



2:3:4 Harmony within the tritave

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In the Pythagorean tuning system, the fifth is used to generate a scale of 12 notes per octave. In this paper, we use the octave to generate a scale of 19 notes per tritave; one can play this scale on a traditional piano. In this system, the octave becomes a proper interval and the 2:3:4 chord a proper chord. We study harmonic properties obtained from the 2:3:4 chord, in particular composition elements using dominants, inversions, major and minor chords, and diminished chords. The *Tonnetz* (array notation) turns out to be an effective tool to visualize the harmonic development in a composition based on these elements. 2:3:4 harmony may sound pure, yet sparse, as we illustrate in a short piece.

Keywords: Harmony; scales; composition; continued fractions; Bohlen-Pierce; tritave; neo-Riemannian *Tonnetz*

Mathematics Subject Classifications: 00A65 (primary); 11A55; 05E99

1. Introduction

The 4:5:6 chord lies at the very center of major mode harmony in the music of the western world. It is ubiquitous in chord sequences involving dominants and subdominants, and its mirror image defines the associated minor chords. Allowing for inversion, the chord appears in many shapes including the 3:4:5 chord with the stunning harmonicity given by its low frequency ratios.

By comparison the 2:3:4 chord, despite being admired since Pythagoras' time for its perhaps unsurpassable purity, appears underused in tonal music. In fact, traditional harmony may not consider the 2:3:4 chord as being proper since chords are defined based on the pitch classes of their notes, up to octave equivalence.

In this paper we take the position that it is the chord, or the chord sequence, which defines what is harmony. If octave equivalence does not fit with the chord, then we have to replace it. This has been done before: the Bohlen-Pierce scale, see ([Advocat 2010](#)), is based on the 3:5:7 chord with equivalence modulo the *tritave* or *duodecime*, the interval given by the frequency ratio 3:1. There, octave-equivalence is not even considered a viable concept.

The system which we propose to study is based on the 2:3:4 chord, modulo tritave-equivalence. Thus we offer an interesting alternative to the traditional octave-equivalence-based triadic system.

Our paper has three parts and some appendices. We discuss the third part first. Here, we take elements of composition from the 4:5:6 system and adapt them to 2:3:4 harmony.

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Recall that the dominant of a chord in the 4:5:6 system is formed by replacing each note by its successor in the circle of fifths, and by adjusting with first and second inversions, which move the base note of a chord up by one octave, or the top note down, respectively. For 2:3:4 harmony, there is a corresponding concept, the *circle of octaves*, which we introduce in Section 2.3. First and second inversions in a tritave-based system are obtained by moving the base note up by one tritave, or the top note down. In each system, for the subdominant, one proceeds in the opposite direction in the circle. Together, tonic, dominant and subdominant form what we call in Section 4.1 the *basic sequence*,

tonic — subdominant — dominant — tonic,

which we picture first in the 4:5:6 system, starting with the C-E-G chord:



A sample 2:3:4 chord is A-E-A' as the E is a fifth (3:2) above the A, and the A' an octave (4:2) above the A. Here is the basic sequence for the A-E-A' chord:



Namely, starting from the 2:3:4 chord A-E-A', the second (subdominant) chord is obtained as follows. Replace each note in A-E-A' by the note one octave lower, this yields A₁-E₁-A. Since A₁ and E are one tritave apart, and so are E₁ and B', we obtain the chord A-E-B', which is *tritave-equivalent* to A₁-E₁-A. For the dominant, we move each note in A-E-A' up by an octave to get A'-E'-A'', which is tritave-equivalent to A-D-A'.

We will see in Section 4.2 that the role of *major and minor chords* is formally and audibly quite different. In Section 4.3 we study the use of *diminished chords* for the development of the harmony.

Throughout the third part of the paper, we refer to a sample piece *Ave Maria in dix-neuf par duodecime* which has been composed by the author. While the harmony explores the above discussed elements of composition in 2:3:4 harmony, the arrangement follows loosely J. S. Bach's C-major *Prelude* from WTK1. We reprint the score in Appendix 4; a recording is available on YouTube at <http://https://youtu.be/Bg1n4jM1n5w>.

We discuss perception of music in the 2:3:4 system in the final Section 4.4 on *purity and sparsity*.

In the first part of this paper, we construct and analyze a Pythagorean scale of 19 notes per tritave in which the 2:3:4 chord can be played and modulated. Harmonically, the notes are arranged in the *circle of octaves*, which we introduce and discuss in Section 2.3.

A nice feature of this scale is that in practice, it can be played on a traditional piano, either in just Pythagorean or in 12 tone equal intonation, as we will see in Section 2.5.

Nevertheless, as the scale is tritave-based, a keyboard which emphasizes periodicity with respect to the tritave would be preferable, as in Figure 1.

In the second part of this paper, we review a method for visualizing harmony, the *Tonnetz*, and adapt it for 2:3:4 harmony. The *Tonnetz* captures the above mentioned elements of composition; sequences of triangles or arcs within the picture visualize how harmony develops throughout a piece.

In the 4:5:6 *Tonnetz*, there are major thirds (5:4) along the up-diagonal, minor thirds (6:5) along the down-diagonal, and hence fifths (6:4) in horizontal direction. By comparison, the 2:3:4

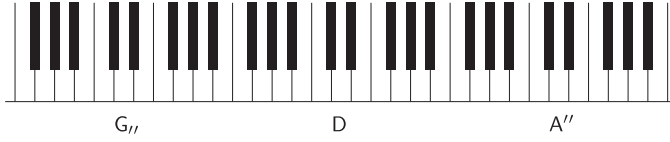


Figure 1. A periodic keyboard for the Tritave.

Tonnetz has fifths (3:2) along the up-diagonal, fourths (4:3) along the down-diagonal and octaves (4:2) in horizontal direction.

In Section 3.3 we adapt the PLR-moves from neo-Riemannian theory to the 2:3:4 *Tonnetz*. While the moves capture elements of 2:3:4 harmony quite well, it turns out that, when compared to 4:5:6 harmony, the number of notes that can be reached with a given number of PLR-moves is smaller.

2. A piano scale based on the tritave

The famous octave-based Pythagorean scale turns out to have a lesser known relative, the tritave-based Pythagorean scale Pyth-3. We briefly review the Pythagorean scale Pyth-2 in order to introduce Pyth-3.

In each Pythagorean scale, a note is given by a relative frequency of the form $2^u 3^v$ where the harmonic degree v describes the position of the note in Pyth-2 in the circle of fifths. The corresponding arrangement of notes by harmonic degree in Pyth-3 is in the circle of octaves; here the position of a note is given by u . Finally, we discuss in this section the playability of Pyth-3 on a traditional piano.

More comprehensive reviews of the Pythagorean scale Pyth-2 can be found in the literature, see for example (Liem 2015). For the construction of well-formed scales and a discussion of desired properties like closure and symmetry we refer to Carey and Clampitt (1989), see Appendix 1.

2.1. Two harmonic degrees for Pythagorean scales

In order to describe notes in Pythagorean scales, two ingredients are needed:

- A base note, which we call D.
- Two intervals, the octave (2:1) and the tritave (3:1).

