

# The Three Lunar Models of Ptolemy

by

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## 1. Introduction

In this paper Ptolemy's theory for the motion of the moon (Almagest IV, 1 to V,9) is expounded. This has previously been done, e.g. by P. Kempf p. 1-35 and O. Neugebauer p. 192-198, 210-214, but contrary to these I shall try to give an examination of the theory of the moon, based on modern results. On the other side, in this work, no attempt has been made to check the internal calculations made by Ptolemy, e.g. trigonometrical calculations and conversion of true solar time into mean solar time. Making such a check would be interesting, but this exceeds the limits of this paper.

The sections 2-7 contain a rather detailed explanation of the theory of the moon, and they may not give much new information to the reader, who is beforehand familiar with the nature of Ptolemy's three geometrical models for description of the motion of the moon. This explanation has been included for the sake of completeness, it is more systematical and it is written in a more modern mathematical language than usually done. The greater part of the investigations relating to modern results are in the sections 3, 8, and 9, and the main result is a proof of the fact that in reality Ptolemy has been able to describe the motion of the moon by means of a complicated model 3 which is superior to not only the simple model 1 handed over by Hipparchus, but also to his own model 2. The calculations made by a computer in section 9 have been done with skilled assistance of stud. scient. Gert Jacobsen and stud. scient. Jens Thyge Kristensen, both University of Copenhagen.

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## Remarks about the notations

The decimal point is used to separate the integral part of a number from the decimal part, whereas semicolons and commas are used when writing a number in the sexagesimal notation.

Whenever an angle is mentioned it is reduced modulo  $360^\circ$ , so that its value is in the principal interval,  $0^\circ \leq a < 360^\circ$ .

Where modern values are to be compared with values from the Almagest, the modern values are marked with a star, \*.

It is a wellknown fact that any calculation involving triangles can be made by means of the Pythagorean theorem and Ptolemy's table of chords. Whenever Ptolemy makes such a calculation it will not be expounded in details here.

The models for the motion of the moon will be given in the following coordinate system: The earth is at the origin  $O$ , the directions of the  $x_1$ -axis and the  $x_2$ -axis are the directions from the earth to the vernal equinox and from the earth to the summer solstice. The angle, which a vector  $\vec{AB}$  makes with the  $x_1$ -axis, is called  $\text{Arg } \vec{AB}$ , the length  $|\vec{AB}|$  of the vector is often called  $AB$ .

## 2. The Epicycle Model for the Description of the Motion of the Moon

When Ptolemy sets up a model for the description of the motion of the moon, it is beforehand certain that only uniform circular motions will be used. One might think that the motion of the moon can be described by an epicycle model, analogous with the one used for the sun. However, from two fundamental facts it appears that this is not the case. These facts have been discovered by steady observations of the moon, made by "the old ones" (i.e. the astronomers before Hipparchus), and mentioned by Ptolemy at the beginning of Almagest IV,2:

1. The moon moves irregularly in such a way that in the course of time it has assumed its maximal velocity for every value of the longitude  $\lambda$ .
2. The moon moves not only in the ecliptic, but has latitudes between approximately  $+5^\circ$  and  $-5^\circ$ , and in such a way that in the course of time it has assumed its maximal latitude for every value of  $\lambda$ .

If the motion of the moon is described by an epicycle model the following facts are valid. The epicycle centre  $C$  moves on the *deferent* and represents the mean position of the moon, so that  $\text{Arg } \vec{OC} = \lambda_m =$  the

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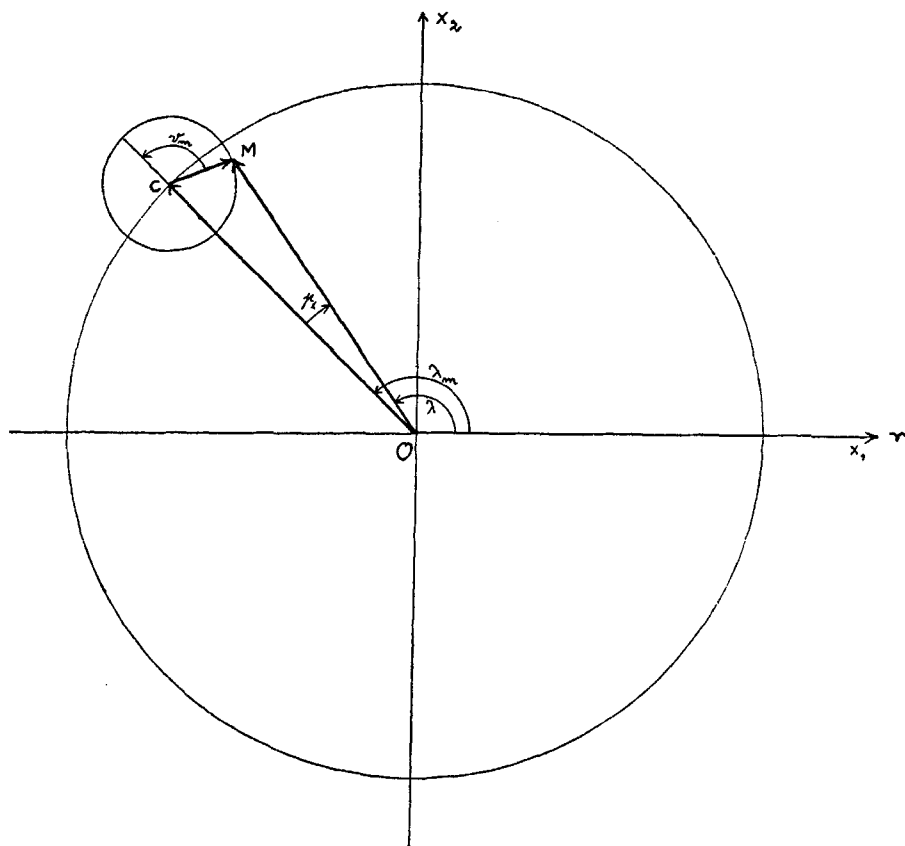


Fig. 1

*mean longitude*. Consequently, the vector  $\vec{OC}$  turns with a constant angular velocity,  $\dot{\lambda}_m$ , relating to the  $x_1$ -axis. The moon  $M$  moves on the *epicycle*. Ptolemy calls  $\angle(\vec{CM}, \vec{OC})$  the *anomaly*  $v_m$ , and the vector  $\vec{CM}$  turns with a constant angular velocity  $\dot{v}_m$  relating to  $\vec{OC}$ , or, since  $\text{Arg } \vec{CM} = \text{Arg } \vec{OC} + \angle(\vec{OC}, \vec{CM}) = \lambda_m - v_m$ , with the angular velocity  $\dot{\lambda}_m - \dot{v}_m$  relating to the  $x_1$ -axis.

If as in the solar theory  $\dot{\lambda}_m = \dot{v}_m$ , then  $v_m(t) = \lambda_m(t) + c$  ( $c$  = a constant). Consequently, the greatest apparent velocity of the moon, which is assumed in the lower point of the epicycle, is always assumed for the

same longitude (namely  $\lambda_m = 180^\circ - c$ ) in contradiction to 1. Hence  $\dot{\lambda}_m \neq \dot{v}_m$ .

From the fact 2 it appears that the plane of the orbit of the moon makes an angle of approximately  $5^\circ$  with the plane of the ecliptic. If the line of intersection of these two planes (the nodal line) is fixed, the moon has always the greatest latitude for the same longitude, namely the longitude of the northern limit point, which is in contradiction with 2., hence, the nodal line turns relating to the  $x_1$ -axis.

Based on such rather rough reflections on the observation material Ptolemy assumes:

The motion of the moon can be described by the above mentioned epicycle model, and the deferent makes an angle  $5^\circ$  with the plane of the ecliptic. However, to make the theory simpler, Ptolemy ignores this angle, so that in the following it is supposed that the epicycle model is in the plane of the ecliptic. Only a minor inaccuracy is involved by this approximation, the projection factor being  $\cos 5^\circ = 0.996 \sim 1$ .

The position vector  $\overrightarrow{OM}$  of the moon  $M$  is a sum of two vectors  $\overrightarrow{OM} = \overrightarrow{OC} + \overrightarrow{CM}$ , where

$$OC = R = \text{a constant}, \quad \text{Arg } \overrightarrow{OC} = \lambda_m = \text{the mean longitude}$$

$$CM = r = \text{a constant}, \quad \text{Arg } \overrightarrow{CM} = \lambda_m - v_m \quad (v_m = \text{the anomaly}).$$

Ptolemy calls the angle  $\angle (\overrightarrow{OC}, \overrightarrow{OM}) = p_1$ , at which the epicycle radius  $\overrightarrow{CM}$  is seen from the earth  $O$ , "prostaphairesis". This is the angle which is to be added to the mean longitude  $\lambda_m$  to give the true longitude  $\lambda$ . So, at any time  $t$  the following relation is valid

$$\text{Arg } \overrightarrow{OM} = \lambda(t) = \lambda_m(t) + p_1(t).$$

Note that  $p_1$  can be regarded as a function of the anomaly  $v_m$ . In the following this model is called "model 1".

Having made this assumption Ptolemy's first problem is to find the constant velocities

$$\dot{\lambda}_m = \text{the mean velocity of the moon in longitude}$$

$$\dot{v}_m = \text{the mean velocity of the moon in anomaly.}$$

If the mean longitude of the sun is called  $\lambda_{\odot m}$ , Ptolemy calls the difference  $\lambda_m - \lambda_{\odot m} = \eta_m$  the "mean elongation".

