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Aristotle and Mathematics

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I. How Mathematics Appears in Aristotle

ARISTOTLE'S writings often make appeals to mathematics, in various ways and for various purposes.¹ Here is a first, approximate classification.

(1) *Incidental allusions and examples.* There are many passing allusions to axioms, theorems, and procedures of the mathematics of Aristotle's own time, used as analogies and illustrations. (One or two such—the incommensurability of the diagonal of the square with its side, the theorem that the angles of a triangle add up to two right angles—are among his stock examples.) They make it clear that Aristotle was just as interested as Plato was in what the mathematicians of his day were actually doing.

(2) *Mathematics as a paradigm of science.* In the *Analytics*, and particularly in *Posterior Analytics* 1, where Aristotle seeks to determine the explanatory structure of any possible science, mathematics is used systematically as a paradigm case. So we find, not just individual mathematical procedures and methods cited as illustrations, but a concern with, and discussion of, the overall logical and explanatory structure of mathematics.²

(3) *Philosophy of mathematics.* Aristotle's own philosophical views on the nature and foundations of mathematics, its objects and its truths, are stated at some points in the *Metaphysics* and *Physics*. So too are his views on how mathematics differs from, and how it is related to, other sciences and especially natural science. Unfortunately, his positive views are mostly given tersely and in passing, though there are extended critiques of rival (mostly Pythagorean and Platonist) views.³

(4) *Mathematics in natural science.* Finally, Aristotle *uses* mathematics

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¹ The pseudo-Aristotelian works *Mechanics*, *On Indivisible Lines*, and *Problems* (presumably later products of Aristotle's school) are not considered here. T. L. Heath, *Mathematics in Aristotle* (Oxford, 1949), gives many of Aristotle's mathematical passages in English translation, with commentary from the point of view of the historian of mathematics.

² Cf. Heath (n. 1), 37–75.

³ Pythagorean and Platonist views of mathematics are discussed at *Metaph. A* 5, 987⁹–21; *A* 8, 989²⁴–990³²; *M* 1–3 and 6–9; *N* 2–6.

in his natural science. Since our theme is *science*, this is the area I shall concentrate on. There are various ways in which, for Aristotle, mathematics enters into natural science, and once again it is useful to start by subdividing the field to be surveyed.

(a) In some places Aristotle claims to determine fundamental questions about the structure of the natural world by an appeal to mathematical truths, thereby implying a kind of subordination of physics to mathematics.

(b) Aristotle uses particular kinds of mathematicized science, in the study of particular physical phenomena: e.g. geometrical optics in the study of the rainbow. Here too he seemingly insists that, in these areas at least, physics is in some sense, and to some extent, subordinate to mathematics.

(c) Aristotle himself carries out mathematical investigations into questions which underlie whole fields of natural science. In particular, he *creates* mathematics (or at least makes an elsewhere unparalleled use of it) in his pioneering study of the structure of continua such as time-stretches, changes, spatial and other quantities. He simultaneously *uses* the results to determine the structure of the natural world.

Finally, (d) he formulates in mathematical terms general principles in which mathematical relationships (proportionalities) are said to hold between physical quantities involved in processes and states in the natural world.

It is this area that I shall explore. But something must be said first, as a foundation, about Aristotle's own substantive philosophy of mathematics.

2. Mathematics and the World of Experience

Everything rests on the original connection which Aristotle makes between mathematics and the world of experience. Unlike Plato, he takes mathematics to be, of its very nature, firmly and necessarily rooted in the world of ordinary experience. Mathematical objects and mathematical truths are seen as somehow in correspondence with objects in this world, and with truths about *them*. So the questions are: *how* do they correspond? and, with *which* objects and truths in the ordinary world?

On *how* they correspond, Aristotle says that to pass from the world to mathematics is to go through a process of 'abstraction'. We may sidestep the highly controversial question about the precise meaning of 'abstraction', since for the purposes of this survey it is not essential to decide it.⁴

On the question of *what* ordinary objects and truths mathematics cor-

⁴ On mathematics as abstracting from the world: *Metaph.* M 3, 1077^b17–1078^a31; also *Ph.* 2, 2, 193^b22–194^a12; *De an.* 3, 4, 429^b18–22; 3, 7, 431^b12–19; *Metaph.* K 3, 1061^a28–b⁷; on sciences generally as always presupposing experience and knowledge of the world, *An. pr.* 1, 30, 46^a17–22; *An. post.* 1, 1, 71^a1–17; 1, 10, 76^a31–6, b³–11, b¹⁸–19; 1, 18, 81^a38–b⁹. For recent discussion on 'abstraction' and mathematical objects in Aristotle see I. Mueller, 'Aristotle on

