

Archive for History of Exact Sciences
1972. 9. p 212-221

The Discovery of the Regular Solids

WILLIAM C. WATERHOUSE

Communicated by B. L. VAN DER WAERDEN

I. The Current View of the Problem

Once upon a time there was no problem in the history of the regular solids. According to PROCLUS, the discoveries of PYTHAGORAS included "the construction of the cosmic solids,"¹ and early historians could only assume that the subject sprang full-grown from his head.² But a better-developed picture of the growth of Greek geometry made such an early date seem questionable, and evidence was uncovered suggesting a different attribution. A thorough study of the testimony was made by E. SACHS,³ and her conclusion is now generally accepted: the attribution to PYTHAGORAS is a later misunderstanding and/or invention.

The history of the regular solids thus rests almost entirely on a scholium to EUCLID which reads as follows:

"In this book, the 13th, are constructed the 5 figures called Platonic, which however do not belong to Plato. Three of these 5 figures, the cube, pyramid, and dodecahedron, belong to the Pythagoreans; while the octahedron and icosahedron belong to Theaetetus."⁴

THEAETETUS lived c. 415-369 B.C., so this version gives a moderately late date; and it has the considerable advantage of seeming unlikely. That is, the details in the scholium are not the sort of history one would naively conjecture, and hence it is probably not one of the stories invented in late antiquity.⁵ As VAN DER WAERDEN says, the scholium is now widely accepted "precisely because [it] directly contradicts the tradition which used to ascribe to Pythagoras anything that came along."⁶

¹ PROCLUS, *In Euc.* (ed. FRIEDLEIN, p. 65).

² Here for example is C. A. BRETSCHNEIDER, *Geometrie und die Geometer vor Euclides* (Leipzig, 1870), p. 86: "Die Kenntnis und Construction der [regelmässigen Körper] wird dem Pythagoras vom ganzen Althertum mit solcher Einstimmigkeit zugeschrieben, dass an der Richtigkeit der Behauptung mit Grund nicht gezweifelt werden kann."

³ E. SACHS, *Die fünf Platonischen Körper* (Berlin, 1917).

⁴ EUCLID XIII, Scholium 1 (ed. HEIBERG, vol. v, p. 654). Some support comes from an entry in the *Suda* saying of THEAETETUS that "he first constructed the so-called five solids". (In both quotations the verb translated "construct" is *γράφειν*.)

⁵ This argument is familiar in textual criticism as the principle of the *lectio difficilior*. See K. VON FRITZ, *Grundprobleme der Geschichte der antiken Wissenschaft* (Berlin, 1971), p. 341.

⁶ B. L. VAN DER WAERDEN, *Science Awakening* (New York, 1961), p. 100.

3503 (p4)
WATERHOUSE, W.C.
1972

The Discovery of the Regular Solids

WILLIAM C. WATERHOUSE

Communicated by B. L. VAN DER WAERDEN

I. The Current View of the Problem

Once upon a time there was no problem in the history of the regular solids. According to PROCLUS, the discoveries of PYTHAGORAS included "the construction of the cosmic solids,"¹ and early historians could only assume that the subject sprang full-grown from his head.² But a better-developed picture of the growth of Greek geometry made such an early date seem questionable, and evidence was uncovered suggesting a different attribution. A thorough study of the testimony was made by E. SACHS,³ and her conclusion is now generally accepted: the attribution to PYTHAGORAS is a later misunderstanding and/or invention.

The history of the regular solids thus rests almost entirely on a scholium to EUCLID which reads as follows:

"In this book, the 13th, are constructed the 5 figures called Platonic, which however do not belong to Plato. Three of these 5 figures, the cube, pyramid, and dodecahedron, belong to the Pythagoreans; while the octahedron and icosahedron belong to Theaetetus."⁴

THEAETETUS lived c. 415–369 B.C., so this version gives a moderately late date; and it has the considerable advantage of seeming unlikely. That is, the details in the scholium are not the sort of history one would naively conjecture, and hence it is probably not one of the stories invented in late antiquity.⁵ As VAN DER WAERDEN says, the scholium is now widely accepted "precisely because [it] directly contradicts the tradition which used to ascribe to Pythagoras anything that came along."⁶

¹ PROCLUS, *In Euc.* (ed. FRIEDLEIN, p. 65).