An arbitrary note in the scale is obtained from the base note by going up a certain number u of octaves and a certain number v of tritaves. Either or both numbers can be negative. We denote by

- Pyth the collection of all notes obtained in this way.

One can also use the octave and the fifth (3:2) for the construction, but the scale and the set Pyth will be the same.

To each note one assigns a frequency ratio φ , it is the quotient of the frequency of the given note by the frequency assigned to the D. This number can be written as

$$\varphi = 2^u \cdot 3^v$$

where $u = \mu_2(\varphi)$ is the 2-adic valuation and $v = \mu_3(\varphi)$ the 3-adic valuation of φ . Thus, the frequency ratios of the notes in Pyth form the subgroup of the multiplicative group $\mathbb{Q}^+$ generated by 2 and 3.

The numbers $\mu_2(\varphi)$ and $\mu_3(\varphi)$ are called the *harmonic degrees* of the note given by frequency ratio φ . We single out two subsets of Pyth:

- Pyth-2 consists of all notes in Pyth of harmonic degree μ_3 between -5 and $+6$.
- Pyth-3 consists of all notes in Pyth of harmonic degree μ_2 between -9 and $+9$.

The bounds of the harmonic degrees will become transparent in the next section.

2.2. The comma and enharmonic notes

So far, we have infinitely many notes, in Pyth there is one for each pair (u, v) of integers. To reduce this number, certain notes are considered to be “enharmonic.” We show that each note is enharmonic to a unique one in Pyth-2 and to a unique one in Pyth-3.

It turns out that the numbers 2^{19} and 3^{12} are very close, their quotient

$$\kappa = \frac{3^{12}}{2^{19}} = 1.01364 \dots$$

is the *Pythagorean comma*. In cents it is about $1200 \cdot \log_2(\kappa) = 23.460 \dots$

We say that two notes are *enharmonic* if their frequency ratio is a power of the Pythagorean comma κ . As a consequence we obtain:

- (1) Each note in Pyth is enharmonic to a unique note in Pyth-2.
- (2) Each note in Pyth is enharmonic to a unique note in Pyth-3.

We explain the second statement for a note of frequency $\varphi = 2^u \cdot 3^v$. Consider the interval $I = \{-9, -8, \dots, 8, 9\}$ of 19 consecutive integers. By the Division Theorem, there is a unique quotient m and a unique remainder $r \in I$ such that $u = 19 \cdot m + r$. The note of frequency ratio $\psi = \varphi \cdot \kappa^m$ is enharmonic with φ and has harmonic degree $\mu_2(\psi) = \mu_2(\varphi \cdot \kappa^m) = u - 19 \cdot m = r$ which is in the range from -9 to $+9$. Hence ψ is in Pyth-3, and it follows from the uniqueness of m that ψ is the unique note in Pyth-3 which is enharmonic with φ . This shows (2).

So the exponent 19 in the Pythagorean comma $\kappa = 3^{12}/2^{19}$ determines the number of consecutive integers which can occur as harmonic degrees in Pyth-3. Similarly, the exponent 12 determines the number of harmonic degrees in Pyth-2.

In [Appendix 2](#), we describe how the numbers 12 and 19 are found, together with other possible pairs.

2.3. The circle of fifths versus the circle of octaves

For each scale degree, we pick one note and assign it a letter name. Then any other note of the same scale degree is equivalent, with respect to the octave or the tritave, to the given note.

The set Pyth-2 is still infinite; the notes with harmonic degree $\mu_3 = h$, where $-5 \leq h \leq 6$, have the following frequency ratios:

$$\dots \quad 2^{-2} \cdot 3^h, \quad 2^{-1} \cdot 3^h, \quad 3^h, \quad 2 \cdot 3^h, \quad 2^2 \cdot 3^h, \quad \dots$$

For any two of them, the quotient is a power of two, so the notes differ by a certain number of octaves. We say they are *octave-equivalent*.

Fix an interval $I_2 = (a, b]$ of positive real numbers such that $b/a = 2$. We call this interval I_2 a *fundamental domain* for Pyth-2. For the 12 harmonic degrees in Pyth-2, we obtain 12 unique

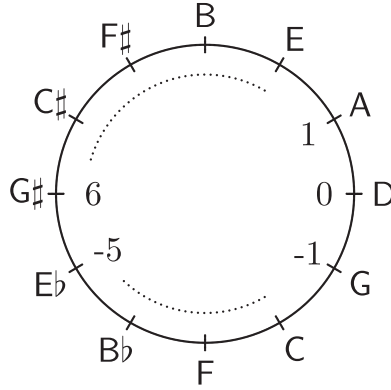


Figure 2. The Circle of Fifths in Pyth-2.

notes in I_2 ; we list the notes, their frequencies and degrees in Table A1 in the Appendix. In the circle of fifths, see Figure 2, the notes are arranged by harmonic degree.

We repeat this process for Pyth-3.

Consider the notes with harmonic degree $\mu_2 = h$, where $-9 \leq h \leq 9$, their frequencies are:

$$\dots \quad 2^h \cdot 3^{-2}, \quad 2^h \cdot 3^{-1}, \quad 2^h, \quad 2^h \cdot 3, \quad 2^h \cdot 3^2, \quad \dots$$

For any two of them, the quotient is a power of three, so the notes differ by a certain number of tritaves. We say they are *tritave-equivalent*.

Here we take as *fundamental domain for Pyth-3* an interval $I_3 = (a, b]$ of positive real numbers such that $b/a = 3$. Among the notes in the list, there is exactly one note with frequency in I_3 . For the 19 harmonic degrees in Pyth-3, we obtain 19 unique notes in I_3 . If we take in particular the interval

$$I_3 = (1/\sqrt{3}, \sqrt{3}],$$

we obtain the following notes:

- the 12 notes A, $\dots$, G# from Pyth-2 in the interval $I_2 = (\kappa/\sqrt{2}, \kappa\sqrt{2}]$
- the Ab
- the notes F, F#, G, which are one octave under F, F#, G, respectively, and
- the notes A', Bb', B' one octave above A, Bb, B, respectively.

In Table A2 in the Appendix we list with each note its frequency, its scale degree, its harmonic degree, and how much it deviates from equal intonation.

Again, up to rotation and tritave-equivalence, the choice of the interval does not matter, so it is best to arrange the notes in a circle, see Figure 3.

We obtain:

- (1) Each note in Pyth-2 is octave-equivalent to a unique note in the circle of fifths.
- (2) Each note in Pyth-3 is tritave-equivalent to a unique note in the circle of octaves.

2.4. Unique notation for each note in Pyth-3

Next we introduce unique notation for each note in Pyth-3. For comparison, recall that the 12 notes in Pyth-2 which lie in the fundamental domain $I_2 = (\kappa/\sqrt{2}, \kappa\sqrt{2}]$, sorted by scale degree,

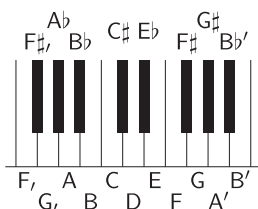


Figure 4. The Fundamental Domain for Pyth-3.

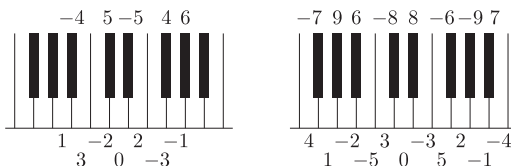


Figure 5. Keyboard with harmonic degrees: Pyth-2 (left) and Pyth-3.