responds to, we must distinguish between arithmetic and geometry. Arithmetic is concerned with what can be *counted* (pluralities of discrete objects), and with the properties that things have, in so far as they are countable. Geometry is concerned with what can be *spatially measured*, and with the properties things have in so far as they are measurable. What is primarily countable or measurable is a quantity, and it is the possibility of counting or measuring quantities that in the end makes mathematics possible as a science bearing on the actual world. The differences, and the interdependence, between counting and measuring, which Aristotle investigates with some care, are the grounds for the difference and the interdependence between arithmetic and geometry.⁵ (Aristotle does sometimes envisage a more general treatment—like the one in Euclid *Elements* 5, attributed to Eudoxus—which can handle both arithmetical and geometrical quantities.⁶ In one place he says that principles of mathematics are the concern of 'first philosophy', i.e. general ontology,⁷ but this in no way denies the close connection between mathematics and the actual world of experience.)

Hence, first, arithmetic and geometry are sciences of exceptional generality: they can be applied to all sorts of things, since they are not confined to being true of only one particular kind of substance. Aristotle seems to want to make this generality the explanation for the characteristic 'exactness' (*akribēia*) of mathematics. At any rate, he supposes that mathematics, unlike natural science, *fits* the ordinary world in an exact and wholly exceptionless way.⁸

While mathematics is, in this sense, concerned with the world of experience, it does not follow that it is a branch of natural science, which for Aristotle is confined to the study of the *natural*, i.e. of natural processes and natural substances. Countability and measurability have no *direct*

Geometrical Objects', in J. Barnes, M. Schofield, and R. Sorabji (eds.), *Articles on Aristotle*, iii. *Metaphysics* (London, 1979), 96–107; J. Lear, 'Aristotle's Philosophy of Mathematics', *Philosophical Review*, 91 (1982), 161–92; J. Annas, 'Die Gegenstände der Mathematik bei Aristoteles', in A. Graeser (ed.), *Mathematics and Metaphysics in Aristotle* (Proceedings of the Tenth Symposium Aristotelicum; Bern, 1987), 131–47; E. Hussey, 'Aristotle on Mathematical Objects', in I. Mueller (ed.), *Περὶ τῶν μαθημάτων* (Apeiron, 24.4; Edmonton, 1991), 105–33; J. J. Cleary, *Aristotle and Mathematics: Aporetic Method in Cosmology and Metaphysics* (Leiden, 1995), chs. 3–5.

⁵ On quantities, numbers, and the nature and presuppositions of counting and measuring: *Cat.* 6, 4^a20–6^a35; *Ph.* 3, 7, 207^b7–10; 4, 11, 219^b5–9; 4, 12, 220^a8–22; 4, 14, 223^a21–9, 224^a2–15; *Metaph.* Δ 6, 1016^b17–24; 13, 1020^a7–11; I 1; and, on the difference between arithmetic and geometry: *An. post.* 1, 7, 75^a38–b²⁰; 1, 27, 87^a35–7; *Metaph.* I 3, 1061^a20–b³. On Aristotle on numbers and arithmetic, M. Mignucci, 'Aristotle's Arithmetic', in Graeser (n. 4), 175–211, is generally useful.

⁶ *An. post.* 1, 5, 74^a17–25; 1, 24, 85^a28–b¹⁵; 2, 17, 99^a1–16; *Metaph.* E 1, 1026^a23–7; K 7, 1064^b8–9; M 2, 1077^a9–10; 3, 1077^b17–22.

⁷ *Metaph.* K 4, 1061^b19–21. cf. F 3, 1005^a19–29.

⁸ Mathematics is 'not (said) of any substrate': *An. post.* 1, 13, 79^a6–10; 1, 27, 87^a31–7; mathematics is 'exact' (*ἀκριβής*): *An. post.* 1, 27, 87^a31–7; *Cael.* 3, 7, 306^a26–30; *Metaph.* A 2, 982^a23–8; M 3, 1078^a9–17.

connection with the definition of nature as 'a principle of change and of rest'. Hence mathematics and natural science must be distinct, and logically independent of one another.⁹ It might seem, then, that any connection or overlap between them will be purely accidental; but it turns out that the situation is more complicated than that.

3. Some Mathematical Branches of Natural Science

In some places Aristotle notes a kind of overlap between mathematical and physical science. In the *Posterior Analytics* he claims that in such cases there are really two sciences in play: one, the subordinate science, is concerned with the *fact*, and is part of natural science; while the higher science is concerned with the *explanation*, and is part of mathematics. The examples given are: optics, mechanics, harmonics, and astronomy.¹⁰ He himself occasionally uses mathematical methods in writing about such areas.¹¹

In other works he seems to give a slightly different analysis, according to which there is but one science involved in each case, and one which is a branch of natural science, though mathematics abstracts from it; so that there is still a branch of mathematics that *corresponds* to it, and its explanations are *derived* from mathematics.