² Here for example is C. A. BRETSCHNEIDER, *Geometrie und die Geometer vor Euclides* (Leipzig, 1870), p. 86: "Die Kenntnis und Construction der [regelmässigen Körper] wird dem Pythagoras vom ganzen Althertum mit solcher Einstimmigkeit zugeschrieben, dass an der Richtigkeit der Behauptung mit Grund nicht gezweifelt werden kann."

³ E. SACHS, *Die fünf Platonischen Körper* (Berlin, 1917).

⁴ EUCLID XIII, Scholium 1 (ed. HEIBERG, vol. v, p. 654). Some support comes from an entry in the *Suda* saying of THEAETETUS that "he first constructed the so-called five solids". (In both quotations the verb translated "construct" is *ποιεῖν*.)

⁵ This argument is familiar in textual criticism as the principle of the *lectio difficilior*. See K. VON FRITZ, *Grundprobleme der Geschichte der antiken Wissenschaft* (Berlin, 1971), p. 341.

⁶ B. L. VAN DER WAERDEN, *Science Awakening* (New York, 1961), p. 100.

But probability arguments can cut both ways, and those scholars who hesitate to accept the scholium do so primarily because it seems *too* unlikely. There have been two main sticking places: first, the earliness of the dodecahedron in comparison with the icosahedron; and second, the surprising lateness of the octahedron. The first objection, however, has been fairly well disposed of. The mineral pyrite (FeS_2) crystallizes most often in cubes and almost-regular dodecahedra; it is quite widespread, being the most common sulphide, and outstanding crystals are found at a number of spots in Italy.⁷ Moreover it regularly occurs mixed with the sulphide ores, and underlying the oxidized ores, of copper; these deposits have been worked since earliest antiquity.⁸ Thus natural dodecahedra were conspicuous, and in fact they did attract attention: artificial dodecahedra have been found in Italy dating from before 500 B.C.⁹ Icosahedral crystals, in contrast, are much less common. Hence there is no real difficulty in supposing that early PYTHAGOREAN geometers in Italy were familiar with dodecahedra but had not yet thought of the icosahedron.

The late discovery of the octahedron has posed a more serious problem. As VON FRITZ writes, "Die Bemerkung ... enthält eine Schwierigkeit darin, dass das Dodekaeder sehr viel schwerer zu konstruieren ist und seine Konstruktion sehr viel grössere Voraussetzungen erfordert als diejenige des Oktaeders, so dass es schon deshalb unwahrscheinlich ist, dass seine Konstruktion früher erfolgt sein sollte."¹⁰ Similarly W. K. C. GUTHRIE: "The tradition is difficult to evaluate, since the construction of the octahedron is a less advanced mathematical feat than that of the dodecahedron, and could certainly have been carried out on principles known long before Theaetetus."¹¹ There has been some dispute whether "construction" should mean use of straightedge and compass or merely construction of a model, but that makes no difference to the basic objection; in either sense the octahedron is undeniably easier to construct.

Furthermore, the octahedron does not seem like a particularly recondite figure. There are minerals, *e.g.* magnetite (Fe_3O_4), with octahedral crystals;¹² and above all, octahedral shapes would seem to occur inevitably to anyone familiar with pyramids. T. L. HEATH, among others, drew the natural conclusion: "the octahedron (which is only a double pyramid with a square base) cannot but have been known to the Egyptians."¹³ Indeed, square-based pyramids were studied in Egypt more than a millenium before THEAETETUS,¹⁴ and the traditional story even has Greek geometry starting with the measurement of a pyramid.¹⁵ Thus the materials for making octahedra were certainly at hand.

⁷ J. D. DANA *et al.*, *System of Mineralogy*⁷, Vol. I (New York, 1944), p. 282-289.

⁸ R. J. FORBES, *Studies in Ancient Technology*, Vol. IX (Leiden, 1964), pp. 6-8, 73-74.

⁹ F. LINDEMANN, *Sitzungsber. Bayr. Akad. Wiss., Math.-Phys. Kl.* 26 (1896), 625-768 (cited in SACHS, *op. cit.*, p. 84).

¹⁰ K. VON FRITZ, in PAULY-WISSOWA's *Realencyclopädie* VA, 1364 (*s.v.* THEAETETOS).

¹¹ W. K. C. GUTHRIE, *History of Greek Philosophy*, Vol. I (Cambridge, 1962), p. 268.

¹² DANA, *op. cit.*, p. 699.

¹³ T. L. HEATH, *The Thirteen Books of Euclid's Elements*, Vol. III (Cambridge, 1908), p. 438.