### 3. The Periods of the Moon

To solve this problem Ptolemy has to use the results of his predecessors. First he mentions how "the old ones" have looked for a period  $L$ , with the following property:

If at the time  $t$  there is a lunar eclipse there is again a lunar eclipse at the time  $t + L$ , and the increase of the longitude of the moon,  $\lambda(t + L) - \lambda(t)$  is a constant.

It is possible to prove that a period  $L$  with this property must contain an integral number of returns in anomaly (under the assumption: that the moon can be described by model 1). Ptolemy mentions this fact without a proof.

Ptolemy states that Hipparchus has found such a period by comparing his own observations with older chaldean observations:

$$L = 126007\frac{1}{24} \text{ days}$$

in which the moon completes

4267 synodic periods  
 4573 returns of anomaly and  
 4612 revolutions in the ecliptic except for  $7\frac{1}{2}^\circ$ .

From these results the mean values of the synodic month, and the anomalistic month

$$\frac{L}{4267}, \text{ and } \frac{L}{4573}$$

and the corresponding daily mean motions in elongation, and anomaly

$$\dot{\eta}_m = \frac{4267}{L} \cdot 360^\circ, \text{ and } \dot{\nu}_m = \frac{4573}{L} \cdot 360^\circ$$

can be found.

Ptolemy finds the daily mean motion in longitude of the moon by means of  $L$  and the daily mean motion in longitude of the sun  $\dot{\lambda}_{\odot m}$ .

The mean motion in longitude of the sun in a syn. month =  $\frac{L}{4267} \dot{\lambda}_{\odot m}$

The mean motion in longitude of the moon in a syn. month =  $\frac{L}{4267} \dot{\lambda}_{\odot m}$   
 +  $360^\circ$

The mean motion in longitude of the moon in a day =

$$\left( \frac{L}{4267} \dot{\lambda}_{\odot m} + 360^\circ \right) / \left( \frac{L}{4267} \right) = \frac{4267}{L} 360^\circ + \dot{\lambda}_{\odot m} = \dot{\eta}_m + \dot{\lambda}_{\odot m}.$$

Consequently, at once he could have said:

The mean velocity in longitude of the moon = the mean velocity in longitude of the sun + the mean velocity in elongation.

The lunar theory is now based on the mean values

$$\dot{\lambda}_m = 360^\circ \frac{4267}{L} + \dot{\lambda}_{\odot m} = 13;10,34,58,33,30,30^\circ/\text{d}$$

$$\dot{\nu}_m = 360^\circ \frac{4573}{L} = 13; 3,53,56,29,38,38^\circ/\text{d}$$

$$\dot{\eta}_m = 360^\circ \frac{4267}{L} = 12;11,26,41,20,17,59^\circ/\text{d}$$

Hipparchus knew these mean values and they were handed over to Ptolemy. However, he re-examines the results and finds a minor correction for  $\dot{\nu}_m$ . Even if this correction is so little (order of magnitude  $12 \cdot 60^{-4} \cong 2'/100$  years) that in practice it plays no role for the lunar theory, it is to Ptolemy's credit that he tries to check and correct the results handed over to him.

The mean motions in longitude, anomaly, and elongation as functions of time are given in four tables in *Almagest* IV,4.

We are here going to examine how accurate  $\dot{\lambda}_m$  and  $\dot{\eta}_m$  are.

According to P. V. Neugebauer [1] I, p. 35 the following formulas are valid

$$\lambda_m^* = \text{const}_1 + 1732564394.50''T + 12.200''T^2 + 0.0066''T^3$$

$$\lambda_{\odot m}^* = \text{const}_2 + 129602764.37''T + 2.600''T^2$$

from which by subtraction

$$\eta_m^* = \text{const}_3 + 1602961630.13''T + 9.600''T^2 + 0.0066''T^3.$$

Here  $T$  means the number of Julian centuries which have elapsed since the year 1800. These are P. V. Neugebauer's original elements, supplied with the corrections based upon the investigations by C. Schoch.

By differentiation we find the mean velocities

$$\dot{\lambda}_m^* = 1732564394.50'' + 24.400''T + 0.0198''T^2$$

$$\dot{\lambda}_{\odot m}^* = 129602764.37'' + 5.200''T$$

$$\dot{\eta}_m^* = 1602961630.13'' + 19.200''T + 0.0198''T^2$$

*Note.* Since the remaining terms ( $T^3$ -term,  $T^4$ -term, etc.) in  $\dot{\lambda}_m^*$  and  $\dot{\lambda}_{\odot m}^*$  are not known, such a differentiation may be risky. But it is reasonable to assume that the mentioned polynomials approximate so well that in a *bounded* time interval the *real* derivatives do not differ essentially from the derivatives of the polynomials.

If  $T = -16.4$  is inserted the mean velocities for the year  $+160$  (the end of Ptolemy's lifetime) are obtained in the unit seconds/100 years. Division by  $3600 \cdot 365.25 \cdot 100$  gives the unit degrees/day.

Results:

$$\dot{\lambda}_m^* = 13.176393640 \text{ }^\circ/\text{d}$$

$$\dot{\lambda}_{\odot m}^* = 0.985646658 \text{ }^\circ/\text{d}$$

$$\dot{\eta}_m^* = 12.190746981 \text{ }^\circ/\text{d}.$$

Ptolemy has the values

$$\dot{\lambda}_m = 13;10,34,58,33,30,30 = 13.176382215 \text{ }^\circ/\text{d}$$

$$\dot{\lambda}_{\odot m} = 0;59, 8,17,13,12,31 = 0.985635278 \text{ }^\circ/\text{d}$$

$$\dot{\eta}_m = 12;11,26,41,20,17,59 = 12.190746937 \text{ }^\circ/\text{d}.$$

The errors in Ptolemy's values for the year  $+160$  are then

$$\Delta \dot{\lambda}_m = \dot{\lambda}_m^* - \dot{\lambda}_m = 0.000011425 \text{ }^\circ/\text{d} = 0.417 \text{ }^\circ/100 \text{ years}$$

$$\Delta \dot{\lambda}_{\odot m} = \dot{\lambda}_{\odot m}^* - \dot{\lambda}_{\odot m} = 0.000011380 \text{ }^\circ/\text{d} = 0.416 \text{ }^\circ/100 \text{ years}$$

$$\Delta \dot{\eta}_m = \dot{\eta}_m^* - \dot{\eta}_m = 0.000000044 \text{ }^\circ/\text{d} = 0.002 \text{ }^\circ/100 \text{ years}.$$

By differentiation of  $\dot{\lambda}_m^*$ ,  $\dot{\lambda}_{\odot m}^*$ , and  $\dot{\eta}_m^*$  the accelerations are found.

$$\ddot{\lambda}_m^* = 0.0066^\circ/100 \text{ years per } 100 \text{ years}$$

$$\ddot{\lambda}_{\odot m}^* = 0.0014^\circ/100 \text{ years per } 100 \text{ years}$$

$$\ddot{\eta}_m^* = 0.0052^\circ/100 \text{ years per } 100 \text{ years}.$$

*Note.* As above.

Consequently, at the beginning of the time of Hipparchus (−160) the following formulas are valid:

$$\Delta \dot{\lambda}_m = 0.417 - 0.0066 \cdot 3.2 \text{ } ^\circ/100 \text{ years} = 0.396 \text{ } ^\circ/100 \text{ years}$$

$$\Delta \dot{\lambda}_{\odot m} = 0.416 - 0.0014 \cdot 3.2 \text{ } ^\circ/100 \text{ years} = 0.412 \text{ } ^\circ/100 \text{ years}$$

$$\Delta \dot{\eta}_m = 0.002 - 0.0052 \cdot 3.2 \text{ } ^\circ/100 \text{ years} = -0.015 \text{ } ^\circ/100 \text{ years.}$$

During the whole period from the beginning of the time of Hipparchus to the end of the time of Ptolemy, −160 to +160, the following is valid with good approximation:  $\Delta \dot{\lambda}_m - \Delta \dot{\lambda}_{\odot m} = 0.41^\circ/100 \text{ years}$ , while  $\Delta \dot{\eta}_m$  is practically equal to zero.

*This examination shows that  $\dot{\eta}_m$  is a good basis for the lunar theory, while  $\dot{\lambda}_m$  is not good. On the whole, the error in the mean velocity in longitude of the moon,  $\dot{\lambda}_m$ , is the same as the error in the mean velocity in longitude of the sun,  $\dot{\lambda}_{\odot m}$ . In *Almagest III*  $\dot{\lambda}_{\odot m}$  was found with a rather great error, originating in an error in the length of the year of 0.0044 days (see Petersen and Schmidt, p. 88–89). This error from the solar theory is undiminished handed over to the lunar theory and causes that the moon according to Ptolemy on an average loses about 0.41 degrees per 100 years.*

#### 4. The Computation of the Longitude of the Moon According to Model 1

To compute the longitude of the moon at a moment  $t$  from the model 1 it is necessary to know two things, namely

- A. The dimensions of the model
- B. The position of the moon at a definite moment.

As only the *proportion*  $\frac{r}{R}$  is important for the computation of *prosthairesis*  $p_1$ , Ptolemy can arbitrarily chose  $R$  to 60 units. In what follows all other distances will be measured in these units. Then the problem A is reduced to fix  $r$ .

Now, the problems A and B are solved using the moments  $T_1$ ,  $T_2$ , and  $T_3$  for 3 lunar eclipses, observed in Babylon shortly after the beginning of the era Nabonassar.