He notes that ordinary usage often does not distinguish between the corresponding branches—that, for example, both empirical and mathematical astronomy are called *ἀστρολογία*—and he himself happily uses *ἁρμονική* or *ἀστρολογία*, in different places, to denote a branch of mathematics, or a branch of physics. But he is consistent in always insisting that the branches of natural science in question are somehow subordinate to, or dependent upon, the corresponding branches of mathematics. The claim is that mathematics supplies something indispensable and fundamental in these fields, since it determines the *explanations* of the physical facts. Since the explanations are for Aristotle the essential part of any science, this already amounts to a strong kind of logical subordination.

⁹ Mathematics distinct from natural science: *Ph.* 2, 193^b23–194^a12; *Metaph.* E 1, 1026^b6–15; *K* 7, 1064^a30–3.

¹⁰ This section is based on *An. post.* 1, 7, 75^b14–17; 1, 9, 76^b9–15, 22–5; 1, 10, 76^b3–11; 1, 13, 78^b32–79^a16; *Ph.* 2, 193^b25–35, 194^a1–12; *Cael.* 2, 14, 297^a2–6; *Metaph.* M 2, 1076^b39–1077^a9; *M* 3, 1078^a14–17.

¹¹ *Mete.* 3, 5, 375^b16–377^a29, appeals to geometrical optics to explain rainbows; the details are obscure. *Metaph.* A 8, 1073^b8–1074^a17, uses a modified version of Eudoxus' concentric-spheres model for the motions of sun, moon, and planets. *Sens.* 3–7 invokes arithmetical ratios, in analogy with musical theory, to explain phenomena of simple and mixed colours, flavours, and odours. There are other incidental appeals to elementary mathematics in the physical works; cf. also §§ 6 and 7 below on mathematical analysis and proportionalities in physical explanations.

4. Mathematics Indicates Fundamental Limitations on the Structure of the Natural World

Further evidence reveals more about this 'subordination' of natural science to mathematics, and makes it clear that Aristotle conceives of the relationship between mathematics and the world of experience in a different way from most modern philosophers.

The most striking example of this 'subordination' is found in Aristotle's rejection of physical atomism, and the grounds he gives for it. Against the possibility that there could be physically indivisible bodies (as asserted by the Atomists), he invokes a fundamental principle of Euclidean geometry: that every line can be divided into two smaller lines, and hence that the division process can proceed *ad infinitum*.¹²

To understand Aristotle's unexpressed reasoning here, we must start from his general view about the truths of mathematics. Since for him they are essentially and basically truths about the world of experience, it follows that the fundamental principles of arithmetic or geometry must have something substantial to say about the world: they must indicate real limitations on what the world can be like.

In what sort of way could it be thought that mathematics 'puts limits' on the nature of the world? Mathematical truths express generalizations from experience. Thus, ' $2 + 2 = 4$ ' expresses the truth that if you add together two non-overlapping countable collections of two *X*s each (where '*X*' corresponds to any concept under which things can be counted), you always get a collection of four *X*s. Here, the *X*s are actual objects in the world (e.g. horses or dogs). The process of adding together the two collections must also correspond to actual processes which might actually occur in the world (e.g. putting all one's horses into the same field), though of course addition may have useful applications to the actual world even when no such merging process actually occurs.

The same is true of geometrical truths. Thus, the truth that every line can be bisected must, if it is really a principle of geometry, correspond to, and be derived from, a fact about the actual world: that every line in the actual world can literally and actually be cut in half. (Aristotle has no room for any distinction between geometry as the science of space, and physics as the science of things that occupy space. As shown by his discussion of *place* in *Ph.* 4, 1–5, there is no such thing for him as a self-subsistent space. Geometry is the science of whatever is spatially extended.)

¹² Divisibility *ad infinitum* of magnitudes used against atomism: *Cael.* 1, 5, 271^b6–11; 3, 4, 303^a20–4; 3, 7, 306^a26–30; *Ph.* 6, 1–2, 231^a21–233^b32; 6, 4, 234^b10–235^b6. (On Aristotle's rejection of atomism see the works cited in n. 18 below, and D. J. Furley, *Two Studies in the Greek Atomists* (Princeton, 1967), 111–30; on magnitudes as 'divisible everywhere' see n. 27 below.) See also § 14 of Milton's paper in this volume.

