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COLOPHON

Music Theory & Analysis (MTA)

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Andrew HICKS

Re-interpreting an Arithmetical Error
in Boethius's *De institutione musica*

Abstract

In an influential 1981 article, “Interpreting an Arithmetical Error in Boethius’s *De institutione musica* (iii.14–16),” André Barbera drew attention to a problematic set of arithmetical proofs. At *Ins. mus.* 3.14ff. Boethius purports to prove that (1) the minor semitone is larger than three commas but small than four, (2) the major semitone is larger than four commas but smaller than five, and (3) the tone is larger than seven commas but smaller than eight. All of these conclusions are correct, but the mathematical procedure seems inherently flawed, for Boethius manipulates the numerical difference between the terms of a ratio as if it were an accurate quantification of the resultant interval. Barbera maintained that the rationale motivating the erroneous mathematics “lies at the heart of Pythagorean cosmogony.” Boethius (Barbera argued) set out to find the numerical truth underlying acoustic phenomena and “seems to have been satisfied by the apparent numerical verification of what he could hear.” The origin of this “arithmetical error” can be specified with greater precision than the vague invocation of “Pythagorean cosmogony” allows. First, I demonstrate that Boethius’s arithmetical error is not nearly as erroneous as Barbera and others would have us believe; rather, it represents an approximate method of calculation used to prove relationships otherwise incalculable. Secondly, I argue that this approximate method was not developed by Boethius but was faithfully translated from his immediate Greek source, Nicomachus of Gerasa’s (now lost) *Eisagoge musikē*. Thirdly, I suggest that this method, or at least its basic principle, was not independently developed by Nicomachus either, for similar arithmetical methods arose within the early stages of the Greek commentary tradition on Plato’s *Timaeus*.

Keywords

Boethius, *De institutione musica*, Nicomachus, Plato, ratio, interval, approximation

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Re-interpreting an Arithmetical Error in Boethius's *De institutione musica*

Andrew HICKS

In an influential article, “Interpreting an Arithmetical Error in Boethius’s *De institutione musica*, iii.14–16,” André Barbera highlighted a problematic set of three arithmetical proofs that conclude Boethius’s extended argument against the Aristoxenian equal division of the tone.¹ At *De institutione musica* 3.14–16 (summarized in Appendix I), Boethius mounts what he claims is a *demonstratio per numeros* (“numerical demonstration”) of three conclusions:

1. the minor semitone is larger than three commas but smaller than four (3.14),
2. the major semitone is larger than four commas but smaller than five (3.15), and
3. the whole tone is larger than eight commas but smaller than nine (3.16).

These results are correct; however, the arithmetical procedure by which Boethius attains these faultless conclusions seems faulty at best: Barbera deems the procedure an “arch arithmetical crime,”² Anja Heilmann calls it a “methodische[r] Schnitzer,”³ and Calvin Bower judges it a flat-out “mistake”—an Aristoxenian mistake at that, a grievous insult to a card-carrying Pythagorean like Boethius.⁴ What occasioned this sound scolding? In short, Boethius’s proof manipulates the numerical difference between the terms of a ratio as if it were a meaningful quantification of the resultant musical interval. This article seeks to absolve Boethius of all charges, and the absolution will proceed in three stages. First, I

My thanks to Calvin Bower, David Cohen, Gabriela Currie, Leofranc Holford-Strevens, and Nathan John Martin for their comments on earlier drafts, the anonymous readers of this journal, and colloquia audiences at Cornell and KU Leuven. Any remaining arch arithmetical crimes, *methodischer Schnitzer*, or flat-out mistakes are mine and mine alone.

1 André Barbera, “Interpreting an Arithmetical Error in Boethius’s *De institutione musica* (iii.14–16),” *Archives Internationales d’Histoire des Sciences* 31 (1981), 26–41.

2 Ibid., 30.

3 Anja Heilmann, *Boethius’ Musiktheorie und das Quadrivium: Eine Einführung in den neuplatonischen Hintergrund von “De institutione musica,”* Hypomnemata: Untersuchungen zur Antike und zu ihrem Nachleben 171 (Göttingen: Vandenhoeck & Ruprecht, 2007), 228.

4 Calvin Bower, *Fundamentals of Music: Anicius Manlius Severinus Boethius*, ed. Claude V. Palisca (New Haven, CT: Yale University Press, 1989), 110.

demonstrate that Boethius's arithmetical "error" is not nearly as erroneous as Barbera, Bower, and Heilmann would have us believe; rather, it represents an approximate method of calculation used to prove relationships that are otherwise incalculable. Secondly, I argue that this approximate method was not developed by Boethius but was faithfully translated from his immediate Greek source, Nicomachus of Gerasa's (now lost) *Εἰσαγωγή μουσική* (*Introduction to Music*). Thirdly, I suggest that this method, or at least its basic principle, was not independently developed by Nicomachus either, for similar arithmetical methods arose within the early stages of the Greek commentary tradition on Plato's *Timaeus*.