¹⁴ VAN DER WAERDEN, *op. cit.*, p. 34.

¹⁵ DIOG. LAERT. I, 27.

Little has been said to counter the objection; defenders of the scholium have generally granted that on this point it seems quite implausible.¹⁶ I hope to show that, on the contrary, the history as given is quite natural when properly understood. The seeming difficulty will prove to be purely imaginary once we realize that a crucial step in the development of the subject has been overlooked.

II. The Importance of Abstraction

Commentators from the scholiast on have discussed the history of the regular solids by considering the discoveries of the individual solids. But obviously the regular solids are not five unrelated bodies chosen at random; they are, precisely, the regular solids. The cube and pyramid may occur separately, but the others are hardly ever mentioned except in contexts involving the regular solids as such. From the Greeks to the present day the regular solids have been studied together, compared with each other, and considered as a group. To discuss the solids one by one misses the point; we must study not just their individual history but above all their joint history.

The real history of the regular solids therefore begins at the point when men realized there was such a subject. The discovery of this or that particular body was secondary; *the crucial discovery was the very concept of a regular solid*. Long familiarity has made the idea seem almost obvious, but it is not. It is hardly a subject arising in everyday life, nor is it an idea strongly suggested by practical geometry. The Egyptians looked at cubes and pyramids without being led to generalize, and the same is true of billions of non-mathematicians down to the present. Indeed, people who have not been told the definition can see models of all five solids and still not isolate their common property. We have the mathematical concept of a regular solid only because some mathematician invented it.

The correct definition, in fact, is not all that easy to formulate. A requirement of convexity is certainly needed, to rule out figures like KEPLER's stellated dodecahedron (constructed from equilateral triangles). Even then one cannot say simply that all faces should be regular; for the cuboctahedron has all faces regular and all solid angles congruent—but faces not all of the same shape.¹⁷ It is worth remembering that the first recorded definition involves not just the figure but its relation to the circumscribed sphere.¹⁸ This might well be the original definition, since it could have been suggested by the familiar process of studying circles via inscribed polygons.¹⁹ In any case it should be clear that the man who first introduced the notion of regular solid made a significant contribution to mathematics.

¹⁶ SACHS (*op. cit.*, p. 80) calls it "eine Angabe, die ... etwas scheinbar Unmögliches gibt". J. L. HEIBERG, before SACHS' arguments on dodecahedra, was tempted to interchange "octahedron" and "dodecahedron" in the text of the scholium (cited in SACHS, p. 81). J. BURNET says simply "We have no right to reject the definite testimony ... on grounds of *a priori* probability"—*Early Greek Philosophy*⁴ (London, 1930), p. 284, n. 1.

¹⁷ This attempted definition may have been considered in antiquity. See the Appendix.

¹⁸ PLATO, *Tim.* 55a.

¹⁹ It is noteworthy that in all cases EUCLID constructs the regular solids together with their circumscribing spheres.

That contribution probably also had an indirect effect in motivating further study of regular solids. The extraordinary Greek interest in them is commonly attributed to the esthetic attractiveness of their regularity, but that can hardly be the full explanation. We do not, after all, find the regular icosagon studied as much as the icosahedron. Surely the particular attraction of the regular solids was that there were only five of them. In purely mathematical terms this meant that certain problems could be solved in finitely many steps, and so might reasonably be attempted; it was possible, for instance, to compute the relative areas of all regular solids inscribed in a given sphere. But the philosophical allure was even more important. We know that Greek philosophical thought showed a strong preference for the limited and the finite,²⁰ and the regular solids revealed an unexpected, seemingly God-given finiteness in the nature of things.

Now the existence of just five regular solids is indeed an attractive theorem, one of the earliest solutions of a classification problem; even today it can startle the uninitiated. But the theorem cannot be proved, cannot even be stated, without an abstract definition of regular solid. It is not the sort of fact which could be known in particular cases and then generalized; it is inherently general. Hence the importance of this fact for the development of the theory demonstrates again the crucial role of the abstract concept.