Given this understanding of mathematical truths, it is not fallacious or irrelevant (as has sometimes been thought) for Aristotle to invoke a geometrical principle to refute physical atomism. It is nevertheless surprising (to a twentieth-century reader) that he shows such unqualified and unexplained confidence in the truth of that particular principle, and in the reliability of 'Euclidean' geometry generally (rather than some other possible geometry, e.g. one allowing indivisible lengths) as a picture of the spatial aspects of the physical world.¹³

What grounded his confidence about this we cannot be sure. But his mention of 'the most *exact* sciences' in this connection suggests that he was invoking the apparently exact fit, in most other respects, between Euclidean geometry and the world. He could point to the elegance, the power, and the empirically confirmed practical usefulness of Euclidean geometry,¹⁴ as giving support to his claim, as well as to the empirical fact that no actually indivisible extended things had ever been discovered. All this would have had to be given up if atomism had been admitted, since, as he notes, the rejection of divisibility *ad infinitum* would amount to a radical revolution in geometry.

If the principle of divisibility *ad infinitum* was, for Aristotle, to be grounded in the ways that have been suggested, that would be in accord with his general theorizing about the discovery and establishment of first principles in the sciences.¹⁵

This appeal to mathematics in the refutation of atomism is not an isolated instance of the subordination of physics to mathematics, though it is the most striking one. Less obvious, perhaps, but equally far-reaching, is the use of (supposed) mathematical truths in *De caelo* to ground fundamental properties of the cosmos as a whole. Its three-dimensionality is explained by the claim that magnitudes in general cannot be more than three-dimensional. Its sphericity and its division into an upper and a lower region, and the properties and motions of the 'simple bodies', are all ultimately based in part on the analysis of all motions as compounds of two simple types: linear and circular. Sphericity of the cosmos is also grounded on arguments using an analysis of geometrical shapes.¹⁶

¹³ It is possible that Democritus had already tried to produce an atomistic alternative to Euclidean geometry, but the evidence is sparse and inconclusive.

¹⁴ Euclidean geometry was in practical use in antiquity in (e.g.) land-surveying, town-planning, map-making, and the construction of tunnels: see O. A. W. Dilke, *Greek and Roman Maps* (Ithaca, NY, and London, 1985); T. E. Rihill and J. V. Tucker, 'Greek Engineering: The Case of Eupalinos' Tunnel', in A. Powell (ed.), *The Greek World* (London, 1995), 403–31. In these applications to the world of experience, it functions as a part of physics (cf. R. Penrose, *The Emperor's New Mind* (Oxford, 1989), 156–62).

¹⁵ On Aristotle's view of how the principles of sciences are arrived at, see M. Burnyeat, 'Aristotle on Understanding Knowledge', in E. Berti (ed.), *Aristotle on Science: The Posterior Analytics* (Padua, 1981), 97–139.

¹⁶ Three-dimensionality of magnitude: *Cael.* 1. 1, 268^b7–^b5; analysis of motions into circular

Above all, there is the use of the so-called 'Axiom of Archimedes' to show the impossibility of *infinitesimal* quantities in physics, a principle just as fundamental for Aristotle as the impossibility of physical atoms.¹⁷

The importance to Aristotle of this last principle is worth underlining. The 'Axiom of Archimedes' (presumably formulated by Eudoxus, if not earlier, and therefore sometimes more reasonably called the 'Axiom of Eudoxus') states that, of any two magnitudes of the same sort, either is less than some finite multiple of the other. It is a linchpin of the general theory of magnitudes and ratios formulated in Euclid *Elements* 5, which is thought to be the creation of Eudoxus. For Aristotle, its importance lies in the fact that it ensures that the proportion between any two magnitudes of the same kind will always be a proportion that one finite quantity bears to another finite quantity. So it guarantees, in particular, that proportionalities in physics (see § 7 below) will never require the introduction of infinite or infinitesimal quantities.

5. The Mathematical Structure of Continuity

Mathematics, in this way, indicates limitations on the structure of the world of experience (and thereby indirectly on the shape of natural science as a whole). Yet one might still doubt whether mathematics, for Aristotle, had to be in any way *directly* and *systematically* relevant to any part of his natural science. The decisive evidence that this is indeed so is given by his treatment of *continuity*.

Aristotle takes change itself, as well as time, to be like a geometrical line in being both continuous and one-dimensional. But he goes further: in a decisive step, he notices and explores the consequences of the fact that *all one-dimensional continua share a common abstract structure*. This common structure can therefore be the subject of a corresponding science; and since the structure inheres in the continua by virtue of their measurability, the science must be closely related to geometry. Aristotle spends much time

and linear: 1. 2, 268^b14–269^a30; analysis of shapes: 1. 2, 268^b19–20; 2. 4, 286^b11–287^a5, cf. *Ph.* 2. 2, 193^b30.