By means of the solar theory Ptolemy computes the longitude,  $\lambda_{\odot}(T_i)$  of the sun at these three moments, and since  $T_1$ ,  $T_2$ , and  $T_3$  are moments

of lunar eclipses (mid-eclipses),  $\lambda(T_i) = \lambda_{\odot}(T_i) + 180^\circ$ . He finds  $\lambda(T_1) = 174^\circ 30'$ ,  $\lambda(T_2) = 163^\circ 45'$ , and  $\lambda(T_3) = 333^\circ 15'$ .

*Note.* The moments  $T_1$ ,  $T_2$ , and  $T_3$  are around the year  $-720$  (see section 8). On account of errors in the solar theory (Petersen and Schmidt, p. 88–90), Ptolemy's sun in the year  $-720$  is about  $2\frac{1}{2}^\circ$  ahead of the real sun. This causes a one-sided error in the longitude of the moon,  $\lambda(T_i)$ , of about  $-2\frac{1}{2}^\circ$ .

The moments  $T_1$ ,  $T_2$ , and  $T_3$  are measured in true solar time. Ptolemy converts the time intervals  $T_2 - T_1$ , and  $T_3 - T_2$  into mean solar time. The results are given in the table I, column I. In column II the differences  $\lambda(T_2) - \lambda(T_1)$ , and  $\lambda(T_3) - \lambda(T_2)$  are given. The one-sided error of  $-2\frac{1}{2}^\circ$  disappears by this subtraction. The column III gives the mean motion of the moon in longitude in any of the two time intervals, calculated by Ptolemy from the moon tables, *Almagest* IV,4. The column IV contains the differences between the true motion of the moon in longitude and the mean motion of the moon in longitude (column II–column III), and finally the column V gives the motion of the moon in anomaly in every of the two intervals, calculated by Ptolemy by means of the moon tables.

|                        | I                                                | II      | III     | IV     | V       |
|------------------------|--------------------------------------------------|---------|---------|--------|---------|
| $T_2 - T_1 \dots\dots$ | 354 <sup>d</sup> 2 <sup>h</sup> 34 <sup>m</sup>  | 349°15' | 345°51' | 3°24'  | 306°25' |
| $T_3 - T_2 \dots\dots$ | 176 <sup>d</sup> 20 <sup>h</sup> 12 <sup>m</sup> | 169°30' | 170°7'  | −0°37' | 150°26' |

Table I

Since the motion of  $M$  on the epicycle (which is the motion in anomaly, column V) determines the difference between the *true* motion in longitude and the *mean* motion in longitude, a close connection between the columns IV and V exists. This solves the problems A and B in the following manner. At first the epicycle is regarded alone, without consideration of its real position in the coordinate system at the three moments  $T_1$ ,  $T_2$ , and  $T_3$ . The position of the moon at these moments are called  $M_1$ ,  $M_2$ , and  $M_3$  (fig. 2). Now, the results of column V give

$$\begin{aligned} \angle(\vec{CM}_2, \vec{CM}_1) &= v_m(T_2) - v_m(T_1) = 306^\circ 25' \\ \widehat{M_3 M_2} = \angle(\vec{CM}_3, \vec{CM}_2) &= v_m(T_3) - v_m(T_2) = 150^\circ 26' \end{aligned}$$

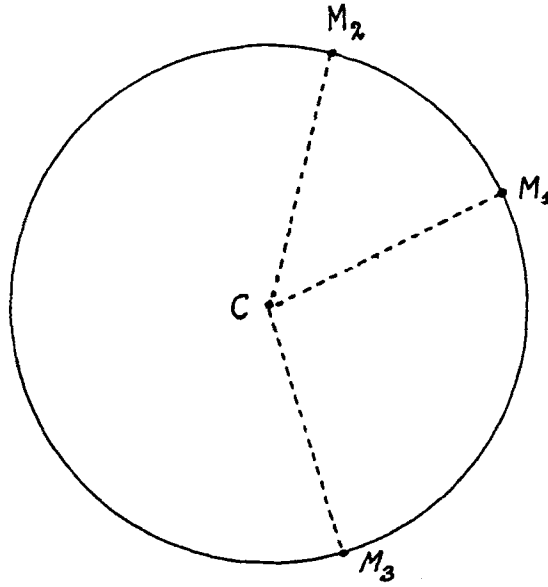


Fig. 2

from which follows

$$\widehat{M_1M_2} = \angle (\vec{CM_1}, \vec{CM_2}) = 360^\circ - 306^\circ 25' = 53^\circ 35'$$

$$\widehat{M_3M_1} = \widehat{M_3M_2} - \widehat{M_1M_2} = 150^\circ 26' - 53^\circ 35' = 96^\circ 51'$$

Since  $\lambda(T_i) = \lambda_m(T_i) + p_1(T_i)$ , the results from column IV give

$$p_1(T_2) - p_1(T_1) = 3^\circ 24' \quad (1)$$

$$p_1(T_3) - p_1(T_2) = -0^\circ 37' \quad (2)$$

and by addition

$$p_1(T_3) - p_1(T_1) = 2^\circ 47' \quad (3)$$

Since prostaphairesis is the angle  $p_1(T_i) = \angle (\vec{OC}, \vec{OM_i})$  this can be inserted in (1)–(3),  $\vec{OC}$  can be eliminated, and this gives

$$\angle (\vec{OM_1}, \vec{OM_2}) = \angle M_1OM_2 = 3^\circ 24'$$

$$\angle (\vec{OM_3}, \vec{OM_2}) = \angle M_3OM_2 = -0^\circ 37'$$

$$\angle (\vec{OM_1}, \vec{OM_3}) = \angle M_1OM_3 = 2^\circ 47'$$

The problem is now equivalent to the following one: "On a circle of radius  $r$  three points  $M_1$ ,  $M_2$ , and  $M_3$  are given, which are seen under given angles from a point  $O$ . Find  $r$  and the position of  $O$  with respect to  $M_1$ ,  $M_2$ , and  $M_3$ ." Neugebauer gives a good and detailed description of the general solution of this problem, see O. Neugebauer, p. 211–214.

The results of the calculations are

$$R = 60 \text{ and } r = 5;13$$

$$\text{the anomaly at the moment } T_2 = v_m(T_2) = 12^\circ 24'$$

$$\text{the prosthaphairesis at the moment } T_2 = p_1(T_2) = -0^\circ 59'$$

Above was found  $\lambda(T_2) = 163^\circ 45'$ , which with  $p_1(T_2)$  gives

$$\text{the mean longitude at the moment } T_2 = \lambda_m(T_2) = 164^\circ 44'$$

*Note.* Of course, here it plays a role that  $\lambda(T_2)$  has an error of about  $-2\frac{1}{2}^\circ$ .

In this way the dimensions of the model and the position of the moon at a definite moment ( $T_2$ ) are found from the moments  $T_1$ ,  $T_2$ ,  $T_3$  of three lunar eclipses. Ptolemy repeats all the calculations from a new set of moments of three lunar eclipses, this time someones, he himself has observed at the moments  $T_4$ ,  $T_5$ , and  $T_6$ . The result is  $r = 5;14$ , after which he goes on with the approximate value  $r = 5;15$ .

From the values of  $\lambda_m(T_2)$  and  $v_m(T_2)$  is found  $\lambda_m(T_0)$  and  $v_m(T_0)$  at the beginning of era Nabonassar (= Nabonassar 1, thoth 1, noon) by first converting  $T_2 - T_0$  into mean solar time and then finding from the moon tables the position of the moon at the moment  $T_0$ :

$$v_m(T_0) = 268^\circ 49'$$

$$\lambda_m(T_0) = 41^\circ 22'$$

From the solar theory is known  $\lambda_{\odot m}(T_0) = 330^\circ 45'$ , from which also the mean elongation at the beginning of the era is known:

$$\eta_m(T_0) = \lambda_m(T_0) - \lambda_{\odot m}(T_0) = 70^\circ 37'.$$

Now, from these starting values the mean longitude  $\lambda_m(t)$  and the anomaly  $v_m(t)$  can be calculated at an arbitrary moment  $t$  by means of the moon tables. Prosthaphairesis  $p_1(t)$  can be calculated from the triangle  $OCM$  (fig. 1), in which  $OC = R$  and  $CM = r$  are known, and  $\angle OCM$  can be found immediately from the anomaly  $v_m$ . To make the calculations easier, Ptolemy has once for ever tabulated  $p_1$  as a function of the anomaly

$v_m$  in Almagest IV,10. Then, in this table  $p_1$  is read, and the true longitude of the moon is given by

$$\lambda(t) = \lambda_m(t) + p_1(t).$$

Ptolemy states in Almagest IV,11, that model 1 was already known by Hipparchus, but declares that Hipparchus had certain errors, which are now corrected.

Finally, it is noted that prostaphairesis  $p_1$  for all  $v_m$  is given by the formula

$$\tan p_1 = \frac{-r \sin v_m}{R + r \cos v_m} \quad (4)$$

which is easy to find using the triangle  $OCM$  (fig. 1).

### 5. First Change, from Model 1 to Model 2

Ptolemy begins Almagest V by stating that model 1 has shown good agreement with observations made by elongations of  $0^\circ$  or  $180^\circ$  (the syzygies), but with other values of  $\eta$  the agreement is sometimes good, sometimes bad, and worst by elongations of  $90^\circ$  or  $270^\circ$  (the quadratures). Ptolemy has masterfully selected the observations on which this discovery is based, partly among the older observations handed over to him by Hipparchus and partly among his own observations.