Table 1. Scale degrees where Pyth-2 and Pyth-3 differ.

Scale degree	Where Pyth-2 and Pyth-3 differ in just intonation							
	-37	-30	-25	-18	-6	25	37	44
Note in Pyth-3	$E\flat^{\sim}$	$B\flat^{\sim}$	$A\flat^{\sim}$	$E\flat^{\sim}$	$A\flat$	$G\sharp^{\flat}$	$C\sharp^{\flat}$	$G\sharp^{\flat}$
Note in Pyth-2	$C\sharp^{\flat\prime\prime}$	$G\sharp^{\flat\prime\prime}$	$C\sharp^{\flat\prime\prime}$	$G\sharp^{\flat\prime\prime}$	$G\sharp^{\flat}$	$E\flat^{\prime\prime}$	$E\flat^{\prime\prime\prime}$	$B\flat^{\prime\prime\prime}$
Frequency quotient	$\frac{1}{\kappa}$	$\frac{1}{\kappa}$	$\frac{1}{\kappa}$	$\frac{1}{\kappa}$	$\frac{1}{\kappa}$	κ	κ	κ

Let us first consider the piano in just Pythagorean intonation. We have seen in Section 2.2 that each note in Pyth-2 is enharmonic to one in Pyth-3, and for most notes on the piano they are equal. They differ only for the eight notes listed in Table 1, and there, the difference is plus or minus one comma.

The first place where they differ is at scale degree -6 , here the note in Pyth-2 is the $G\sharp^{\flat}$ of frequency $2^{-10} \cdot 3^6$, while Pyth-3 has at this position the $A\flat$ of frequency $2^9 \cdot 3^{-6} = 2^{-10} \cdot 3^6/\kappa$ (in Pyth-2, we give preference to a note of harmonic degree $\mu_3 = 6$ over harmonic degree $\mu_3 = -6$, while in Pyth-3, we prefer harmonic degree $\mu_2 = 9$ over $\mu_2 = -10$). Table 1 lists all notes on the 88-key piano where Pyth-2 and Pyth-3 differ.

Next we consider the piano in equal temperament. The note at scale degree n is commonly tuned to frequency $2^{n/12}$. This system is called 12-EDO (equal division of the octave).

For equal intonation in a system of 19 notes per tritave, we simply assign to the note at scale degree n the ratio $3^{n/19}$. We call this system 19-EDT. We have seen in Table A2 that each note in Pyth-3 in just intonation differs by at most ± 11.11 cents from the corresponding note in 19-EDT.

Let us compare 19-EDT with 12-EDO. Note that the numbers $3^{1/19}$ and $2^{1/12}$ are very similar. Since

$$\left(\frac{3^{1/19}}{2^{1/12}}\right)^{12 \cdot 19} = \frac{3^{12}}{2^{19}} = \kappa,$$

the difference between $3^{1/19}$ and $2^{1/12}$ in cents is just $(1/(12 \cdot 19) = 1/228)$ times (κ in cents), that is, around $23.46/228 \approx 0.103$ cents.

We restrict this comparison to the notes on a piano with 88 keys. If the middle notes are in tune, then for any scale degree the note in 12-EDO differs by less than ± 5 cents from the note in 19-EDT of the same scale degree.

We have already seen that each note in 19-EDT differs by at most ± 11.11 cents from its corresponding note in Pyth-3.

In this sense, both 19-EDT and Pyth-3 in just intonation can be called “piano scales,” as in the title of this section. In particular, we will be using the usual music notation for all scales.

For the convenience of the reader, we attach in [Appendix 3](#) key labels for Pyth-3 for an 88-key piano keyboard to exhibit the notes with hats and check marks.

3. Visualizing harmony: the *Tonnetz*

First, we briefly revisit the *Tonnetz* for 4:5:6 harmony, then adapt it to our situation.

3.1. 4:5:6 harmony

The *Tonnetz* (German: tone-network) has been first described by Leonhard Euler, see in particular ([Euler 1774](#)), and then was used in 19th century musicology as the space in which harmonic development and key changes can be traced. It plays an important role in neo-Riemannian theory, see [Section 3.3](#).

The original *Tonnetz* is infinite in each direction in the plane. In horizontal direction, the notes progress as given by the circle of fifths, corresponding to the ratio 6:4 or 3:2. The horizontal step is divided into two parts given by the two small intervals in the chord: the major third (5:4), pictured as an upwards step, and the minor third (6:5), represented by a downwards step. We picture in [Figure 6](#) one fundamental tile. The *infinite Tonnetz* can be obtained by connecting different copies of the fundamental tile along parallel solid lines. The note labels need to be adjusted to reflect that there are perfect fifths in horizontal direction (so the note on the left of the E^b is to be labeled A^b , not $G^\sharp$) and major thirds along the upwards diagonals.

In neo-Riemannian theory it is common to use equal intonation (12-EDO) and to identify octave-equivalent notes. This yields the *finite Tonnetz* which, topologically, is best represented as the torus obtained by identifying parallel solid lines in the fundamental tile in [Figure 6](#).

3.2. The *Tonnetz* for 2:3:4 harmony

We introduce the *Tonnetz* for 2:3:4 harmony, more precisely, the finite 2:3:4 *Tonnetz* and the infinite 2:3:4 *Tonnetz*. In horizontal direction, the notes progress by octaves (ratio 4:2 or 2:1), each step is divided into an upwards move given by the fifth (3:2) and a downwards move given

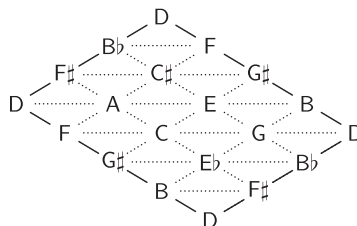
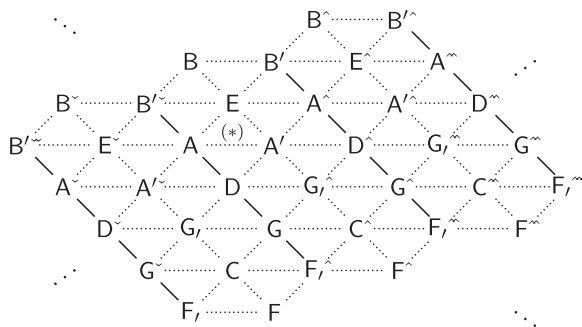
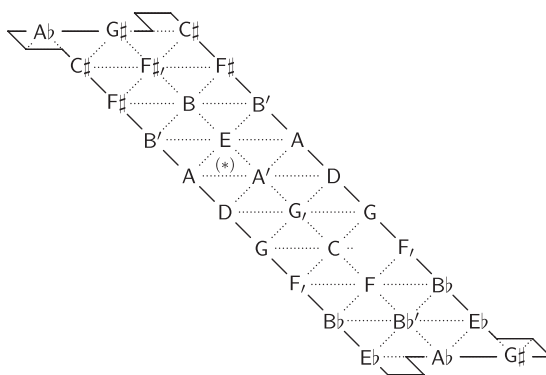


Figure 6. *Tonnetz* for 4:5:6 harmony (one fundamental tile).