This importance of abstraction should not be surprising; for the theory of regular solids is purely Greek, and the introduction of generality and abstraction is probably the best-known characteristic of classical Greek thought. In mathematics the Greeks habitually formulated general theorems where pre-Greek writers merely worked out examples. The same trait is equally prominent in philosophy, where ARISTOTLE in fact attributes to SOCRATES a specific emphasis on general definitions.²¹ Indeed, the developments in the two fields are related, as many recent studies have emphasized. The relation in our particular case may be unusually close—for general definitions are what SOCRATES supposedly praised in the mathematics of the young THEAETETUS.²²

III. The Individual Solids

The knowledge of regular solids developed in two stages, first a study of some particular solids and then a general investigation of regular solids as such. With this in mind, one can attempt to specify the places of the individual solids. Obviously, for instance, the cube and the pyramid were studied in the first stage, though some of their properties may not have been found until later. The dodecahedron also must have attracted attention by itself; for although this is not *a priori* obvious, it is precisely the conclusion to be drawn from the evidence mentioned in Section I.

The place of the icosahedron is less certain. It might well have been constructed in a search for all regular solids, but it might equally well have come earlier. Indeed,

²⁰ As one example out of many, recall that limit is good and unlimited is bad in the PYTHAGOREAN table of opposites given by ARISTOTLE (*Metaph.* A, 986a).

²¹ *Metaph.* A, 987b and M, 1078b.

²² PLATO, *Theat.* 147–148.

one can even picture a mathematician trained in abstraction who constructs an icosahedron, is intrigued by the new shape, asks himself what its key property is, and thus is led to the general concept. But this is pure speculation. We can say only that the icosahedron is surely not much later than the abstract concept, and that there is no reason to expect it to be much earlier.

The octahedron, finally, can be placed without much doubt in the second stage. For, as HEATH quite correctly said, the octahedron "is only a double pyramid with a square base"; and that is a very good reason why no one would have bothered with it. We can readily grant that a man who in some sense could construct a dodecahedron could in the same sense construct an octahedron—but why should he? Someone thoroughly familiar with pyramids would attach no special importance to this particular combination of them. He could assemble an octahedron, he might even admire its appearance; but *mathematically he would have nothing to say about it*. Only someone possessing the general concept of regular solid would have reason to single it out.²³

The discovery of the octahedron thus was rather like the discovery of, say, the fifth perfect number: what required discovery was not so much the object itself as its significance. Some Babylonian accountant may well have written down the number 33,550,336; but he did not thereby discover the fifth perfect number, because he did not observe the property which distinguishes this number from others. Similarly, the octahedron became an object of special mathematical study only when someone discovered a role for it to play.

The arguments in this section have carefully been kept independent of the scholium, but clearly they tend to show that it is more reasonable than it looks at first glance. The scholium does reflect two stages of discovery, and its placement of the individual solids is roughly what might be expected. In particular, the late date of the octahedron turns out to be a good sign of reliability. It has seemed incomprehensible only because of a tacit and unjustifiable assumption that geometers from the very start shared our interest in finding regular solids.

IV. The Terminological Evidence

Further evidence for the two-stage development of the subject can be derived from the names of the regular solids. Briefly, the facts are these. The cube and pyramid have names from common speech, and there are scattered indications that the dodecahedron was once called something like "the sphere of twelve pentagons".²⁴ The octahedron and icosahedron apparently never had any other names. These facts were brought out by SACHS to support the earliness of the dodecahedron and lateness of the octahedron. But one can actually draw a stronger conclusion.

²³ Consider for comparison the quartz crystal. Quartz (SiO_2) is the most common mineral on earth; its crystals are large, unmistakable hexagonal pyramids and prisms; the Greek term for it has given us the very word "crystal". Yet in all of Greek geometry there is no special study of hexagonal prisms or pyramids. The shapes were familiar enough, but there simply was nothing particular to say about them.

²⁴ IAMBlichus, *Vit. Pyth.* 88; similar phrases in PLATO, *Phaedo* 110b, and PLUTARCH, *Quaes. Plat.* V, 1003d. (References and discussion in SACHS, *op. cit.*, p. 83.)

Obviously "octahedron" and "icosahedron" are technically descriptive names. What is more, they are part of a series of terms: "dodecahedron" became standard, "tetrahedron" in this context ultimately replaced EUCLID's "pyramid", and even "hexahedron" is found as a name for the cube.²⁵ Now these names are not just technical, they are *systematic*. Clearly they were designed as a uniform terminology for the regular solids. They were used only in connection with regularity; a hexagonal prism has eight faces, for instance, but it was never called an octahedron.²⁶ Whoever coined the names also must have known a fact not obvious in advance, namely, that no two regular solids have the same number of faces. Thus the terminology strongly suggests that whereas the cube, the pyramid, and probably the "sphere of twelve pentagons" were considered by themselves, the octahedron and the icosahedron were first studied in connection with a systematic survey of regular solids.