After an analysis of these observations he finds:

A. If prostaphairesis  $p_1 > 0$ , the moon is observed to be ahead of the position, determined by model 1, and the error increases with  $p_1$ .

B. If prostaphairesis  $p_1 < 0$ , the moon is observed to be behind the position, determined by model 1, and the error increases with  $|p_1|$ .

Since then model 1 does not correspond to the observations, it must be changed into a model, say "model 2". In this the mean longitude of the moon,  $\lambda_m$ , is provided with a new prostaphairesis  $\bar{p}$  to give the true longitude  $\lambda$ , and compared with prostaphairesis  $p_1$  the following must be valid.

$$\text{For } \eta \cong 0^\circ, 180^\circ : \bar{p} = p_1$$

$$\text{For } \eta \cong 90^\circ, 270^\circ : |\bar{p}| > |p_1|$$

These demands are satisfied by inserting a mechanism into the model, bringing the epicycle closer to the earth at elongations  $\eta \cong 90^\circ$  or  $270^\circ$ , while the distance remains unchanged at elongations  $\eta \cong 0^\circ$  or  $180^\circ$ .

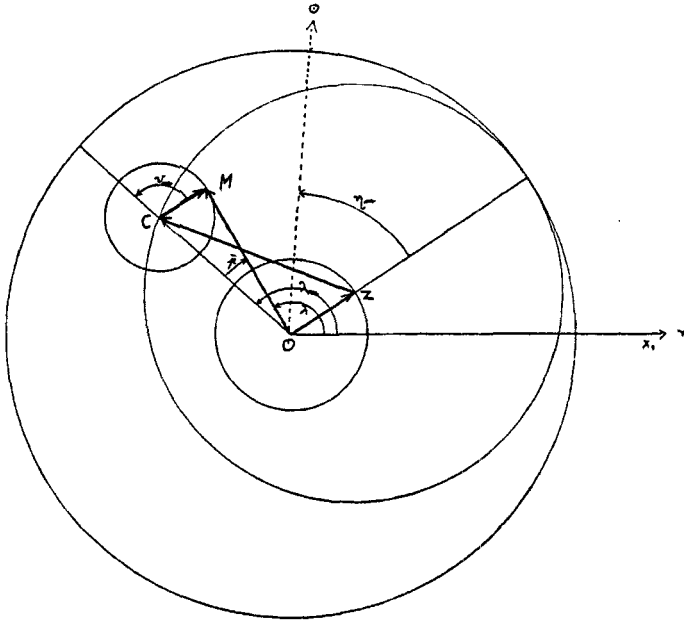


Fig. 3

Ptolemy inserts this mechanism assuming that the motion of the moon can be described by model 2, fig. 3.

Now, the epicycle centre does not move on the deferent of radius  $R$  and centre  $O$ , but on an excentre of radius  $R-s$  and centre  $Z$  which moves on a small circle of radius  $s$  and centre  $O$  (the constant  $s$  is found below). The epicycle centre steadily represents the mean moon, consequently  $\text{Arg } \vec{OC} = \lambda_m$ , while the centre  $Z$  of the excentre moves on the small circle in such a way that  $\angle(\vec{OZ}, \vec{OC}) = 2\eta_m =$  the double of the mean elongation. The motion on the epicycle of the moon  $M$  is with respect to  $\vec{OC}$  unchanged from model 1.

Now, the position vector of the moon is a sum of *three* vectors

$$\vec{OM} = \vec{OZ} + \vec{ZC} + \vec{CM}, \text{ where}$$

$$|\vec{OZ}| = s = \text{a constant. } \text{Arg } \vec{OZ} = \text{Arg } \vec{OC} - \angle(\vec{OZ}, \vec{OC}) = \lambda_m - 2\eta_m$$

$$|\vec{ZC}| = R-s = \text{a constant. } \text{Arg } \vec{ZC} = \text{Arg } \vec{OC} + \angle(\vec{OC}, \vec{ZC}) = \lambda_m + \alpha$$

$$|\vec{CM}| = r = \text{a constant. } \text{Arg } \vec{CM} = \text{Arg } \vec{OC} - \angle(\vec{CM}, \vec{OC}) = \lambda_m - \nu_m$$

The small angle  $\alpha = \angle (\vec{OC}, \vec{ZC})$  is given by

$$\sin \alpha = \frac{s}{R-s} \sin 2\eta_m, \quad (5)$$

which is easily found from the law of sines applied to the triangle  $OCZ$ . Then the position vector  $\vec{OM}$  is a sum of three vectors with constant lengths of which the two  $\vec{OZ}$  and  $\vec{CM}$  turn with a constant angular velocity, while the third turns more irregularly.

$\vec{OM}$  can also be regarded as a sum of *two* vectors:

$$\vec{OM} = \vec{OC} + \vec{CM},$$

where  $\vec{OC}$  has a *constant* angular velocity  $\dot{\lambda}_m$ , but variable length  $OC$ , given by

$$\begin{aligned} OC &= ZC \cdot \cos \alpha + OZ \cdot \cos 2\eta_m \\ &= (R-s) \cdot \sqrt{1 - \sin^2 \alpha} + s \cdot \cos 2\eta_m \end{aligned} \quad (6)$$

Now, (6) shows that

$$\text{for } \eta_m = 0^\circ, 180^\circ: OC = (R-s) \cdot 1 + s = R$$

$$\text{for } \eta_m = 90^\circ, 270^\circ: OC = (R-s) \cdot 1 - s = R - 2s$$

This shows that the epicycle has the least possible distance  $R - 2s$  from the earth at the quadratures, while at the syzygies it has the same distance  $R$  from the earth as in model 1. Thus model 2 agrees with the observations.

The remaining problem, to find the constant  $s$ , is solved by Ptolemy in *Almagest*, V 3–4 in the following way.

Since the numerical value of prostaphairesis  $\bar{p}$  has maximum when

1.  $\eta_m \cong 90^\circ, 270^\circ$  (the epicycle as close as possible to the earth)

2.  $\nu_m \cong 90^\circ, 270^\circ$  ( $\vec{OM}$  nearly is a tangent to the epicycle),

he observes the moon at a moment  $T_{16}$ , where this is the case. He finds

$$\text{The longitude of the moon } \lambda(T_{16}) = 219^\circ 40'$$

$$\text{The longitude of the sun } \lambda_\odot(T_{16}) = 318^\circ 50'$$

$$\text{The elongation } \eta(T_{16}) = 260^\circ 50'$$

The time interval  $T_{16} - T_0$  is converted into mean solar time, and from the moon tables the following mean values are calculated

|                                                           |                                     |
|-----------------------------------------------------------|-------------------------------------|
| The mean longitude of the moon $\lambda_m(T_{16})$        | $= 227^\circ 20'$                   |
| The mean longitude of the sun $\lambda_{\odot m}(T_{16})$ | $= 316^\circ 27'$                   |
| The mean elongation $\eta_m(T_{16})$                      | $= 270^\circ 53' (\cong 270^\circ)$ |
| The anomaly $\nu_m(T_{16})$                               | $= 87^\circ 19' (\cong 90^\circ)$   |

Then the conditions 1 and 2 are satisfied, and  $\bar{p}_{max}$  can be found:

$$\bar{p}_{max} = |\bar{p}(T_{16})| = |\lambda(T_{16}) - \lambda_m(T_{16})| = 7^\circ 40'.$$

Ptolemy ends *Almagest* V,3 by stating that he has reached the same result from many other observations, and in *Almagest* V,4 the radius from this result is calculated. Since 1 and 2 are satisfied, with good approximation the following is valid:  $OC = R - 2s$  and  $\angle CMO = 90^\circ$ . Then, in the rectangular triangle  $OMC$  (fig. 4)  $MC = r$  and  $\angle COM = 7^\circ 40'$  are known, from which the hypotenuse  $OC = R - 2s$  can be found. Finally, this gives (in the units of which  $R = 60$ ):

$$s = 10;19.$$

Model 2 is now completely determined, and Ptolemy ends his description of this part of the complicated lunar theory ascertaining that model 2 agrees well with observations at  $\eta \cong 0^\circ, 90^\circ, 180^\circ$ , and  $270^\circ$ .

Briefly, the change from model 1 to model 2 consists in replacing the *constant* radius  $R$  with the *variable* length  $OC$  given by (4). Therefore, the formula (4) can be used for model 2 with  $OC$  instead of  $R$ , and gives

$$\tan \bar{p} = \frac{-r \sin \nu_m}{OC + r \cos \nu_m} \quad (7)$$

## 6. The Second Change, from Model 2 into Model 3

Now, in true scientific way Ptolemy starts examining whether model 2 gives good results compared with Hipparchus' and his own observations. As mentioned above this is the case for the elongations  $\eta \cong 0^\circ, 90^\circ, 180^\circ$ , and  $270^\circ$  but it appears *not* to be the case for  $\eta \cong 45^\circ, 135^\circ, 225^\circ$ , and  $315^\circ$ , i.e. around the octants.