Figure 7. Part of the infinite 2:3:4 *Tonnetz*.Figure 8. *Tonnetz* for 2:3:4 harmony (one fundamental tile).

by the fourth (4:3). In Figure 7, we picture the notes around the A-E-A' chord which we label by an (*).

The *infinite 2:3:4 Tonnetz* is obtained by placing copies of the fundamental tile in Figure 8 next to each other, as indicated by the solid lines. Hooks are used to indicate how the solid lines are to be shifted against each other. In horizontal direction, hats and checkmarks need to be added so that in each step, the frequency increases by an octave, as in Figure 7. In vertical direction, the notes and their labels need to be adjusted by Pythagorean commas; for example, the top left note in the fundamental tile should be a G $\sharp$, not an A $\flat$, and the bottom right note an A $\flat$, not a G $\sharp$.

For the *finite 2:3:4 Tonnetz*, we use equal intonation (19-EDT) and identify tritave-equivalent notes. Topologically, the *Tonnetz* is best represented as the torus obtained by identifying opposite sides of the fundamental tile in Figure 8.

Major and minor chords. One can define *major* and *minor* chords for 2:3:4 harmony as chords where the fifth is followed by a fourth, or the fourth followed by a fifth, respectively. So the 2:3:4 chord A-E-A' is a major chord, and the 3:4:6 chord A-D-A' a minor chord. Similarly, an *augmented* or *diminished* chord consists of two fifths or two fourths, respectively.

There is an important difference for major and minor 2:3:4 chords, when compared with 4:5:6 harmony: the first inversion of a major chord turns out to be a minor chord. For example, the first inversion of A-E-A' is the minor chord E-A'-A, pictured on the right hand side of (*) in the *Tonnetz*.

We will discuss this further in Section 4.2.

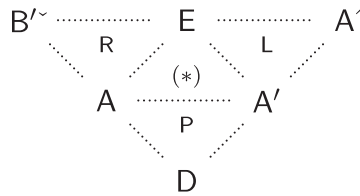


Figure 9. PLR-moves in the infinite 2:3:4 *Tonnetz*.

3.3. Neo-Riemannian theory for 2:3:4 chords

In neo-Riemannian theory, chord progressions are analyzed in terms of elementary moves or *transformations* in the *Tonnetz*. For example, the process of going from tonic to dominant (which differs from the tonic chord in two notes) is decomposed as a product of two elementary moves of which each changes only one note.

The three elementary moves considered are the P-, L-, and R-transformations; they map a major or minor triad to the minor or major triad adjacent to one of the edges of the triangle representing the chord in the *Tonnetz*. See (Cohn 1998).

In this section we describe the three transformations when adapted to 2:3:4 harmony. Here we need to work in the infinite 2:3:4 *Tonnetz* since major and minor triads are not invariant under first and second inversions modulo the tritave.

The P-transformation (parallel major or minor) maps a major triad to the minor triad with which it has the octave in common, and conversely. In the example, the major A-E-A' chord and the minor A-D-A' chord correspond to each other under P-moves, see Figure 9.

Note that A-D-A' is the dominant chord for A-E-A' so a single move in 2:3:4 harmony suffices to go from tonic to dominant, while two moves (R and L) are needed in 4:5:6 harmony. Also note that the middle notes of the two chords differ by two semitones, while in the 4:5:6 system, a P-transformation moves the middle note up or down by one semitone.

Under the R-transformation (relative major or minor), a major and a minor chord correspond to each other whenever they have the fifth in common. In the example, an R-move maps the A-E-A' chord to B'~A-E.

Note that the R-transformation for (*) is the second inversion of the subdominant chord A-E-B'. In 2:3:4 harmony, it moves the last note of the major chord up by two semitones, up to a tritave; this is similar to 4:5:6 harmony.

Finally, the L-transformation (lead tone exchange) maps a given 2:3:4 major or minor chord to the one with which it shares the fourth. This results in an inversion of the chord. By comparison, in 4:5:6 harmony, the lead tone of a major chord is replaced by a note one semitone lower.

Voice-leading parsimony. An important application of neo-Riemannian theory is to voice leading since the PLR-moves provide a measure for harmonic distance between chords.

In this paragraph we suppose that the harmony is given by a sequence of triads which are to support a melody. We ask: which notes may possibly occur in the melody provided that changes in the harmony are limited to a small number of PLR-moves?

We compare how many notes can be reached using a given number of PLR-moves in 4:5:6 harmony with the corresponding numbers for 2:3:4 harmony. Table 2 shows that in 4:5:6 harmony, with one exception (the F# if C-E-G is the tonic), any note is part of a triad that can be reached from the tonic in at most two moves.

By comparison, in 2:3:4 harmony, only seven notes of 19 can be reached with up to two moves, up to tritave-equivalence, see Table 3. The notes which can be reached from the A-E-A' chord and the corresponding PLR-moves are pictured in Figure 10.

Table 2. Notes in the 4:5:6 system reachable by PLR-moves.

Number of transformations	0	1	2	3
Notes reachable up to octave-equivalence	3	6	11	12

Table 3. Notes in the 2:3:4 system that can be reached by PLR-moves.

Number of transformations	0	1	2	3	4	5	6	7	8
Notes reachable up to tritave-equiv.	3	5	7	9	11	13	15	17	19

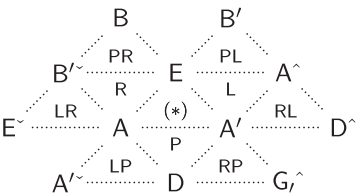


Figure 10. Notes in the 2:3:4 system reachable with up to two PLR-moves.

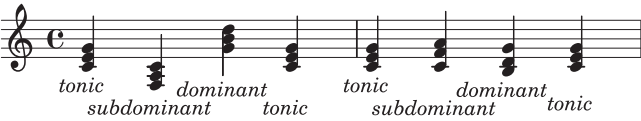


Figure 11. The basic sequence in 4:5:6 harmony.

In summary, the PLR-moves of neo-Riemannian theory can be adapted to 2:3:4 harmony. They mimic closely composition elements related to dominants and subdominants, inversions, and major and minor chords. When compared to 4:5:6 harmony, fewer notes can be reached with a given number of PLR-moves.

4. Elements of composition for 2:3:4 harmony

We study basic elements of composition, which we adapt to a tonal system that is modulated with respect to the tritave. We discuss perception of 2:3:4 harmony in the concluding Section 4.4 on purity and sparsity.

4.1. Tonic, dominant and subdominant

We revisit the sequence tonic to subdominant to dominant to tonic in 4:5:6 harmony. Subdominant and dominant of a triad are obtained by replacing each note by its predecessor or successor, respectively, in the cycle of fifths. Starting from the major chord C-E-G, we obtain in Figure 11 the following sequence, which we repeat in the second bar using first and second inversions of the triads modulo the octave.

We recreate this sequence in the 2:3:4 system. Here, subdominant and dominant of a chord are given by substituting each note by its predecessor or successor, respectively, in the cycle of octaves. Taking as tonic the A-E-A' chord, we obtain in Figure 12 the following sequence. In the second bar, we use inversions modulo the tritave to bring the triads into one fundamental domain.