In one important respect there is a less close fit than might have been expected between mathematics and the world: the cosmos is spatially finite and of fixed size, although geometry is happy to consider arbitrarily large spatial magnitudes. On Aristotle's modified finitism (expounded in *Ph.* 3. 4–8), akin to intuitionism, in his philosophy of mathematics, and its relation to the world of experience, see e.g. J. Lear, 'Aristotelian Infinity', *Proceedings of the Aristotelian Society*, 80 (1980), 187–210; E. Hussey, *Aristotle: Physics III and IV* (Oxford, 1982), xviii–xxvi and 72–98; M. J. White, *The Continuous and the Discrete: Ancient Physical Theories from a Contemporary Perspective* (Oxford, 1992), 133–87.

¹⁷ 'Axiom of Archimedes': (e.g.) *Ph.* 1. 4, 187^a25–6; *Cael.* 1. 6, 273^a27–32; and see H. J. Waschkies, *Von Eudoxus zu Aristoteles: Das Fortwirken der Eudoxischen Proportionentheorie in der Aristotelischen Lehre vom Kontinuum* (Amsterdam, 1977), 308–18; White (n. 16), 62–9.

investigating this structure.¹⁸ He freely uses geometrical axioms, and refers to lettered diagrams and gives general abstract proofs in geometrical style.

Here, then, Aristotle is aware of a mathematical structure which lies at the very heart of all natural process. For him this can hardly be just an accident. He takes the continuity of natural changes as fundamental (in spite of certain partial exceptions),¹⁹ and as something that helps to make possible the overall unity and coherence of the natural world.

In exploring the common structure of one-dimensional continua, Aristotle, it seems, is extending the scope of Greek mathematics, for we know of no previous exploration of this kind. He considers not only spatially extended continua (bodies and their bounding surfaces, lines and points), but temporal ones (time-stretches, changes), and other physical quantities neither spatial nor temporal. In treating time-stretches as wholly analogous to lines, Aristotle is formally 'spatializing' time—an indispensable step on the way to a truly mathematical physics.²⁰ Just as noteworthy is Aristotle's willingness to draw physical quantities generally within the scope of his analysis. Such quantities as 'power' or 'weight' are 'continuous' and 'one-dimensional' in the extended sense that they can be represented by positive numbers, and admit of continuous increase and decrease; and these quantities too may therefore be represented by lines.

These initial steps towards a truly mathematical physics, and the accom-

¹⁸ (a) Definition of 'continuous': *Ph.* 5. 3, 227^a10–17, cf. 226^b34–227^a6 and 6 1, 231^a21–3; (b) role of continua in mathematics: *Ph.* 6. 1, 231^a24–^b18; *Metaph.* K 4, 1061^b21–4; (c) physical magnitudes, changes, time-stretches as continua, as divisible everywhere, divisible *ad infinitum*: *Ph.* 3. 6, 207^a21–3; 3. 7, 207^b15–17; 4. 11, 219^a10–14; 6. 1–2, and 6. 4. Structural correspondence between continua given by the 'following' (*ἀκολουθεῖν*) relation: *Ph.* 4. 11, 219^a10–220^a10; 8. 7–8, 261^a31–265^a12. Part of Aristotle's motivation in all this (but only part) is to answer Zeno's paradoxes.

On Aristotle's theory of the continuum, the best study of the mathematical aspects is Waschki (n. 17); see also White (n. 16), 133–87; and on some broader related questions W. Korr, 'Infinity and Continuity: The Interaction of Mathematics and Philosophy in Antiquity', in N. Kretzmann (ed.), *Infinity and Continuity in Ancient and Mediaeval Thought* (Ithaca, NY, and London, 1982), 112–45. Some philosophical aspects are treated in R. Sorabji, 'Aristotle on the Instant of Change', in Barnes, Schofield, and Sorabji (n. 4), 159–77. S. Waterlow, *Nature, Change and Agency in Aristotle's Physics* (Oxford, 1982), 131–58; R. Sorabji, *Time, Creation and the Continuum* (London, 1983), chs. 21, 24, and 26; D. Bostock, 'Time and the Continuum', *OSAP* 6 (1988), 255–70, id., 'Aristotle on Continuity in *Physics* VI', in L. Judson (ed.), *Aristotle's Physics: A Collection of Essays* (Oxford, 1991), 179–212.

¹⁹ But some (non-central) types of change may be *instantaneous*: see *Ph.* 6. 4, 235^a13–18, 24–7, 236^b1–18; 6. 9, 240^a19–^b7; 8. 3, 253^b23–6; *Sens.* 3–6.