After an analysis of a large number of observations Ptolemy has found that the deviations relating to the moon positions determined by model 2 are largest when

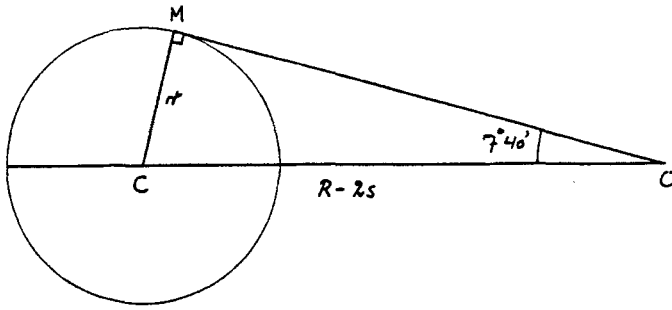


Fig. 4

1.  $\eta \cong 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ$  and
2.  $\nu_m \cong 0^\circ, \text{ or } 180^\circ$ .

Since model 2 does not agree completely with the observations it must be changed into a new model, say "model 3". In this the mean longitude  $\lambda_m$  of the moon is provided with a new prostaphairesis  $p$ , which compared with the prostaphairesis  $\bar{p}$  from model 2 satisfies the following conditions

- A. For  $\eta \cong 0^\circ, 90^\circ, 180^\circ, 270^\circ$ :  $p = \bar{p}$
- B. For  $\eta \cong 45^\circ, 225^\circ$  (i.e.  $2\eta \cong 90^\circ$ ) and  $\begin{cases} \nu_m \cong 0^\circ: p < \bar{p} \\ \nu_m \cong 180^\circ: p > \bar{p} \end{cases}$
- C. For  $\eta \cong 135^\circ, 315^\circ$  (i.e.  $2\eta \cong 270^\circ$ ) and  $\begin{cases} \nu_m \cong 0^\circ: p > \bar{p} \\ \nu_m \cong 180^\circ: p < \bar{p} \end{cases}$

*Note.* Ptolemy does not formulate these demands so sharply. But in his construction of model 3 in *Almagest* V,5 he applies two observations which satisfy two of the conditions mentioned above.

These conditions are satisfied if model 2 is changed in such a way that  $\angle(\vec{CM}, \vec{OC})$  is slightly larger than  $\nu_m$  for  $\eta \cong 45^\circ, 225^\circ$ , and slightly smaller than  $\nu_m$  for  $\eta \cong 135^\circ, 315^\circ$ . Ptolemy makes this change by assuming that the motion of the moon can be described by model 3, fig. 5.

In model 3 the points Z and C move as in model 2, i.e. the vectors  $\vec{OC}$  and  $\vec{ZC}$  are determined as in model 2. M is still to move on the epicycle of radius  $r$  and centre C, but now the anomaly is to be counted

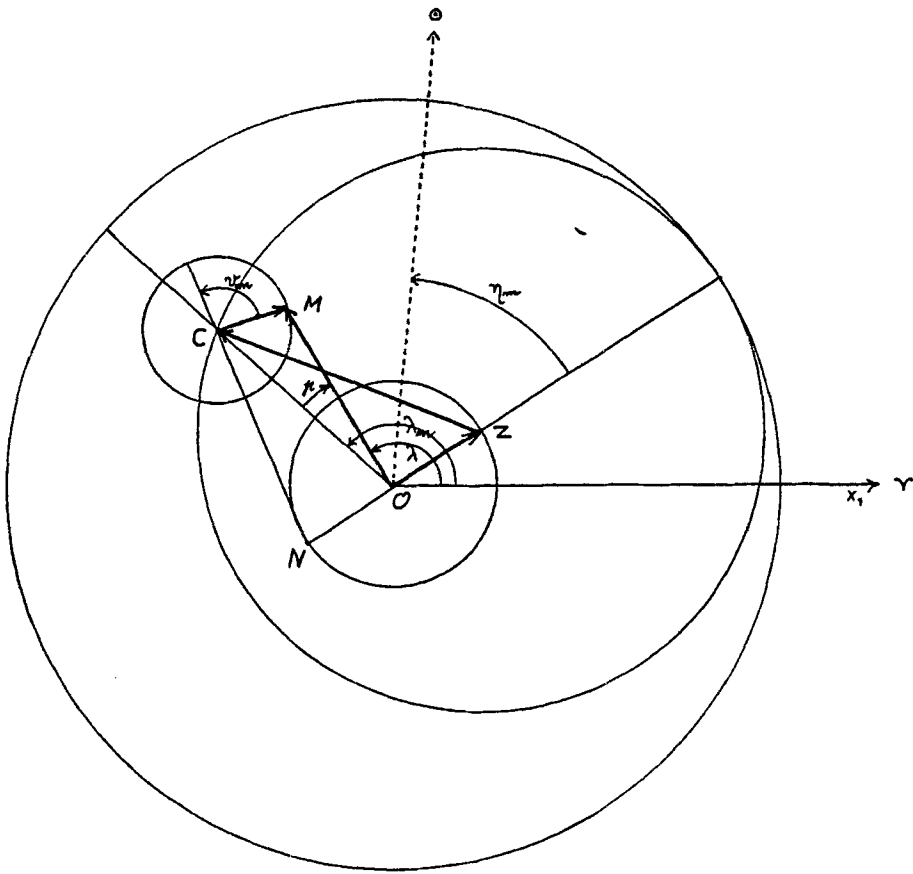


Fig. 5

from the variable direction  $CN$ , where  $N$  is placed diametrically opposite to  $Z$  on the small circle of radius  $s$  and centre  $O$ . Since the motion of  $M$  in model 3 on the epicycle is somewhat different from the motion of  $M$  in model 2 on the epicycle it follows that the prostaphairesis of model 3 may differ from the prostaphairesis of model 2.

In what follows the angle  $\angle(\vec{CM}, \vec{OC})$  is called the *true anomaly*,  $v$ , and then

$$v = \angle(\vec{CM}, \vec{NC}) + \angle(\vec{NC}, \vec{OC}) = v_m + \beta \quad (8)$$

From fig. 5 it follows that:

for  $\eta_m = 0^\circ, 90^\circ, 180^\circ, 270^\circ$ :  $\angle ZOC = 0^\circ$  or  $180^\circ$  and then  $\beta = 0^\circ$

for  $\eta_m = 45^\circ, 225^\circ$ :  $\angle ZOC = 90^\circ$  and then  $\beta$  positive

for  $\eta_m = 135^\circ, 315^\circ$ :  $\angle ZOC = 270^\circ$  and then  $\beta$  negative

consequently, complete agreement with the conditions A, B, and C.

The position vector  $\vec{OM}$  of the moon is a sum of three vectors:

$$\vec{OM} = \vec{OZ} + \vec{ZC} + \vec{CM}, \text{ where}$$

$\vec{OZ}$  and  $\vec{ZC}$  are as in model 2, while

$$|\vec{CM}| = r = 5;15, \text{ Arg } \vec{CM} = \lambda_m - (v_m + \beta) = \lambda_m - v.$$

The small angles  $\alpha$  and  $\beta$  are given by

$$\sin \alpha = \frac{s}{R-s} \sin 2\eta_m \quad (5)$$

$$\sin \beta = \frac{\sin \alpha}{\sqrt{1 + 4\left(\frac{s}{R-s}\right)^2 + 4\left(\frac{s}{R-s}\right) \cos(2\eta_m + \alpha)}} \quad (9)$$

The last formula is found by at first computing  $NC$  by the triangle  $NCZ$  and then applying the law of sines to triangle  $NCO$ .

Ptolemy begins by mentioning how model 3 is constructed and then he shows that the so constructed model agrees with two observations. One of these is made by Hipparchus at the moment  $T_{18}$  where the moon is around an octant. The following was observed:

The longitude of the moon  $\lambda(T_{18}) = 351^\circ 27'$

The longitude of the sun  $\lambda_\odot(T_{18}) = 37^\circ 45'$

The elongation  $\eta(T_{18}) = 313^\circ 42'$

The time interval  $T_{18} - T_0$  is converted into mean solar time, and from the moon tables the following mean values are calculated:

The mean longitude of the moon  $\lambda_m(T_{18}) = 352^\circ 13'$

The mean longitude of the sun  $\lambda_{\odot m}(T_{18}) = 36^\circ 41'$

The mean elongation  $\eta_m(T_{18}) = 315^\circ 32' (\cong 315^\circ)$

The anomaly  $v_m(T_{18}) = 185^\circ 30' (\cong 180^\circ)$

From these results prostaphairesis  $p(T_{18})$  is found:

$$p(T_{18}) = \lambda(T_{18}) - \lambda_m(T_{18}) = -0^\circ 46'$$

*Note.* Since  $v_m(T_{18}) > 180^\circ$ , model 2 gives a positive prostaphairesis,  $\bar{p}(T_{18})$ , while in fact the observation shows a negative prostaphairesis, corresponding to the condition C.

By a detailed calculation Ptolemy shows that this negative prostaphairesis is obtained, if the anomaly  $v_m$  is counted from the direction  $NC$ , where  $N$  is opposite to  $Z$  on the line through  $O$  and  $Z$ , and  $ON = 10;18$ . Now, in model 2  $OZ = 10;19$  was found, hence, approximately  $ON = OZ$ , which agrees with the construction of model 3.

Ptolemy makes quite analogous calculations from another observation, made by Hipparchus at the moment  $T_{19}$ , and finds  $ON = 10;20 \cong OZ$ . He finishes by saying that many other observations have given the same result and states that the model now set up gives good results in syzygies, quadratures, and octants, and with this he is satisfied.

Briefly, the change from model 2 into model 3 consists in replacing the anomaly  $v_m$  with the *true* anomaly  $v$ . Therefore, the formula (7) can be used in model 3 with  $v$  instead of  $v_m$ , and gives

$$\tan p = \frac{-r \sin v}{OC + r \cos v} \quad (10)$$

The next problem is to find simple methods for computing the longitude of the moon according to model 3 at an arbitrary moment. The solution of this problem is shortly expounded in the next section.