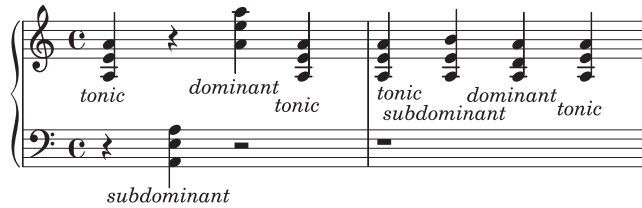


Figure 12. The basic sequence in 2:3:4 harmony.

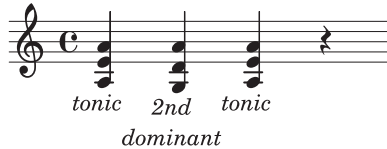


Figure 13. Creating a sense of finality in 2:3:4 harmony.

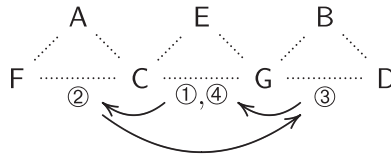


Figure 14. The basic sequence in 4:5:6 harmony.

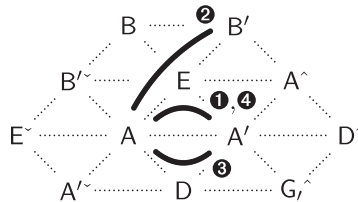


Figure 15. The basic sequence in the infinite 2:3:4 Tonnetz.

Cadences in 2:3:4 harmony. It appears to the author that the step from dominant to tonic in the above basic sequence in 2:3:4 harmony can be perceived as a cadence, much like the corresponding step in 4:5:6 harmony. A perhaps stronger sense of finality can be obtained by using the step from the second dominant back to tonic, as in the example in Figure 13.

Dominant and subdominant in the Tonnetz. The basic sequence in 4:5:6 harmony

$$\text{C-E-G (①)} - \text{C-F-A (②)} - \text{B-D-G (③)} - \text{C-E-G (④)}$$

appears as follows in the *Tonnetz* in Figure 14:

Note that the 4:5:6 *Tonnetz* does not distinguish octave-equivalent, but different, notes. Hence the notes in the C-F-A-triad (②) appear as F-A-C.

Here is the basic sequence for 2:3:4 harmony, before applying tritave-equivalence:

$$\text{A-E-A' (①)} - \text{E'-B'-A (②)} - \text{A'-A'-D (③)} - \text{A-E-A' (④)}$$

We picture the sequence in the infinite 2:3:4 *Tonnetz*. Note that the pitch increases in the direction of the arc.

An example. Here are bars 3 and 4 from our sample piece which is reprinted in [Appendix 4](#). The right hand plays the four chords from the basic sequence while the left hand emphasizes the tritave-based structure by repeating the upper two notes of each chord one tritave lower. The voice in the top line consists of notes taken from those chords.

4.2. Inversions

The term *inversion* refers to first or second inversion in which the bottom note of a chord is substituted by a new top note, or conversely (here, the term inversion does not refer to mirror inversion). In octave-based harmony, the two notes differ by an octave, so we speak of inversions modulo the octave. In the 2:3:4 system, we use inversions modulo the tritave.

Major and minor chords. We observe that the role of major and minor chords is quite different in 2:3:4 harmony, when compared to 4:5:6 harmony.

There are three intervals which may occur in a — possibly inverted — major or minor triad in 4:5:6 harmony: a major third (④), a minor third (③) or a fourth (⑤).

There are six possibilities to pick two different intervals from the above. Up to inversion, there are two types:

$$\textcircled{4} + \textcircled{3}(\text{major}) - \textcircled{3} + \textcircled{5}(\text{1st inv. major}) - \textcircled{5} + \textcircled{4}(\text{2nd inv. major})$$

or

$$\textcircled{3} + \textcircled{4}(\text{minor}) - \textcircled{4} + \textcircled{5}(\text{1st inv. minor}) - \textcircled{5} + \textcircled{3}(\text{2nd inv. minor}).$$

Tritave-based 2:3:4 harmony is different. The 2:3:4 chord consists of a fifth (⑦) and a fourth (⑤), but to complement the chord to a tritave, another fifth (⑦) is needed.

Hence there are only three possibilities to pick two intervals from the above:

$$\textcircled{7} + \textcircled{5}(\text{tonic}) - \textcircled{5} + \textcircled{7}(\text{1st inv.}) - \textcircled{7} + \textcircled{7}(\text{2nd inv.}).$$

Note that the first inversion appears as a minor chord, while the second inversion is an augmented chord.

A remark regarding perception. Octave-based 4:5:6 harmony appears to be richer as there are two modes (major and minor) and three inversions; by comparison in tritave-based 2:3:4 harmony, there is only one mode with two inversions, or — looking at it differently — three modes (major, minor, augmented) but no inversions. It would be interesting to find out if a listener perceives the three chords ⑦ + ⑤, ⑤ + ⑦, and ⑦ + ⑦ as being inversions of the same mode, or rather as representing three different modes.

Acoustically, that is, when considering the overtone sequences of notes, some evidence seems to support the first interpretation (one mode with two inversions). Consider two numbers associated to a chord: one is the distance d_B from a base note for which all notes in the chord are overtones. For the 2:3:4 chord A-E-A', the base note is an octave below the A, which is the A, or E, so $d_B = 2$ as in “2:3:4.”

The second measure is the distance d_O to the first common overtone for all the notes in the chord. For this, we rewrite the chord using reciprocals with simplified denominators: the 2:3:4 chord is the $\frac{1}{6} : \frac{1}{4} : \frac{1}{3}$ chord. From this notation we see that the first common overtone of the A-E-A' chord is the A' or E'', which is one tritave above the A', hence $d_O = 3$ as in “ $\frac{1}{3}$.”

In the table in [Appendix 5](#) we list for each chord both numbers. It turns out that for major 4:5:6 chords, d_O is large and d_B is small; for minor 4:5:6 chords, d_O is small and d_B is large. But for each of the three 2:3:4 chords (⑦ + ⑤, ⑤ + ⑦, and ⑦ + ⑦), both numbers are small.



Figure 16. Bars 3 and 4 from the Ave.

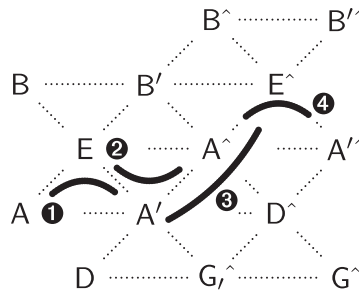


Figure 17. A 2:3:4 chord with three successive inversions.



Figure 18. Bars 15 and 16 from the Ave.

Inversions in the Tonnetz. In 4:5:6 harmony, inversions are difficult to picture in the *Tonnetz*: in the finite *Tonnetz*, octave-equivalent notes are identified, hence inversions modulo the octave are invisible. In the infinite 4:5:6 *Tonnetz*, it is not clear which shift to use for the octave (for example, 3 major thirds or 4 minor thirds?), but in each case, the inverted chord is no longer contiguous.