V. Conclusion

The study of the regular solids started with a sort of "prehistory"; in that stage men investigated some individual bodies but did not yet recognize the connection between them. The history proper begins with the general definition and includes comparative studies and the listing of all possible regular solids. The change is accurately reflected by the ancient terminology and the data in the scholium, and once its importance is grasped they become part of a coherent picture of historical development.

The arguments leading to this result, besides explaining an obscure bit of history, may serve to point a moral; for there is a fundamental issue involved. Mathematics is, above all, a collection of true statements, yet it is never a disordered assortment. Each mathematician in his mind's eye sees not individual facts but a closely-knit fabric of concepts and theories. Hence the historian of mathematics must have a double vision. Obviously he must have a thorough modern knowledge of his subject; otherwise he cannot judge which proofs are valid, which discoveries are significant, which lines of thought prove to be productive. But he must also imagine how the subject appeared to men in the past. For, as with the octahedron, something quite simple can be overlooked merely because no one has a reason for finding it.

History of mathematics thus has a good deal in common with other parts of the history of ideas, and can profit from warnings like the following:

"[C]ommon and ... devastating is *misinterpretation by abstraction*. Your author mentions *X*; and *X* appears to you to be a case of *Y*; and on the strength of that you say that your author 'was well aware of *Y*', or even that he 'explicitly mentions *Y*'. Because you have abstracted *Y* from *X*, you assume that your author did so too. But such an assumption must not be made on general grounds, for no man has ever made or ever will make all the abstractions possible from any one object present to his consciousness... ."

²⁵ C. MUGLER, *Terminologie Géométrique des Grecs* (Paris, 1958), p. 180.

²⁶ Similarly "tetrahedron" in ancient usage refers only to the regular triangular pyramid: MUGLER, *op. cit.*, p. 420. Cf. HERON, *Def.* 99 (ed. HEIBERG vol. iv, p. 62).

"Closely related to the above is *misinterpretation by inference*. 'Plato says p , and p implies q ; therefore Plato meant q .' The conclusion does not follow; for Plato may have thought that p did not imply q ; or more probably, the suggestion that ' p implies q ' may never have occurred to him at all; or, most probably of all, even the proposition q itself may never have occurred to him. Every proposition implies an indefinite multiplicity of others; and no one ever perceives all the implications of any proposition. Even those consequences which now seem to us to follow most obviously and directly from a given proposition were often not realized by the acutest of earlier thinkers, as the history of thought shows again and again."²⁷

This was written by a student of ancient philosophy; but there is no difficulty illustrating it from modern mathematics. Reading our current abstractions into earlier work is an ever-present danger. It is all too easy, for example, to see "group theory" in authors who, as we know from the work of H. WUSSING,²⁸ did not possess the notion of an abstract group. And overlooked inferences are even more common. Here for instance is JACQUES HADAMARD, looking back at his early work: "Two theorems, important to the subject, were such obvious and immediate consequences of the ideas contained therein that, years later, other authors imputed them to me, and I was obliged to confess that, evident as they were, I had not perceived them."²⁹

The dangers are especially acute in ancient mathematics. In modern work we at least possess most of the data necessary to trace the history of a theorem or a concept; in addition, the ideas are learned late enough in our lives that we usually do not come to think them self-evident. In reconstructing ancient mathematics, however, we must continually remind ourselves that concepts cannot be studied until someone has thought of them. The concept of a regular solid had to be invented at some point in history. Men before that time may have studied certain solids, and those solids may have been regular, but in a fundamental sense men were not yet studying regular solids.

Finally, we may draw one specific conclusion from these generalities. The scholiast to EUCLID, whatever his merits, can hardly be credited with a modern understanding of the history of ideas. He takes the general concept for granted and focuses on the individual bodies. Hence the data he transmits must have been as inexplicable to him as to subsequent scholars. Since the data nevertheless seem to be correct, we can only conclude that he drew them from a reliable source.³⁰ Therefore his specific attributions, which we cannot check *a priori*, can also be accepted. Hence we can reasonably conclude that the general theory of regular solids is due to THEAETETUS.

²⁷ R. ROBINSON, *Plato's Earlier Dialectic* (Ithaca, 1941), p. 2.