²⁰ This 'spatialization' of time, for the purposes of mathematical physics, does not imply any philosophical thesis about the nature of time in itself. Aristotle's theory of time (*Ph.* 4. 10–14) in fact insists on the reality of temporal 'flow', and generally on the differences between temporal and spatial continua as well as the analogies. On Aristotle's philosophy of time: G. E. L. Owen, *Logic, Science and Dialectic: Collected Essays in Greek Philosophy* (London, 1986), 295–314; Hussey (n. 16), xxxvi–xlix, 138–75; Sorabji, *Time, Creation, and the Continuum* (n. 18), chs. 1, 4, 6, 7; Bostock, 'Time and the Continuum' (n. 18); White (n. 16), ch. 2.

In studying the systematic correspondence between times, changes, distances travelled, etc., by means of the relation of 'following', Aristotle comes close to formalizing explicitly the notion of *function* in the mathematical sense, which is implicit in his work.

panying mathematical investigation of the continuum, are one of Aristotle's greatest achievements. It must be admitted that Aristotle himself never explains clearly what he is doing. His exposition, in *Ph.* 4 and 6, is often tangled and obscure. Further, Aristotle does not *label* what he is doing as 'mathematics' or 'mathematical physics'. It is obviously not part of arithmetic, nor, quite, of geometry, though it includes geometrical truths within a more general framework. It contains the first beginnings of the mathematical discipline now known as 'topology'.

6. Mathematical Analysis and Synthesis in Natural Science

These steps taken by Aristotle, it may be said, are only *steps in the direction* of a mathematical physics: they do not actually *constitute* a mathematical physics, and Aristotle never constructed any such thing. It is certainly true that natural science, as understood by Aristotle, was not *completely* subordinate to mathematics in the way in which some of its specialized branches were. We have only to think of his biology. But even within the more restricted area of what we would call 'physics' and 'chemistry', Aristotle shows no sign of wishing to make a complete subordination of physics to mathematics. A thoroughly mathematical physics, then, of the modern kind, was never his aim.

So the interesting questions are: (1) how important in Aristotle's natural science were the mathematical, i.e. the countable and measurable, properties and relationships of natural substances and processes? and (2) how far did Aristotle suppose that the scientific study of those properties and relationships had to be subordinated to mathematics?

The evidence shows, I claim, both that countable and measurable properties were (not surprisingly) important in certain branches of Aristotelian natural science, and that Aristotle at least envisaged, though he did not carry through, a wholly mathematical treatment of the relationships holding among those properties in natural processes. In this sense he is (at least in intention) the 'first mathematical physicist'. Not only are his investigations of natural continua (§ 5) difficult to understand as anything other than *preparations* for a thoroughly mathematical treatment. There is more substantial evidence that that is just what they were intended to be.²¹

First, there is good evidence of Aristotle's readiness to use mathematical

²¹ Aristotle's use of mathematics in physics naturally cannot be separated from his substantive physical theories, and the concepts with which he operates: in particular, the theory of the motions of bodies (as intermittently expounded in *Ph.* 4 and 8 and *Cael.* 3–4) and the concepts of 'power' (*δύναμις*) and 'impulse' (*ῥοπή*), and of 'weight (heaviness)' (*βάρος*) and 'lightness' (*κοιφότης*). Twentieth-century scholarship in this area has mostly followed the lead of Henri Carteron's brilliant book (*La Notion de force dans le système d'Aristote* (Paris, 1923; repr. New York and London, 1979)) in minimizing the mathematical aspects and the analogies and connections with Newtonian concepts. Against Carteron's view, I. E. Drabkin's

operations in the physicist's analysis and synthesis of natural states and processes. In every well-developed physical theory there has to be some systematic recognition of complex physical situations which are 'compounds' or 'superpositions' of simpler situations: for example, two or more forces acting on an object at once, or two or more elemental ingredients in it. The physical theory has to give a description of the resultant thing or process, and it is one of the hallmarks of a mathematical theory that it invokes mathematical relationships in such cases: the analysis and synthesis involved are reduced to mathematical procedures such as addition or multiplications of various quantities.