By comparison, inversions are faithfully represented in the infinite 2:3:4 *Tonnetz*. Starting from the major A-E-A' chord (❶) in Figure 17, the first inversion is the minor E-A'-A^ chord (❷). Inverting again leads to the augmented chord A'-A^E' (❸); a third inversion brings us to the A^E'-A'^ chord (❹), a tritave higher than the chord we started with.

We briefly discuss the role of inversions in our sample piece.

Note that throughout the score of the Ave, the pitch increases by one tritave. We compare the four chords in bars 3-4 in Figure 16 played by the right hand with those in bars 9-10, in bars 15-16, and in bars 21-22.

The chords in bars 3-4 are the basic sequence, shown in Figure 15. Each of those chords is played in first inversion in the corresponding chord in bars 9-10. Another first inversion yields the chord sequence in bars 15-16 (see Figure 18 for the score); finally, the basic sequence is played one tritave higher in bars 21-22 (in variation). The first chords in each pair of bars are pictured in Figure 17.

Thus the sequence of first inversions in bars 9-10, and the sequence of second inversions in bars 15-16 divide the Ave into three parts. They are intended to correspond to the three parts of the lyrics (greeting; adoration; prayer request).

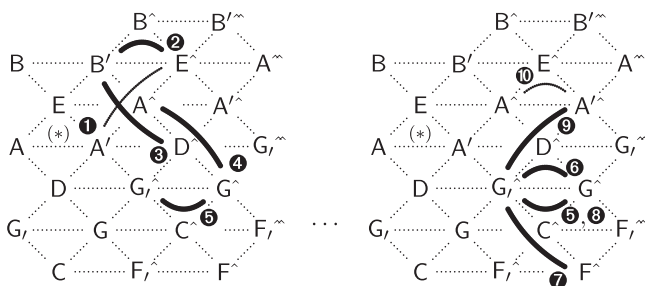
Figure 19. *Tonnetz* for Ave, bars 16b – 21a.

Figure 20. Bars 17 to 20 from the Ave.

4.3. Diminished chords

In our sample piece, we use diminished chords to move from a given chord to its inversions in a variety of ways. As mentioned in the previous section, the inversions of the chords in the basic sequence divide the Ave into three parts. In this subsection, we discuss the third part.

The first and the last chord in the basic sequence is the A-E-A', so the third part of the Ave lies between the second inversion of this chord, which is the augmented chord A'-A'-E' (❶) in bars 15a and 16b, see Figure 18, and the chord one tritave higher, A'-E'-A'^ (❷) in bars 21a and 22b.

Figure 19 shows the *Tonnetz* for the third part of the Ave; in Figure 20 we reprint the corresponding bars 17 - 20.

In bars 17 - 18, we start from the major chord (❷), then move to the diminished chord B'-A'-D' (❸), push the diminished chord up to A'-D'-G' (❹) and resolve to minor G'-C'-G' (❺). The S-shaped figure ❷-❺ contains the notes E', D', C'; hence the notes played by the left hand support the melody E-D-C one tritave lower.

The third part of the Ave concludes with bars 19 - 20.

The minor chord (❺) is followed by its subdominant (❻), then the harmony descends in the *Tonnetz* to the diminished triad (❼), returns to the minor (❸), moves up to its second subdominant (❾), and continues the upwards movement to the second subdominant for the previous chord, which is the final major chord A'-E'-A'^ (❷).

Throughout this part, the melody line has W-shape (E-D-C-D-C-D-E) which is reflected in the up-and-down movement of the harmony in the *Tonnetz* in Figure 19.

4.4. Purity and sparsity

When asking listeners about their impression of the *Ave Maria in dix-neuf par duodecime*, the piece has been characterized as “reminding of Gregorian chants,” as “sounding Celtic,” and as having a sound that is “pure and sparse.” Hence this section on purity and sparsity.

Purity. For a chord, two measures for purity come to mind; the numbers d_O and d_B from Section 4.2 measure the distance between the highest note in the chord and the first common overtone, and the distance between the lowest note and a base note for which all notes in the chord are overtones, respectively.

In Appendix 5 we compute both measures for the 4:5:6 system and for 2:3:4 harmony. Major chords in both systems stand out for their small distance from the base note, while minor chords are very close to the first common overtone. Also in both systems, augmented chords do better in both measures than diminished chords.

Overall, all chords are much purer with respect to both measures in 2:3:4 harmony than in the 4:5:6 system. On the other hand, as we have discussed in Section 4.2, there may be less variety in 2:3:4 harmony as the distinction between major and minor chords is less pronounced.

Sparsity. It is possible that the impression of sparsity arises since the intervals 5:4 and 6:5 in 4:5:6 harmony are missing in the 2:3:4 system (because any frequency ratio between two notes has the form $2^u \cdot 3^v$, so no prime factor 5 occurs). With the major and minor third missing, chords are spaced further apart and hence may sound “thinner.”

The other reason for sparsity is that for a piece in 2:3:4 harmony it is necessary to emphasize the tritave to create the proper frame. Throughout the *Ave* for example, the left hand reproduces the top two notes of the right hand, but one tritave lower. Clearly, by playing notes in parallel, the density of overtones is reduced, which the listener may perceive as pure yet sparse.

Further comments regarding perception.

- We have seen in Section 3.3 that harmonic development along fundamental moves in the *Tonnetz* limits the voice to fewer options than in the 4:5:6 system, perhaps adding to the piece being perceived as pure yet sparse.
- The author would find it interesting to study perception of chord sequences in 2:3:4 harmony in a project similar to the research in (Walker 2001) for the Bohlen-Pierce scale.
- Since the 2:3:4 chord consists of a fourth, a fifth, and an octave, we cannot omit a quote from the text book (Loy 2006, Section 3.8.2) with which we conclude this section.

The interval of the third in the Pythagorean scale was considered a dissonance in the Middle Ages, and as a result compositions would typically omit the third in the final chord of a composition so as to end only with perfect intervals — fourths, fifths, and octaves — an effect that sounds hollow to modern ears.

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Thanks to Elaine Walker for her guidance and advice, in particular regarding the relevance and the role of the frame given by the octave or the tritave. The author is grateful for many helpful discussions with Feruza Dadabaeva, his piano teacher. He would like to thank Co-Editors-in-Chief Emmanuel Amiot and Jason Yust and the referees, for comments which have led to substantial improvements of the manuscript.

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Appendices

Appendix 1. Comparison of Pythagorean vs. equally tempered scales

For the Pythagorean scales Pyth-2 and Pyth-3, we list in Tables A1 and A2 for each note the scale degree, the harmonic degree, the frequencies associated in just and in equal intonation, and their quotient, measured in cents.