### 7. Computation of the Longitude of the Moon According to Model 3

To compute the longitude of the moon at an arbitrary moment  $t$ , the mean values  $\lambda_m(t)$ ,  $v_m(t)$ , and  $\eta_m(t)$  can at once be found from the moon tables. Since

$$\lambda(t) = \lambda_m(t) + p(t)$$

the problem is to calculate  $p(t)$ , which is given by (10), where  $v = v_m + \beta$ ,  $\beta$  is given by (5) and (9), and  $OC$  is given by (5) and (6), such that  $p = p(v_m, \eta_m)$ .

In principle, it is no problem to find  $\lambda(t)$  according to model 3, but in *Almagest* V,7-9 Ptolemy sets himself the problem to set up *few* and *simple* tables, by means of which  $p$  can be found from the mean values  $v_m$  and  $\eta_m$ .

A table for computing  $p(v_m, \eta_m)$  could be set up as a table with two entries. If the prostaphairesis is tabulated for 45 values of  $v_m$ , and 45 values of  $\eta_m$  (and one might expect about 45 values since Ptolemy in model 1 has tabulated prostaphairesis for 45 values of the anomaly) then the table would contain altogether 2025 values.

Ptolemy succeeds in avoiding that many values by using (without proving) the fact that approximately

$$p(v_m, \eta_m) \cong p_1(v) + f(\eta_m) \cdot [p_2(v) - p_1(v)]$$

where  $f(\eta_m)$  and  $p_2(v)$  will be defined later.

In this way he constructs 4 tables, (namely the values of  $\beta$ ,  $p_1$ ,  $p_2 - p_1$ , and  $f$ ) with a total of  $4 \cdot 45 = 180$  values. These 4 tables are the columns 3, 4, 5, and 6 in the collection of tables *Almagest* V,8.

*Column 4* contains the function  $p_1$ , given by

$$\tan p_1 = \frac{-r \sin v}{R + r \cos v}$$

corresponding to prostaphairesis as a function of  $v$ , when the epicycle centre has the maximum distance  $R$  from the earth, consequently precisely the prostaphairesis known from model 1.

*Column 5.* The function  $p_2$  is defined by

$$\tan p_2 = \frac{-r \sin v}{(R - 2s) + r \cos v}$$

corresponding to prostaphairesis as a function of  $v$ , when the epicycle centre has the minimum distance  $R - 2s$  from the earth. *Column 5* contains the values of

$$p_2(v) - p_1(v)$$

which is the angle by which prostaphairesis increases, when  $OC$  changes from the maximum to the minimum value.

*Column 6.* If  $\eta_m$  has such a value that  $OC$  is between its maximum value  $R$  ( $p(\eta_m, v) = p_1(v)$ ) and its minimum value  $R - 2s$  ( $p(\eta_m, v) = p_2(v)$ ), prostaphairesis has a value between  $p_1(v)$  and  $p_2(v)$ .

A factor  $\varphi$  exists, so that

$$p(\eta_m, v) = p_1(v) + \varphi \cdot (p_2(v) - p_1(v)),$$

namely

$$\varphi = \frac{p(\eta_m, \nu) - p_1(\nu)}{p_2(\nu) - p_1(\nu)}$$

This factor  $\varphi$  is a function of  $\nu$  and  $\eta_m$ . But it can be shown that it is approximately independent of  $\nu$ . This implies that for any value of  $\eta_m$  it is sufficient to compute  $\varphi(\eta_m, \nu)$  for one suitable value of  $\nu$ , and the values of  $\varphi$  can be tabulated in a table with one entry.

Obviously, Ptolemy has been aware of these circumstances, and by means of this he finishes the solution of the problem in the following way.

In column 6 a function  $f$  given by

$$f(\eta_m) = \frac{\max |p(\eta_m, \nu)| - \max |p_1(\nu)|}{\max |p_2(\nu) - p_1(\nu)|}$$

(where maximum is made for  $0^\circ \leq \nu < 360^\circ$  and  $\eta_m$  fixed) is tabulated.

Column 4 shows:  $\max |p_1(\nu)| = 5^\circ 1'$

Column 5 shows:  $\max |p_2(\nu) - p_1(\nu)| = 2^\circ 39'$

so that  $f$  is given by

$$f(\eta_m) = \frac{1}{2^\circ 39'} (\max |p(\eta_m, \nu)| - 5^\circ 1')$$

Finally Ptolemy states that the prostaphairesis is given by

$$p = p_1 + f \cdot (p_2 - p_1)$$

Perhaps, the connection between Ptolemy's function  $f$  and my function  $\varphi$  is not quite obvious. The connection between  $f$  and  $\varphi$  appears from the fact that maximum of prostaphairesis is obtained for  $\nu \cong 90^\circ$  or  $270^\circ$ . This means that the function  $f$  is approximately the function  $\varphi(\eta_m, 90^\circ)$ . By a closer examination it can be proved that for all  $\eta_m$

$$|\varphi(\eta_m, 90^\circ) - f(\eta_m)| \leq 5 \cdot 10^{-3}.$$

Further it can be proved that for all  $\eta_m, \nu$

$$|\varphi(\eta_m, \nu) - \varphi(\eta_m, 90^\circ)| < 2 \cdot 10^{-2}$$

so that on the whole

$$|\varphi(\eta_m, \nu) - f(\eta_m)| < 2.5 \cdot 10^{-2}.$$

Since the maximum value of  $p_2 - p_1$  is  $2^\circ 39' = 159'$ , the error made in the computation of  $p$  by replacing  $\varphi(\eta_m, \nu)$  with  $f(\eta_m)$  is rather unimportant, namely smaller than  $2.5 \cdot 10^{-2} \cdot 159' \sim 4'$ . It is a very small error, compared with the accuracy of the model, see section 9.

### 8. An Examination of the Observations in the Lunar Theory

In connection with setting up of *model 1*, 15 observations of lunar eclipses made at the moments  $T_1, \dots, T_{15}$  are mentioned. These moments Ptolemy has converted into the Egyptian calendar, Alexandrian time. By means of P. V. Neugebauer's [2] tables, the moments  $T_1, \dots, T_{15}$  can be converted into the civil Julian calendar, Alexandrian time. (Concerning details of this conversion, see Petersen and Schmidt p. 85–87.)

Exactly computed moments  $T_1^*, \dots, T_{15}^*$  for the corresponding eclipses can be found in Oppolzer's "Canon der Finsternisse", these moments are given in Greenwich time, mean solar time. Addition of two hours converts  $T_1^*, \dots, T_{15}^*$  into Alexandrian time, mean solar time, and finally the true solar time is obtained by applying the equation of time, also found in Oppolzer's tables.

The absolute errors  $\Delta T_i$  in the moments  $T_1, \dots, T_{15}$  are given by  $\Delta T_i = T_i^* - T_i$ . The results of these subtractions appear from table 2, where  $\Delta T_i$  is given in minutes.

| $T_i$ | Date             | $\Delta T_i$ | $T_i$    | Date             | $\Delta T_i$ | $T_i$    | Date             | $\Delta T_i$ |
|-------|------------------|--------------|----------|------------------|--------------|----------|------------------|--------------|
| $T_1$ | –720<br>March 19 | 14           | $T_6$    | +136<br>March 6  | –30          | $T_{11}$ | –381<br>June 18  | 22           |
| $T_2$ | –719<br>March 8  | 6            | $T_7$    | +125<br>April 5  | 31           | $T_{12}$ | –381<br>Dec. 12  | –3           |
| $T_3$ | –719<br>Sept. 1  | –36          | $T_8$    | –490<br>April 25 | –41          | $T_{13}$ | –200<br>Sept. 22 | –8           |
| $T_4$ | +133<br>May 6    | –2           | $T_9$    | –501<br>Nov. 19  | 50           | $T_{14}$ | –199<br>March 20 | –13          |
| $T_5$ | +134<br>Oct. 20  | 14           | $T_{10}$ | –382<br>Dec. 23  | 33           | $T_{15}$ | –199<br>Sept. 12 | 16           |

Table 2

All the errors are smaller than 1 hour, half of them even smaller than  $\frac{1}{4}$  hour. The average of the numerical values of the errors is 22 min.

This shows that just from the beginning of era Nabonassar it has been possible to observe lunar eclipses with great accuracy. It is especially noted that the error in  $T_2$ , used for starting the moon in the mean motion (section 4), is only 6 minutes.

The setting up of *model 2* and *model 3* is based on observations of the longitude  $\lambda$  of the moon, the longitude  $\lambda_\odot$  of the sun, and the elongation  $\eta = \lambda - \lambda_\odot$ . In *Almagest* V, 1–9 altogether 4 such observations made at the moments  $T_{16}, \dots, T_{19}$  are mentioned. The first one Ptolemy has made himself, the three others are made by Hipparchus.