Before beginning the argument proper, I must lay out the relevant background by summarizing the two different conceptions of musical relations in Greek antiquity: λόγος (ratio) and διάστημα (interval).⁵ These in turn reflect the basic theoretical approaches of the two primary schools of Greek harmonics, the Pythagorean and the Aristoxenian.⁶ Pythagorean harmonics championed the thesis that musical relations are fundamentally and essentially quantitative; musically meaningful pitch relations (e.g., the octave, the fifth, and the fourth) are reducible to whole numerical ratios (2 : 1, 3 : 2, and 4 : 3, respectively), and any pitch relations that are *irreducible* to whole-number ratios are not (and cannot be) musically meaningful relations. What Pythagorean harmonics cannot readily quantify, however, is the distance (διάστημα) or difference (ὑπεροχή; lit., "excess") between such pitch relations on a numerical (or geometrical) continuum. Hence, when describing pitch relations as intervals, the primary reference is not the distance between the terms in the ratio, but a spread across (διὰ-) strings: διὰ τεσσάρων ("across four [strings]" or a fourth), διὰ πέντε ("across five [strings]" or a fifth), and διὰ πασῶν ("across all [strings]" or an octave). The interval of, say, the minor semitone, expressed by the ratio 256 : 243, is not (or at least, not in any meaningful way) reducible to the distance or difference between the terms in its simplest numerical expression (i.e., the interval of the semitone is not 13, the numerical "distance" between 256 and 243).

Intervals as measurable (if not, strictly speaking, quantifiable) distance find their most natural expression in the Aristoxenian tradition, for Aristoxenus took as primary not numerical quantity but the sense-perceptible continuity (or discontinuity) of vocal sound,

5 See Albrecht Riethmüller, "Logos und Diastema in der griechischen Musiktheorie," *Archiv für Musikwissenschaft* 42 (1985), 19–36, doi: 10.2307/930684.

6 The best introduction is Andrew Barker, *The Science of Harmonics in Classical Greece* (Cambridge: Cambridge University Press, 2007), 19–30, doi: 10.1017/CBO9780511482465; for a concise summary, see Andrew Barker, "Early *Timaeus* Commentaries and Hellenistic Musicology," in *Ancient Approaches to Plato's Timaeus*, ed. Robert W. Sharples and Anne Sheppard, *Bulletin of the Institute of Classical Studies* 78 (London: Institute of Classical Studies, University of London, 2003), 73–75, doi: 10.1111/j.2041-5370.2003.tb02135.x.

subjected to rigorous Aristotelian empiricism.⁷ Central to the Aristoxenian enterprise is the notion of musical motion (κίνησις) across an abstract pitch space (τόπος), analogous to the traversal of distances from point to point. The vocal motion of speech is continuous (συνεχής), for it does not articulate discrete points within a well-defined pitch space, whereas the vocal motion of a singing voice is intervallic (διαστηματική), moving discretely from note to note, each note (φθόγγος) articulating a single “incidence of the voice on one pitch” (φωνῆς πτώσις ἐπὶ μίαν τάσιν).⁸ Hence, an Aristoxenian interval (διάστημα) is “what is bounded by two notes that do not have the same pitch, since an interval appears, so to speak, to be a certain difference [διαφορά] between pitches and a space [τόπος] capable of receiving notes higher than the lower of the pitches bounding the interval, and lower than the higher of them.”⁹ Here we not only can but must speak of intervals as pitch distances, for Aristoxenus, in line with Theophrastus and other Peripatetics,¹⁰ claimed that numerical quantity had nothing to do with the sense-perceptible presentation of sound to the ears (or to the mind). To say that the tone is 9 : 8 has little bearing on the sense-perceptible reality of music, and thus Aristoxenus and his adherents had no qualms about dividing the tone into two, three, or four equal parts—a position impossible in Pythagorean harmonics, as the whole tone (9 : 8) cannot be generated by squaring (or cubing, etc.) any rational proportion.¹¹ As any Pythagorean worth his salt knew, the whole tone was properly divided into a minor semitone, the λείμμα, in the ratio 256 : 243 (the “remainder” when two tones are subtracted from a fourth), and a major semitone, the ἀποτομή, in the ratio 2,187 : 2,048 (the “remnant” of subtracting the λείμμα from the tone). Finally, the comma, in the ratio 531,441 : 524,288, can be obtained by subtracting the λείμμα from the ἀποτομή. The third book of Boethius’s *De institutione musica* is devoted to one of the favorite pastimes of music theorists in the Pythagorean camp: the refutation of the Aristoxenian position that the whole tone can be divided into equal semitones.