Both scales satisfy the symmetry condition in (Carey and Clampitt 1989) and hence can be considered well-formed. In (Carey and Clampitt 1989, Table 2), Pyth-2 is the chromatic scale with group $\mathbb{Z}_N = \mathbb{Z}_{12}$ in which the fifth has scale degree $b = 7$. The scale Pyth-3 is not listed there explicitly (since the scale generator is an octave, not a fifth); it is based on the group $\mathbb{Z}_N = \mathbb{Z}_{19}$ in which the octave, which is the scale generator, has degree $b = 12$. Hence in Table A2, the scale degree is obtained from the harmonic degree by multiplication by $b = 12$, for example harmonic degrees $0, 1, 2, \dots$ correspond to scale degrees $0, 12 \equiv -7 \pmod{19}, 24 \equiv 5 \pmod{19}, \dots$. Conversely, since $b^{-1} \equiv 8 \pmod{19}$, scale degrees $0, 1, 2, \dots$ correspond to harmonic degrees $0, 8, 16 \equiv -3 \pmod{19}, \dots$

Appendix 2. On the number of notes per tritave

What is the best number of notes in a scale? Suppose our system will have q notes per tritave, and the octave will be at the p th note, $0 < p < q$. Then we have the approximation

$$2^q \approx 3^p$$

or, equivalently,

$$q \log(2) \approx p \log(3), \quad \text{or} \quad \frac{\log(2)}{\log(3)} \approx \frac{p}{q}.$$

We need to approximate $\log(2)/\log(3)$ by a rational number, this can be done using continued fractions, see (Benson 2007, Section 6.2), (Krantz and Douthett 2011) and, for an application to the Bohlen-Pierce scale, (Advocat 2010, Section 2.6).

$$\frac{\log(2)}{\log(3)} = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{5 + \dots}}}}}}}} \approx 0.630930$$

Table A1. Two octave-based scales: Pyth-2 vs. 12-EDO.

Pythagorean vs. 12 tone equal tuning					
Scale degree n		Pyth-2 tuning $\pi(n)$	Harm. degree $\mu_3(\pi(n))$	12-EDO tuning $2^{n/12}$	$\pi(n)/2^{n/12}$ in cents
(-6)	A $\flat$	$\frac{512}{729} = 2^9 \cdot 3^{-6}$	-6	$2^{-6/12} \approx 0.7071$	-11.73
-5	A	$\frac{3}{4} = 2^{-2} \cdot 3^1$	1	$2^{-5/12} \approx 0.7492$	+1.96
-4	B $\flat$	$\frac{64}{81} = 2^6 \cdot 3^{-4}$	-4	$2^{-4/12} \approx 0.7937$	-7.82
-3	B	$\frac{27}{32} = 2^{-5} \cdot 3^3$	3	$2^{-3/12} \approx 0.8409$	+5.87
-2	C	$\frac{8}{9} = 2^3 \cdot 3^{-2}$	-2	$2^{-2/12} \approx 0.8909$	-3.91
-1	C $\sharp$	$\frac{243}{256} = 2^{-8} \cdot 3^5$	5	$2^{-1/12} \approx 0.9439$	+9.78
0	D	$1 = 2^0 \cdot 3^0$	0	$2^0 \approx 1.0000$	± 0
1	E $\flat$	$\frac{256}{243} = 2^8 \cdot 3^{-5}$	-5	$2^{1/12} \approx 1.0595$	-9.78
2	E	$\frac{9}{8} = 2^{-3} \cdot 3^2$	2	$2^{2/12} \approx 1.1225$	+3.91
3	F	$\frac{32}{27} = 2^5 \cdot 3^{-3}$	-3	$2^{3/12} \approx 1.1892$	-5.87
4	F $\sharp$	$\frac{81}{64} = 2^{-6} \cdot 3^4$	4	$2^{4/12} \approx 1.2599$	+7.82
5	G	$\frac{4}{3} = 2^2 \cdot 3^{-1}$	-1	$2^{5/12} \approx 1.3348$	-1.96
6	G $\sharp$	$\frac{729}{512} = 2^{-9} \cdot 3^6$	6	$2^{6/12} \approx 1.4142$	+11.73

Using the first five terms (omitting the summand after the second number 2) yields the approximation

$$\frac{\log(2)}{\log(3)} \approx \frac{12}{19}.$$

It is not a surprise that this is a good approximation since the number $3^{12}/2^{19}$, being the Pythagorean comma κ , is very close to one. It gives rise to a scale of just intonation of 19 notes per tritave with the octave at scale degree 12. This is the scale Pyth-3 studied in Section 1.

Incidentally, the approximation at degree 7 (given by omitting the fraction containing the 5) is an excellent one: $\log(2)/\log(3) \approx 53/84$. It yields a scale of 84 notes of which the octave is at scale degree 53. This scale corresponds to the octave-based scale of 53 notes generated by the fifth at scale degree 31 (Loy 2006, 3.14.1).

Appendix 3. Key labels for the 88-key piano keyboard

In Figures A1 and A2 we present the key assignment for an 88-key piano keyboard. Either equal intonation (12-EDO) or just intonation (Pyth-2) can be used to play 19 notes per tritave, see Section 2.5.

In our discussion of the equal tempered scale 19-EDT we mentioned already that, in principle, the scale can be played on the piano with keyboard and tuning as usual. Yet the keyboard in Figure A3 better reflects harmonic properties of the scale: in particular, the periodicity given by the tritave and the assignment of notes with lower harmonic degrees to white keys. Since the harmonic degree μ_2 is invariant under the shift by a tritave, the white keys all have notes with harmonic degrees between -5 and 5. For each note symbol, the harmonic degree is as indicated in Figure 5.

Table A2. Two tritave-based scales: Pyth-3 and 19-EDT.

Pythagorean vs. 19 tones equal in the tritave					
Scale degree n		Pyth-3 tuning $\pi_3(n)$	Harm. degree $\mu_2(\pi_3(n))$	19-EDT tuning $3^{n/19}$	$\pi_3(n)/3^{n/19}$ in cents
(−10)	E $_7$	$\frac{9}{16} = 2^{-4} \cdot 3^2$	−4	$3^{-10/19} = 0.5609$	+4.94
−9	F $_7$	$\frac{16}{27} = 2^4 \cdot 3^{-3}$	4	$3^{-9/19} = 0.5943$	−4.94
−8	F $\sharp_7$	$\frac{81}{128} = 2^{-7} \cdot 3^4$	−7	$3^{-8/19} = 0.6297$	+8.64
−7	G $_7$	$\frac{2}{3} = 2^1 \cdot 3^{-1}$	1	$3^{-7/19} = 0.6671$	−1.23
−6	A $\flat$	$\frac{512}{729} = 2^9 \cdot 3^{-6}$	9	$3^{-6/19} = 0.7069$	−11.11
−5	A	$\frac{3}{4} = 2^{-2} \cdot 3^1$	−2	$3^{-5/19} = 0.7489$	+2.47
−4	B $\flat$	$\frac{64}{81} = 2^6 \cdot 3^{-4}$	6	$3^{-4/19} = 0.7935$	−7.41
−3	B	$\frac{27}{32} = 2^{-5} \cdot 3^3$	−5	$3^{-3/19} = 0.8408$	+6.17
−2	C	$\frac{8}{9} = 2^3 \cdot 3^{-2}$	3	$3^{-2/19} = 0.8908$	−3.70
−1	C $\sharp$	$\frac{243}{256} = 2^{-8} \cdot 3^5$	−8	$3^{-1/19} = 0.9438$	+9.88
0	D	$1 = 2^0 \cdot 3^0$	0	$3^0 = 1.0000$	± 0
1	E $\flat$	$\frac{256}{243} = 2^8 \cdot 3^{-5}$	8	$3^{1/19} = 1.0595$	−9.88
2	E	$\frac{9}{8} = 2^{-3} \cdot 3^2$	−3	$3^{2/19} = 1.1226$	+3.70
3	F	$\frac{32}{27} = 2^5 \cdot 3^{-3}$	5	$3^{3/19} = 1.1894$	−6.17
4	F $\sharp$	$\frac{81}{64} = 2^{-6} \cdot 3^4$	−6	$3^{4/19} = 1.2602$	+7.41
5	G	$\frac{4}{3} = 2^2 \cdot 3^{-1}$	2	$3^{5/19} = 1.3352$	−2.47
6	G $\sharp$	$\frac{729}{512} = 2^{-9} \cdot 3^6$	−9	$3^{6/19} = 1.4147$	+11.11
7	A $'$	$\frac{3}{2} = 2^{-1} \cdot 3^1$	−1	$3^{7/19} = 1.4989$	+1.24
8	B $\flat'$	$\frac{128}{81} = 2^7 \cdot 3^{-4}$	7	$3^{8/19} = 1.5882$	−8.64
9	B $'$	$\frac{27}{16} = 2^{-4} \cdot 3^3$	−4	$3^{9/19} = 1.6827$	+4.94
(10)	C $'$	$\frac{16}{9} = 2^4 \cdot 3^{-2}$	4	$3^{10/19} = 1.7829$	−4.94

