have not carried out these details but have given only a "recipe" for proving the theorem. Note that we have given infinitely many such "recipes", since we can choose any one of infinitely many places for a vertex of the second array and hence can select the location of the square ϵ from an infinity of possibilities.

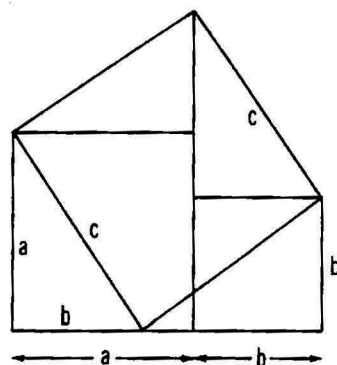


Figure 7. Simple proof of the Pythagorean theorem by reassembly

Were we to fill in all details of our argument to derive a “proof” from a “recipe”, the chain of reasoning would become somewhat long and tedious. If one is primarily interested in a concise and complete proof of the Pythagorean theorem, one may adopt any of the customary arguments such as the first presented here*; or one may specialize our second proof to the case in which, with reference to Figure 6, the lowest vertex of the square $\mathcal{C}$ lies on the lower side of the square $\mathcal{A}$, at a distance b from its left endpoint (see Figure 7). On the other hand, in the approach taken here certain essential features of the Pythagorean theorem are more clearly illuminated than in other approaches. Also, some steps of the present approach have counterparts in the approach that we shall develop next, after we have discussed two basic notions: “signed numbers” and “vectors”.

in: FROM PYTHAGORAS
TO EINSTEIN

* or that indicated in the footnote of p. 28.