7 See Sophie Gibson, *Aristoxenus of Tarentum and the Birth of Musicology*, Studies in Classics 9 (New York: Routledge, 2005), 23–30.

8 All of these terms are initially defined and discussed in Aristoxenus’s *Elementa Harmonica* 1.8.14–10.10 (Rosetta Da Rios, ed., *Aristoxeni Elementa Harmonica*, Scriptores Graeci et Latini consilio Academiae Lynceorum editi [Rome: Typis publicae officinae polygraphicae, 1954], 13.7–15.5); the formal definition of φθόγγος occurs at 1.15.14–24 (Da Rios, *Aristoxeni Elementa Harmonica*, 20:15–19).

9 *El. Har.* 1.15.25–33 (Da Rios, *Aristoxeni Elementa Harmonica*, 20.20–21.4): “διάστημα δ’ ἐστὶ τὸ ὑπὸ δύο φθόγγων ὠρισμένον μὴ τὴν αὐτὴν τάσιν ἔχόντων. φαίνεται γάρ, ὡς τύπῳ εἰπεῖν, διαφορά τις εἶναι τάσεων τὸ διάστημα καὶ τόπος δεκτικὸς φθόγγων ὁξυτέρων μὲν τῆς βαρυτέρας τῶν ὀρίζουσών τὸ διάστημα τάσεων, βαρυτέρων δὲ τῆς ὁξυτέρας.” I follow here Barker’s translation with minimal modifications (Andrew Barker, *Greek Musical Writings*, vol. 2, *Harmonic and Acoustic Theory* [Cambridge: Cambridge University Press, 1989], 136, doi: 10.1017/CBO9780511585753).

10 E.g., Theophrastus, *De musica* (preserved in Porphyry’s *Commentary on Ptolemy’s Harmonics*): “καὶ γὰρ εἰ πᾶν διάστημα πληθὺς τι, τὸ δὲ μέρος ἐκ διαφορῶν φθόγγων, τὸ μέρος ὅτι ἀριθμὸς τοιόνδε ἂν εἴη· ἀλλ’ εἰ μηδὲν ἄλλο (ἢ) ἀριθμὸς, πᾶν ἀριθμητὸν μετέχει ἂν καὶ μέλους, ὅσον καὶ ἀριθμοῦ” (Ingemar Düring, *Porphyrios Kommentar zur Harmonielehre des Ptolemaios*, Göteborgs Högskolas Årsskrift 38 [Göteborg: Elanders Boktryckeri, 1932], 62.7–10).

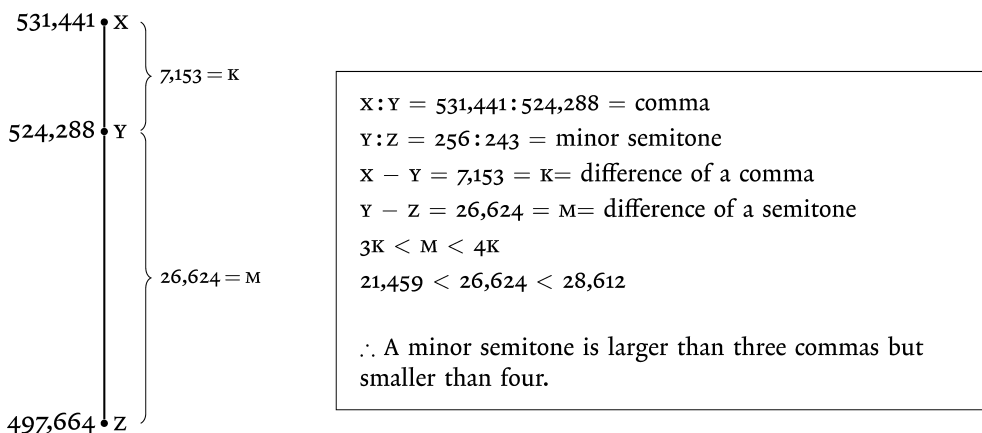
11 A true semitone would be 3 : 2√2.

Boethius's concluding demonstration of the interrelationships between these *unequal* microtonal intervals is not, strictly speaking, part of that proof, but it is a logical extension of his broader arithmetical refutation of the Aristoxenians.