Now, the moments  $T_{16}, \dots, T_{19}$  are converted from Egyptian calendar, Alexandrian (or Rhodesian) time, true solar time, into the civil Julian calendar, Greenwich time, mean solar time, the results are given in table 3, column I. Column II contains Ptolemy's (or Hipparchus') values for  $\lambda(T_i)$ , column V contains their values for  $\lambda_\odot(T_i)$ . In the columns III and VI the corresponding modern values  $\lambda^*(T_i)$  and  $\lambda_\odot^*(T_i)$ , computed according to Tuckerman, are given. The columns IV and VII contain the absolute errors  $\Delta\lambda(T_i) = \lambda^*(T_i) - \lambda(T_i)$  and  $\Delta\lambda_\odot(T_i) = \lambda_\odot^*(T_i) - \lambda_\odot(T_i)$ .

| $T_i$    | I                   | II<br>$\lambda(T_i)$ | III<br>$\lambda^*(T_i)$ | IV<br>$\Delta\lambda(T_i)$ | V<br>$\lambda_\odot(T_i)$ | VI<br>$\lambda_\odot^*(T_i)$ | VII<br>$\Delta\lambda_\odot(T_i)$ |
|----------|---------------------|----------------------|-------------------------|----------------------------|---------------------------|------------------------------|-----------------------------------|
| $T_{16}$ | +139<br>Febr. 9.210 | 219.7°               | 221.9°                  | 2.2°                       | 318.83°                   | 319.43°                      | 0.60°                             |
| $T_{17}$ | –127<br>Aug. 5.162  | 42.3°                | 41.0°                   | –1.3°                      | 128.58°                   | 128.73°                      | 0.15°                             |
| $T_{18}$ | –126<br>May 2.176   | 351.5°               | 351.3°                  | –0.2°                      | 37.75°                    | 37.64°                       | –0.11°                            |
| $T_{19}$ | –126<br>July 7.583  | 149.0°               | 149.2°                  | 0.2°                       | 100.90°                   | 100.96°                      | 0.06°                             |

Table 3

The values  $\lambda^*(T_i)$  are found by linear interpolation in Tuckerman's tables, which can give an inaccuracy of 1.6° (Tuckerman I, p. 5–6). On account of this rather great error no definite conclusion can be made

about the accuracy of  $\lambda(T_i)$ ,  $i = 17, 18, 19$ . But as for  $\lambda(T_{16})$  we are justified in maintaining that it is in error of at least  $+0.6^\circ (= 2.2^\circ - 1.6^\circ)$ .

The values  $\lambda_\odot^*(T_i)$  are also found by linear interpolation, but here the accuracy is  $0.01^\circ$ , so that  $\Delta\lambda_\odot$  is quite precisely defined.

In particular we note that at the time  $T_{16}$  Ptolemy has observed the sun  $0.60^\circ$  behind its real place, and he has observed the moon at least  $0.60^\circ$  behind of its real place. This may indicate that his instruments had a systematical error.

Hipparchus' values for  $\lambda_\odot$  are unquestionable better than Ptolemy's. The average of the numerical values of Hipparchus' errors is  $0.32^\circ/3 = 6.4'$ , which confirms the generally accepted statement that Hipparchus was a good observer and that the accuracy of observations in antiquity is approximately 5–10 arc minutes.

### 9. An Examination of the Quality of the Models

Having expounded the three models and examined the basic observations an answer is given below to the following question: How accurate can these models describe the motion of the moon at Ptolemy's time, and are Ptolemy's complicated models 2 and 3 in reality better than model 1, known by Hipparchus?

From Tuckerman's tables the longitude  $\lambda^*$  of the moon during the years  $+150$  to  $+169$  at moments with five days intervals can be found, totally 1461 values. This time interval begins in Ptolemy's last years of activity and its length, 20 years, is selected to have a fairly large material. By means of the above expounded methods,  $\lambda$  can be computed according to the three models at the same 1461 moments and then the absolute errors  $\Delta\lambda$  in  $\lambda$  can be found. To make these computations quickly and safely, they have been made by means of a computer (GIER, Københavns Universitets Matematiske Institut) in the following way.

From Ptolemy's tables is at first manually calculated  $\lambda_m(t_1)$ ,  $v_m(t_1)$ , and  $\eta(t_1)$  at the moment  $t_1 = +150$ , Jan. 5.75, Alexandrian local time. The computer is fed with these mean values and the mean velocities  $\dot{\lambda}_m$ ,  $\dot{v}_m$ ,  $\dot{\eta}_m$ . The true longitudes  $\lambda$  of the moon, according to the three models, are given by the formulas

$$\text{Model 1: } \lambda = \lambda_m + \tan^{-1} \frac{-r \sin v_m}{R + r \cos v_m}$$

$$\text{Model 2: } \lambda = \lambda_m + \tan^{-1} \frac{-r \sin v_m}{OC + r \cos v_m}$$

$$\text{Model 3: } \lambda = \lambda_m + \tan^{-1} \frac{-r \sin v}{OC + r \cos v}$$

where  $OC$  and  $v$  are given by the formulas (5), (6), (8), and (9).

Now, by means of this information the values for  $\lambda$  according to the three models at the 1461 moments are computed, and by comparing these with Tuckerman's values  $\lambda^*$ , the absolute errors  $\Delta\lambda = \lambda^* - \lambda$  can be found at once. The computer has also found  $\Delta\lambda$ , being fed with Tuckerman's values before running the program. All results are computed in degrees with one decimal, since Tuckerman calculated with this accuracy.

In fig. 6 pillar diagrams of the result of this examination are drawn, the height of the pillars for each value of  $\Delta\lambda$  signifies how often this error appears.

It is at once noted that in model 1  $-2.0^\circ \leq \Delta\lambda \leq 4.1^\circ$ , in model 2 the corresponding interval is somewhat smaller  $-1.3^\circ \leq \Delta\lambda \leq 3.4^\circ$ , and model 3 has the smallest interval  $-0.3^\circ \leq \Delta\lambda \leq 2.5^\circ$ . This shows that Ptolemy actually has improved the lunar theory by constructing model 2 and model 3, but it is remarkable that the longitude of the moon computed according to the three models has a tendency to be smaller than the real longitude of the moon. This is especially clear by model 3, where only about 1% of the errors are negative.

This distortion of the errors must be a result of a systematical error in the lunar theory, originating in the following facts. The mean motion of the moon in longitude according to Ptolemy is too slow (section 3) and the moon is started with an error of  $2\frac{1}{2}^\circ$  (section 4). Both errors originate in corresponding errors in the solar theory. At a definite moment  $t_0$  in history, Ptolemy's moon coincides in the mean with the real moon. Before this moment Ptolemy's moon is ahead of the real moon, after this moment behind it. Since we know that the longitude of Ptolemy's moon diminishes about  $0.41^\circ/100$  years, the moment  $t_0$  can be fixed if only the systematical error at one moment (e.g. +160) is known.

If there was not a systematical error in the computation of the longitude of the moon according to Ptolemy, but only casual errors, the mean values of the errors would be about  $0^\circ$ . If, on the other hand, the mean

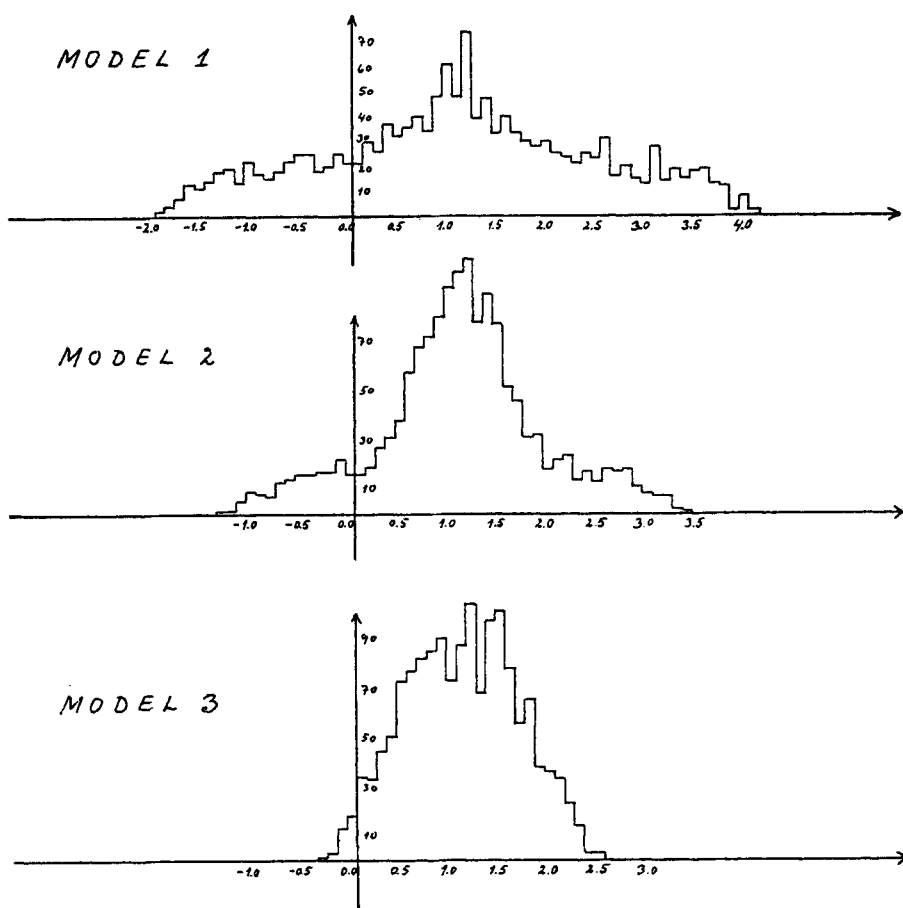


Fig. 6

value of the errors differ essentially from  $0^\circ$ , the size of the mean value must originate in a systematical error, the size of which is then fixed. A calculation shows the following mean values of the 1461 errors:

$$\left. \begin{array}{l} \text{Model 1: } 1.110^\circ \\ \text{Model 2: } 1.108^\circ \\ \text{Model 3: } 1.107^\circ \end{array} \right\} \text{average } 1.108^\circ \sim 1.11^\circ.$$