Aristotle in several places shows that he is thinking mathematically in such cases. One example has already been mentioned (§ 4): in *De caelo* the discussion of simple and complex motions, which shapes his whole treatment of the cosmos, appeals to mathematical analysis to show that there are two kinds of simple motion, into which all others can be analysed. Then there is the extended treatment of 'mixture' of elementary bodies in *De generatione et corruptione*, particularly at 2. 7, 334^b8–30, where properties of compounds are explained as 'mixtures' of properties of their simple components. The analysis of the concepts of *mixture*, *nourishment*, and *elemental change* is interwoven with a physico-mathematical analysis of what must be actually going on in mixtures: an example of the inseparability, in Aristotle's practice and thinking, of 'science' and 'philosophy'.²² In one place (*Mete.* 1. 4, 342^a24–6) he appeals to the law of vector addition for speeds and movements. The mathematical addition of the motions of concentric spheres is also the basis of the analyses of planetary motions by Eudoxus and Callippus, which Aristotle draws on and extends in *Metaph.* A 8, 1073^b17–1074^a18.²³

pioneering attempt ('Notes on the Laws of Motion in Aristotle', *AJP* 59 (1938), 60–84) to find 'mathematical laws of physics' in Aristotle is not wholly satisfactory, as shown by Owen (n. 20), 315–33. An attempt to rework Drabkin's interpretation is E. Hussey, 'Aristotle's Mathematical Physics', in Judson (n. 18), 213–42, on which I draw here.

²² On the problems of Aristotle's theory of mixture see H. H. Joachim, *Aristotle on Coming-To-Be and Passing-Away* (Oxford, 1922), 175–89; R. Sharvy, 'Aristotle on Mixtures', *Journal of Philosophy*, 80 (1983), 439–57; K. Fine, 'The Problem of Mixture', in E. A. Lewis and R. Bolton (eds.), *Form, Matter and Mixture in Aristotle* (Oxford and Malden, 1996), 82–182.

²³ On Eudoxus' model and Aristotle's use of it, some recent publications are: G. E. R. Lloyd, *Aristotelian Explorations* (Cambridge, 1996), ch. 8; 'Heavenly Aberrations: Aristotle the Amateur Astronomer'; id., 'Metaphysics A 8', in M. Frede and D. Charles (eds.), *Aristotle's Metaphysics Lambda* (Oxford, 2000), 245–73; H. Mendell, 'Reflections on Eudoxus, Callippus and their Curves: Hippopedes and Callippopedes', *Centaurus*, 40 (1998), 177–275; I. Yavetz, 'On the Homocentric Sphere of Eudoxus', *Archive for the History of the Exact Sciences*, 51 (1998), 221–78.

7. Proportional Relationships in Natural Science

The final, but again disputable, evidence is provided by those places where Aristotle assumes relationships of proportionality (as *A* is to *B*, so is *C* to *D*) between physical quantities in natural processes. These are prominent in *Physics* 7 and in some parts of *De caelo*.²⁴ Many scholars have denied that these statements are meant as anything like 'mathematical laws of physics', and it is true that they often appear with, in the first instance, a negative, dialectical purpose: to reduce an adversary's position to a contradiction.

It need not follow, though, that Aristotle himself put no faith in them. In fact, there are good reasons for thinking that he took them as at least rough guides to the truth of the matter. (1) The dialectical use occurs in so many places that we cannot suppose that his adversaries were the same set of people in all cases. Therefore it is just not plausible to see these assumptions of proportionality as merely *ad hominem*, and the natural conclusion is that Aristotle appealed to them as to something that within his own school, at any rate, would not be questioned. (2) In one case the appeal to a proportionality, though made in a polemical context, is expressly presented as an appeal to a fact of ordinary observation.²⁵ (3) There are places where the proportionalities occur, outside any argumentative context, ostensibly as part of Aristotle's own thinking. So, above all, at *Ph.* 7. 5, 249^b27–250^a28, where there is a general statement about the proportionality between the input of 'power' to an object, and the speed and amount of its resulting change.²⁶

More work needs to be done on this question. But, if we may assume that Aristotle meant the statements of proportionality seriously, that at least has the merit of giving an intelligible motivation for his investigation of the structure of physical continua. For there are certain things that need to be settled before one can confidently apply proportional relationships to any particular kind of quantity. Above all, one must know that there will always exist a fourth proportional. That is, if we say that as quantity *A* is to quantity *B*, so is quantity *C* to quantity *D*, we must know in advance that, whatever the particular quantities *A*, *B*, and *C* may be in particular cases, there will always exist a possible quantity *D* bearing that relationship to the other three. This is *not* a routine matter; we must know that the types of

²⁴ *Ph.* 7. 5, 249^b27–250^a28 (on which see Hussey (n. 21), 215–20: the theory of physical change); *Ph.* 4. 8, 215^a24–216^a11; 8. 10, 266^b6–24 (Hussey 227–39: the theory of motion); and e.g. *Cael.* 1. 5–7; 2. 8, 289^b15–16; 2. 9, 290^a34–291^a4; 2. 10, 291^a32–^b10; 2. 12, 293^a10–11; 2. 13, 294^a15, ^b5–6; 3. 2, 310^a26–^b16; 3. 5, 304^a24–^b9, ^b15–19; 3. 6, 305^a6–7, 11–13; *Mete.* 1. 3, 340^a3–19, 341^a35–6. For the view that the proportionalities are not stated as general truths of physics, see Carteron (n. 21), 1–32; Owen (n. 20), 329–32.