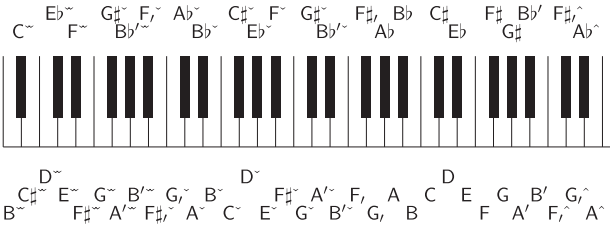


Figure A1. Key labels for the 88-key keyboard (left part).

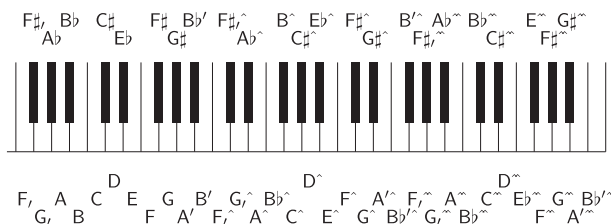


Figure A2. Key labels for the 88-key keyboard (right part).

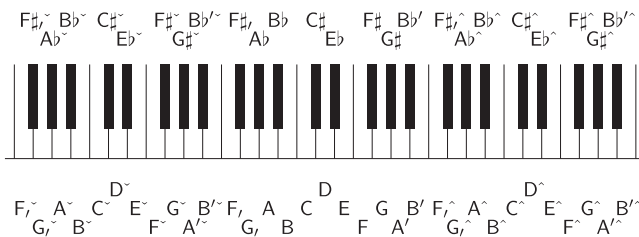


Figure A3. Ideal Keyboard for Pyth-3 or 19-EDT.

Appendix 4. The score of the Ave Maria in dix-neuf par duodecime

For a recording please visit <https://youtu.be/Bg1n4jM1n5w>.

Andante.

Voix

Piano

The musical score is written for Voice and Piano. The Voice part is in treble clef with a common time signature (C). The Piano part is in bass clef with a common time signature (C). The score consists of two measures. The first measure has a whole note for the Voice and a half note for the Piano. The second measure has a whole note for the Voice and a half note for the Piano. The Piano part includes markings for 'Andante', 'Ped.', and 'simile'.

3
A - ve, a - ve Ma - ri - i - a,
gra - ti - a ple - na, do - mi - nus te - cum, te - cum. Be - ne -
8
dic - ta tu in mu - li - e - ri - bus,
11
et be - ne - dic - tus fruc - tus ven - tris tu - i, Je - sus, Je - sus. Sanc - ta Ma -
14
ri - a ma - ter de - i, o - ra pro no - o - bis,

17

O - ra pro no - bis, no-bis pe- ca- to- ri- bus, nunc, nunc,

et in ho- ra mor- tis nos- trae, a - men. A - a - a - men.

rubato

Appendix 5. Two tables about chord purity

We refer to Sections 4.2 and 4.4 for the two measures of purity exhibited for the chords in the following two tables.

The *common base note* of a chord is the highest note such that each note in the chord is an overtone; its *distance* d_B from the lowest note in the chord is given by the first entry in the integer frequency ratios. The *common over tone* of a chord is the lowest note which occurs as an overtone for each note in the chord. Its *distance* d_O from the highest note in the chord can be read off from the last entry in the reciprocal frequency ratios.

Table A3. Two measures of purity for 2:3:4 chords.

Proximity of common base note and overtone for 2:3:4 chords					
Type of chord	Example	Frequency ratios		Base note, distance d_B	Over- tone, distance d_O
Major	A-E-A'	2:3:4	$\frac{1}{6} : \frac{1}{4} : \frac{1}{3}$	E' = A _r , 2	A'^ = E'', 3
Minor	A-D-A'	3:4:6	$\frac{1}{4} : \frac{1}{3} : \frac{1}{2}$	A' = D _r , 3	D' = A'', 2
Augmented	A-E-B'	4:6:9	$\frac{1}{9} : \frac{1}{6} : \frac{1}{4}$	B' = A _r , 4	A'' = B''', 4
Diminished	A-D-G	9:12:16	$\frac{1}{16} : \frac{1}{12} : \frac{1}{9}$	A'' = G _r , 9	G' = A''', 9

Table A4. Two measures of purity for 4:5:6 chords.

Proximity of base note and overtone for 4:5:6 chords					
Type of chord	Example	Frequency ratios		Base- note, distance d_B	Over- tone, distance d_O
Major	C-E-G	4:5:6	$\frac{1}{15} : \frac{1}{12} : \frac{1}{10}$	$C_{//}, 4$	$B'''' , 10$
Major, 1st inv.	E-G-C'	5:6:8	$\frac{1}{24} : \frac{1}{20} : \frac{1}{15}$	$C_{//}, 5$	$B'''' , 15$
Major, 2nd inv.	G-C'-E'	3:4:5	$\frac{1}{20} : \frac{1}{15} : \frac{1}{12}$	$C_{/}, 3$	$B'''' , 12$
Minor	A-C-E	10:12:15	$\frac{1}{6} : \frac{1}{5} : \frac{1}{4}$	$F_{////}, 10$	$E'' , 4$
Minor, 1st inv.	C-E-A'	12:15:20	$\frac{1}{5} : \frac{1}{4} : \frac{1}{3}$	$F_{////}, 12$	$E'' , 3$
Minor, 2nd inv.	E-A'-C'	15:20:24	$\frac{1}{8} : \frac{1}{6} : \frac{1}{5}$	$F_{////}, 15$	$E''' , 5$
Augmented	C-E-G $\sharp$	16:20:25	$\frac{1}{25} : \frac{1}{20} : \frac{1}{16}$	$C_{////}, 16$	$G\sharp'''' , 16$
Diminished	B-D-F	25:30:36	$\frac{1}{36} : \frac{1}{30} : \frac{1}{25}$	$E\flat_{////}, 25$	$C\sharp'''' , 25$

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