Now, it is reasonable to suppose that on an average Ptolemy's moon

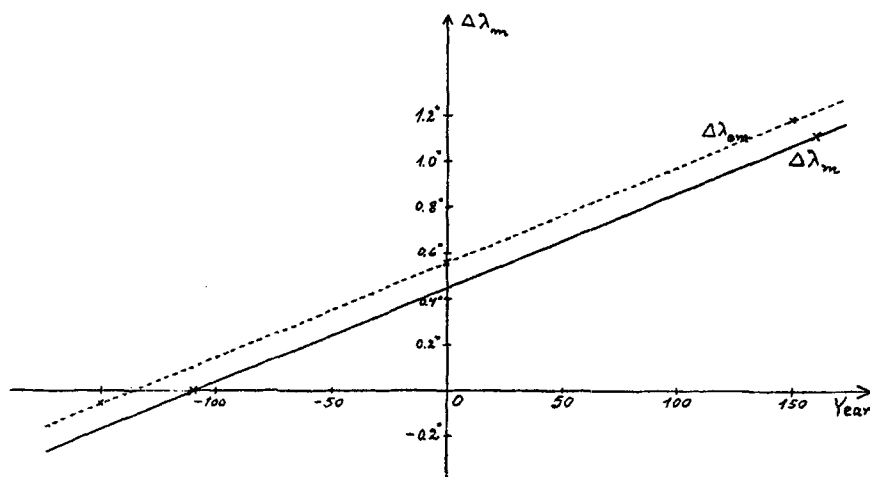


Fig. 7

is  $1.11^\circ$  behind the real moon in the year  $+160$ . The mentioned moment  $t_0$  is then given by the solution of the equation

$$\begin{aligned} (160 - t_0) 0.0041 &= 1.11 \\ 0.656 - 1.11 &= 0.0041 \cdot t_0 \\ t_0 &= -\frac{0.454}{0.0041} \cong -110 \end{aligned}$$

If the mean error in Ptolemy's moon is called  $\Delta\lambda_m$ , then  $\Delta\lambda_m(160) = 1.11^\circ$  and  $\Delta\lambda_m(-110) = 0.00^\circ$  is now known. This gives rise to drawing a graph (a straight line, fig. 7), from which the mean error  $\Delta\lambda_m$  is read at any time in the time interval  $-160$  to  $+160$  (where  $\Delta\lambda_m$  is regarded as a constant:  $\Delta\dot{\lambda}_m = 0.0041^\circ/\text{year}$ ).

*Note.* It is remarkable that  $\Delta\lambda_m$  has very near the same size as the corresponding mean error  $\Delta\lambda_{\odot m}$  in the solar theory, found and described by Petersen and Schmidt p. 88–90. The graph of  $\Delta\lambda_{\odot m}$  is given as a dotted line in fig. 7. So Ptolemy has a fine computation of the mean elongation  $\eta_m$ ; during the whole period from  $-160$  to  $+160$  it is true that  $\Delta\eta_m = \Delta\lambda_m - \Delta\lambda_{\odot m} \cong -0.1^\circ$ .

Corresponding to the accuracy, with which the errors are computed (1 dec.) it is assumed that  $\Delta\lambda_m = 1.1^\circ$  during the years from  $+150$  to

+169. The three pillar diagrams in fig. 6 also show symmetry relating to this value. If a true picture is wanted of the value of Ptolemy's lunar theory it is necessary to correct for this systematical error. After such a correction the following intervals are obtained:

Model 1:  $-3.1^\circ \leq \Delta\lambda \leq 3.0^\circ$

Model 2:  $-2.4^\circ \leq \Delta\lambda \leq 2.3^\circ$

Model 3:  $-1.4^\circ \leq \Delta\lambda \leq 1.4^\circ$

which definitely shows the superiority of model 3, and also shows that *model 3 is much better than model 2*.

An other measure for the quality of the models can be obtained by finding the "standard deviation"  $S$ , given by

$$S = \sqrt{\frac{1}{1460} \sum (\Delta\lambda)^2}$$

The result of such a computation is

Model 1:  $S = 1.37^\circ$ , model 2:  $S = 0.83^\circ$ , model 3:  $S = 0.57^\circ$ ,

consequently, again model 3 is the best one.

It is interesting to get an answer to another question: Do the causes, on account of which the models are determined, make themselves known in the numerical material, i.e. have the three models approximately the same (small) errors at syzygies, is model 2 better than model 1 at quadratures, and is model 3 better than model 2 in the octants?

The answer is given in fig. 8, made in the following way: When the computer ran the above sketched program, it also gave the mean elongation  $\eta_m$  for each of the 1461 moments. Now, it is possible to go through the material and for each tabulated value of  $\eta_m$  read the corresponding errors  $\Delta\lambda$ . For practical reasons the interval of  $\eta_m$  from  $0^\circ$  to  $360^\circ$  is divided into intervals (decades) of the length  $10^\circ$ , and the result of this is that then in any of these decades the smallest and the largest value of  $\Delta\lambda$  is found. These extremes are marked in fig. 8.

From this it is at once seen that all three models have the smallest errors around the syzygies, while model 2 is definitely better than model 1 at the quadratures and further that model 3 has smaller errors than model 2 at the octants (even if model 3 has still the largest errors at the octants). In this examination the systematical error of  $1.1^\circ$  is taken into

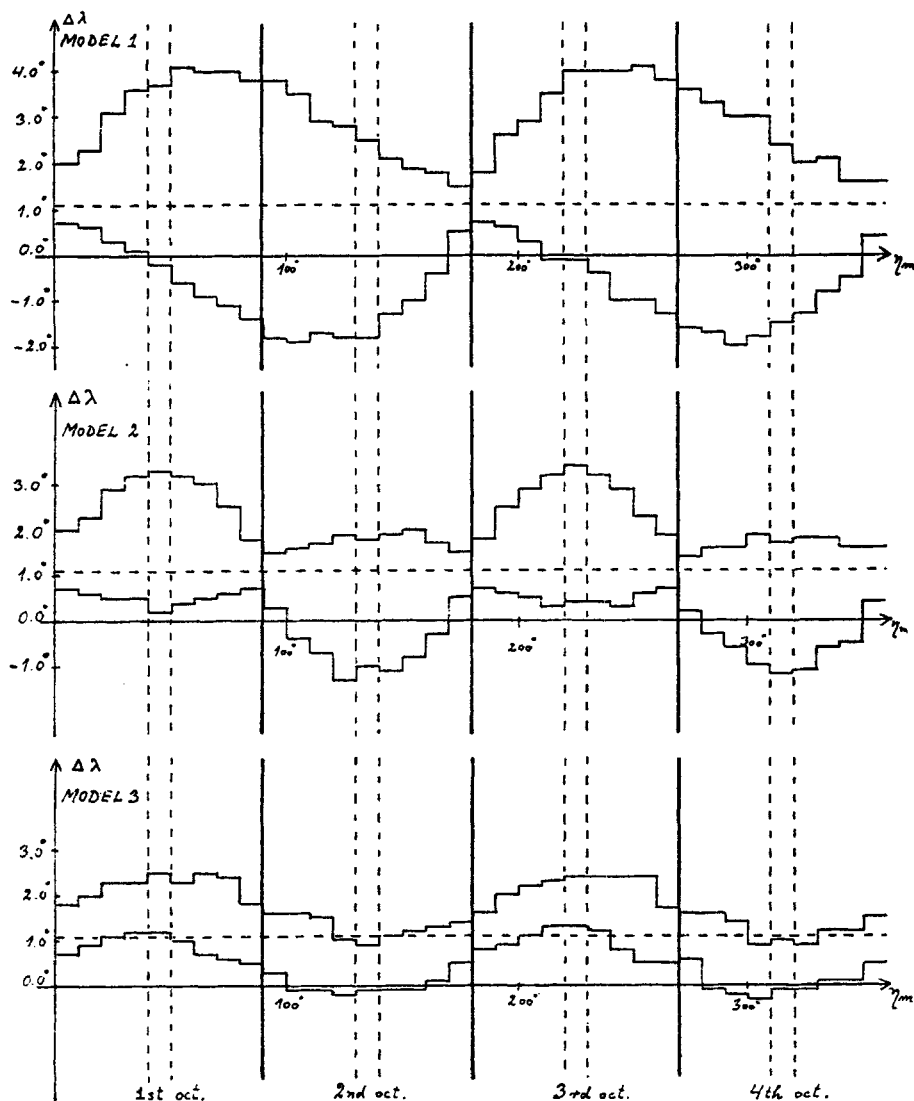


Fig. 8

account so that regarding fig. 8 the dotted horizontal line is to be used as starting point.

*Note.* In all three models Ptolemy's moon has a tendency to be behind the real moon for  $\eta_m \cong 45^\circ, 215^\circ$ , while it is ahead of the real

moon for  $\eta_m \cong 135^\circ, 315^\circ$ . This is due to the fact that the models do not include the *variation* which causes that the moon in 1st and 3rd octant can be nearly 40' ahead of the ecliptical position, while in 2nd and 4th octant it can be nearly 40' behind of the ecliptical position.

Finally, it is noted that all the results in this section are based on Tuckerman's tables and are therefore as accurate as these tables. However, the main result is firm: Ptolemy's complicated model 3 is superior to not only Hipparchus' model 1, but also to his own model 2.

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