²⁸ H. WUSSING, *Die Genesis des abstrakten Gruppenbegriffes* (Berlin, 1969).

²⁹ J. HADAMARD, *An Essay on the Psychology of Invention in the Mathematical Field* (Princeton, 1945), p. 51.

³⁰ The natural source would have been the (lost) history of mathematics written by EUDEMUS sometime shortly before 300 B. C.; for it would certainly have discussed the regular solids.

Appendix. Semi-Regular Solids

ARCHIMEDES wrote a work on semi-regular solids; it has not survived, but PAPPUS gives a summary of its results.³¹ It is also mentioned briefly in HERON'S *Definitions*, where there is an additional bit of historical information perhaps relevant to the regular solids. Unfortunately the passage as it stands is full of errors—it survives only as part of a Byzantine compilation—and a careful analysis is necessary. The text is as follows:

Εὐκλείδης μὲν οὖν ἐν τῷ ιγ' τῶν Στοιχείων ἀπέδειξε, πῶς τῇ σφαίρᾳ τὰ πέντε ταῦτα σχήματα περιλαμβάνει· μόνα γὰρ τὰ Πλάτωνος οἶται. Ἀρχιμήδης δὲ τριακαίδεκα ὅλα φησὶν εὐρίσκεισθαι σχήματα δυνάμενα ἐγγραφῆναι τῇ σφαίρᾳ προστιθεὶς ὀκτὼ μετὰ τὰ εἰρημένα πέντε· ὧν εἶδέναι καὶ Πλάτωνα τὸ τессαρεσκαιδεκάεδρον, εἶναι τε τοῦτο διπλοῦν, τὸ μὲν ἐξ ὀκτῶ τριγώνων καὶ τετραγώνων ἐξ σύνθετον, ἐκ γῆς καὶ ἀέρος, ὅπερ καὶ τῶν ἀρχαίων τινὲς ᾔδεσαν, τὸ δὲ ἕτερον πάλιν ἐκ τετραγώνων μὲν ὀκτὼ, τριγώνων δὲ ἕξ, ὃ καὶ χαλεπώτερον εἶναι δοκεῖ.³²

“Euclid has shown in Book 13 of the *Elements* how he surrounds these five figures—for he considers only the Platonic ones—with a sphere. But Archimedes—adding eight to the five mentioned—says that there are thirteen complete figures capable of being inscribed in a sphere, of which Plato knew the tessareskaidekaedron [*i.e.*, fourteen-sided one]; it moreover has two forms, one composed of eight triangles and six squares, of earth and air, which some of the ancients also knew, and the other on the contrary of eight squares and six triangles, which seems to be harder.”

We can begin by eliminating from consideration the phrases which I have set off with dashes in the translation. Obviously they are the author's attempt to connect this material with the discussion of regular solids which precedes it. They are in fact a misunderstanding: ARCHIMEDES discussed thirteen new solids, not eight new ones for a total of thirteen. This could be carelessness on the part of HERON, but I think it is more likely to be the later compiler's attempt to harmonize the two bits of information available to him. In any event the mistake is not serious, since its motivation is obvious and it does not affect the remaining material.

The rest of the second sentence is in indirect discourse, and thus claims to tell us what ARCHIMEDES said. But there are several signs that it does not. First of all, only in late antiquity were men of PLATO'S time called “the ancients”. ARCHIMEDES, who was little more than a century later, naturally speaks rather of his “predecessors” or “earlier geometers”.³³ And ARCHIMEDES can hardly be held responsible for the malapropos reference to “earth and air”. This is an allusion to PLATONIC physics, where earth is a cube (six squares) and air an

³¹ PAPPUS V, 34. A semi-regular solid is one in which the solid angles are congruent and the faces are all regular polygons but not all of the same shape. ARCHIMEDES discovered thirteen convex semi-regular solids. (There are also two trivial infinite families, consisting of prisms and prismoids.)

³² HERON, *Def.* 104 (ed. HEIBERG vol. iv, 64–6).

³³ See for instance the preface in *De Quad. Parab.*

octahedron (eight triangles); and it is an unthinking allusion, because PLATO does not allow his cube to change to any other form.³⁴ Even HERON is unlikely to be guilty of such a remark—especially since his preceding discussion of regular solids avoids any mention of the PLATONIC interpretations.