²⁵ *Ph.* 4. 8, 215^a25–^b12.
²⁶ The context is indeed partly 'dialectical', since it makes a counter (250^a19–28) to one of Zeno's arguments; but the counter is founded, not on the proportionality itself, but on a restriction to its applicability (250^a9–19).

quantity involved are mathematically well-behaved in certain ways. Now, if they have the structure of Aristotelian continua, they are guaranteed to be well-behaved enough for this purpose. For, like the quantities of Eudoxus' general theory as given in Euclid's *Elements* 5, they will obey the 'Axiom of Archimedes' (which bans infinitesimals: above, § 4), and they will also be 'divisible everywhere'. Whatever exactly 'divisible everywhere' means, it must be strong enough to guarantee the existence of the fourth proportional, and should therefore be listed as yet another mathematical principle with a direct bearing on the nature of the world of experience.²⁷

All of this suggests that Aristotle put some faith in his proportionalities, but does not tell us why he did so. It is a reasonable guess, though, that he saw in them, as in other principles of mathematics, the product of a dialectical negotiation between experience and generalization, leading to a reflective equilibrium.²⁸ In one case at least, as pointed out, he deduces them from observation. In many others they can be seen as formalizations of a thought intuitively 'obvious' or 'natural'. Thus the commonplace thought that the more 'power' put in, the greater or the swifter the change that results, is what underlies the formulation of *Ph.* 7. 5.

8. Summary and Conclusion

I have been looking at the way mathematics and the natural world are related, in Aristotle's theory and in his practice. The central points I have underlined are (1) Aristotle's characteristic philosophy of mathematics as derived from the world of experience; (2) his consequent conception of the partial subordination of natural science to mathematics; and, (3) on the detail of that subordination, his general study of the mathematical structure of continua; (4) the 'spatialization' of time and change as one-dimensional continua analogous to lines; and consequently (5) the creation of a mathematical theory of the intrinsic structure of changes of measurable quantities.

I have not tried to tell a chronological story. It may well be that there was a process of development in his thinking on these matters,²⁹ but the first requirement is to make sense of the texts, so far as possible without making any particular hypothesis about chronology.

²⁷ 'Divisible everywhere' may possibly indicate something analogous to the modern concept of *order-completeness*; if so, Aristotle (or someone earlier) anticipated Dedekind's construction of the 'real numbers'. On this possibility, see O. Becker, *Das mathematische Denken in der Antike* (Göttingen, 1966), 15, 108; White (n. 16), 133–87.

²⁸ See above, n. 15.

²⁹ It may be a sign of developmental change that in *Posterior Analytics* 1 Aristotle shows much enthusiasm for mathematics as a paradigm of science, but little interest in the application of mathematics to the natural world, while in other (possibly later) works it is the other way round.

The making of moves (1) to (5) was in itself epoch-making. In the end, it must be admitted, Aristotle did not get much further beyond them towards the construction of a substantive 'mathematical physics'. His steps in that direction look tentative. (It may quite well be that he saw what had to be done, better than he could himself do it—though this is a possibility Aristotelian scholars are always reluctant to consider.) But about the direction in which he was headed, there is no room for doubt.

That Aristotle's achievements and ambitions in this field have gone without full understanding and recognition for so long is something of a reproach to scholarship. Apart from the difficulties presented by the texts themselves, one might identify several factors. One is the exclusivity and 'retrospective imperialism' of different modern traditions. For example: philosophers whose thinking on the nature of mathematics has been formed by Frege's *Foundations of Arithmetic* will assume, rightly, that Frege can help us to grasp the sense and importance of things that Aristotle says about number. Too often they will assume also, and wrongly, that there cannot be any non-Fregean thoughts about number in Aristotle, or that any that there are are unimportant or just mistaken. A second factor is the prevalence of modern assumptions about the nature of mathematics, science, and philosophy and the relations between them. A third is the consequent compartmentalization of scholarly work. Philosophers study Aristotle 'as a philosopher', historians of science (or mathematics) study him 'as a scientist' (or as a source for the history of mathematics). Yet it is evident that, for Aristotle himself, mathematics, natural science, and 'first philosophy', though they are distinct fields of knowledge, are systematically and organically interconnected, and cannot be understood in isolation from one another. To explore and understand those interconnections, it is necessary first of all to take Aristotle seriously as someone who was, equally and simultaneously, all three: philosopher, scientist, and mathematician.

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