Above all, the statement beginning “it has two forms” cannot be due to ARCHIMEDES because it is thoroughly wrong. ARCHIMEDES actually constructed *three* fourteen-sided semi-regular solids. One of them, now called the cuboctahedron, does indeed have six squares and eight triangles. Of the other two, however, one has six squares and eight hexagons, and the other has six octagons and eight triangles. The solid with eight squares and six triangles, “which seems to be harder”, is in fact impossible.

At this point one might be tempted to give up the whole passage in disgust, but it is not actually hopeless. The serious errors occur in the part beginning “it has two forms”; the author of those words does not know what ARCHIMEDES did. But he also does not realize that the regular solids were discussed without reference to PLATONIC physics. Indeed, he hardly seems to know the first part of the sentence, since he vaguely attributes to “some of the ancients” knowledge which is assigned specifically to PLATO a few lines earlier. In short, this part must be an independent remark which has been added to the text. And there is no difficulty seeing how the addition came to be made: it follows immediately after the uncommon word “tessareshkaidekaedron”, the natural place to put an explanatory gloss. Thus we can conclude that this part of the text is a brief note on tessareshkaidekaedra which the compiler has inserted from some other (obviously ill-informed) source.

The two erroneous additions to the text have now been explained and isolated. If we eliminate them, we are left with the assertion “Archimedes says that there are thirteen complete figures capable of being inscribed in a sphere, and that of them Plato knew the tessareshkaidekahedron.” This is a perfectly reasonable statement; it inspires trust because it has the number of solids right, and I see no reason to doubt that something like it was written by HERON. All of it, including the reference to PLATO, could well have come from ARCHIMEDES’ own introduction to his work, for “his manner is to state simply what particular discoveries made by his predecessors had suggested to him the possibility of extending them in new directions”.³⁵ All in all, then, we have good authority for believing that PLATO was familiar with one semi-regular solid, the cuboctahedron.³⁶

This is as far as the text will take us; the rest can only be conjecture. But it is reasonable to ask in what mathematical context this single solid was introduced.

³⁴ PLATO, *Tim.* 55–56, esp. 56d.

³⁵ T. L. HEATH, *The Works of Archimedes* (Cambridge, 1897), p. xxxix.

³⁶ The passage has sometimes been interpreted as saying that PLATO knew two fourteen-sided solids; see MUGLER, *op. cit.*, p. 299, n. 1, and E. J. DIJKSTERHUIS, *Archimedes* (Copenhagen, 1956), p. 405, n. 2. But the only mention of two solids is in the erroneous second part of the sentence; the first part by itself clearly refers to a single solid. The fact that the erroneous addition does contain a correct description of the cuboctahedron suggests that this was the one known before ARCHIMEDES; in any case it is the simplest of the three and so most likely. (Cuboctahedra dating back to antiquity have been found—see LINDEMANN, *op. cit.*, p. 635–636.)

It is unlikely to have been a search for semi-regular solids; for while it might take an ARCHIMEDES to discover all thirteen, anyone seeing a cuboctahedron and looking for semi-regular solids could easily see how to construct a few more by the same obvious process of cutting off corners. Indeed, it is safe to assume that as usual ARCHIMEDES was the first to raise the question which he then settled.³⁷ Thus the cuboctahedron must have been introduced originally for some other purpose.

In most mathematics there is just one role played by isolated examples: they serve as "counter-examples", showing that some suggested statement is wrong. And this is not a recent logical development; in fact, we know specifically that the use of counter-examples was common practice in the Academy. The most famous is doubtless DIOGENES' plucked chicken, produced in response to PLATO's definition of man as a featherless biped.³⁸ We can observe the same process at work more seriously in the SOCRATIC dialogues. And ARISTOTLE in the *Topics* explains the use of counter-examples immediately after describing what a definition is.³⁹

Thus it is mathematically natural and historically appropriate to suggest that the cuboctahedron was first introduced as a counter-example. And indeed, as was mentioned in Section II, it is a counter-example to a quite natural attempted definition of regular solid. Thus it could well have been introduced for that purpose. Its association with PLATO, then, is another piece of evidence suggesting that the definition of regular solid was first worked out in the Academy.

³⁷ Since the concept here is less familiar than that of regular solid, it is easier to see that it took mathematical genius to invent a definition leading to a satisfactory theory.

³⁸ DIOG. LAERT. VI, 40. This of course may be apocryphal, but that does not affect its value as evidence that the practice was familiar.

³⁹ *Top.* A, 102, esp. 102b 29-33.

Mathematics Department
Cornell University

(Received August 12, 1972)