PART ONE: HOW ERRONEOUS IS THE "ERROR"?

Figure 1 presents an arithmetical reduction of the "offending" proof. Because all three proofs—for the minor semitone, major semitone, and whole tone, respectively—employ the same so-called erroneous method, I will deal only with the first.

Figure 1: Boethius's proof that a minor semitone is larger than three commas but smaller than four (*Ins. mus.* 3.14)



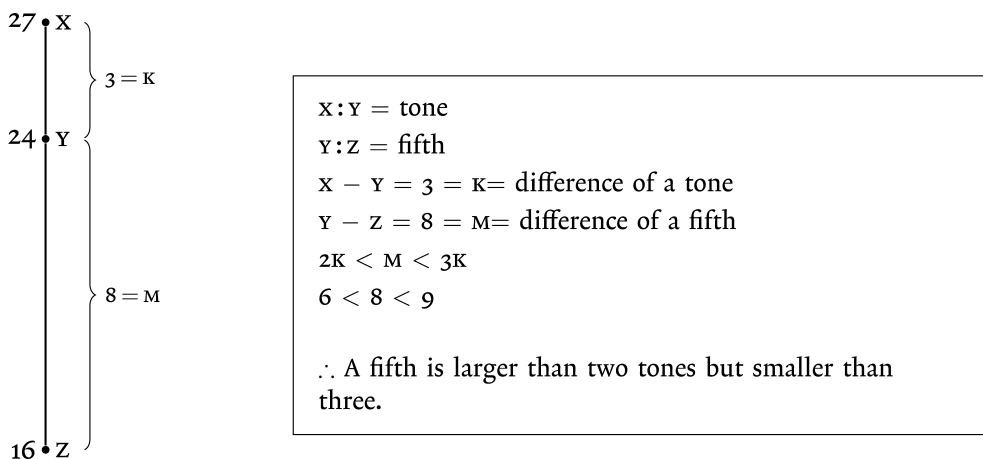
1. Let X, Y, and Z be arranged such that X is a comma from Y, and Y, in turn, is a minor semitone from Z.
2. Calculate the difference between the terms in the ratio of the comma ($X:Y$) and store the result as K. Likewise, calculate the difference between the terms in the ratio of the minor semitone ($Y:Z$) and store the result as M.
3. Since M (26,624) is greater than 3K (21,459) and less than 4K (28,612), the minor semitone is larger than three commas but smaller than four, *quod erat demonstrandum*.

This acoustical relationship is correct, as I have already noted, but Barbera sensed something fraudulent in its calculation, and thus he attempted to lay bare the "fallacy" in Boethius's reasoning by offering a counter-proof, a *reductio ad absurdum*.¹² Using the very same method, Barbera proved that the fifth is larger than two whole tones but smaller

¹² Barbera, "Interpreting an Arithmetical Error," 31.

than three (as in Fig. 2): Let X, Y, and Z be arranged such that X is a whole tone from Y, and Y, in turn, is a fifth from Z. Perform the same calculations as above, and since M (8) is greater than 2K (6) but less than 3K (9), the fifth is larger than two whole tones but smaller than three.

Figure 2: Barbera's counter-proof that a fifth is larger than two tones but smaller than three (1981, p. 31)



A fifth, however, is larger than three tones but smaller than four. Thus the conclusion of Barbera's counter-proof is, as he intended, absurd, and it convicts the method of equal absurdity. Ergo, Boethius's demonstration is unreliable. Or so Barbera argues, and in the remainder of his paper he attempts to prove that Boethius knew the correct relationships only because he could hear them. According to Barbera, the numerical demonstration is merely a *post hoc* justification of relationships previously determined by empirical investigation: good acoustics, bad arithmetic. Heilmann has argued against Barbera's conclusion, noting that it's highly unlikely that Boethius devised a numerical proof to verify previous empirical observations, for two reasons.¹³ First, such a manner of proceeding suits neither Boethius's usual methodology nor his understanding of music theory in general, and Heilmann cites as an example the eleventh, which Boethius denies is a consonance strictly on the basis of its multiple superbipliant ratio (8 : 3): arithmetic trumps acoustics, good or bad. Secondly, Heilmann counters that Boethius could have deduced the relationships among the comma and the semitones from two earlier demonstrations in the third book—that the comma is greater than 75 : 74 and less than 74 : 73 (3.12) and that the minor semitone is greater than 20 : 19 and less than 19½ : 18½

13 Heilmann, *Boethius' Musiktheorie und das Quadrivium*, 226–27.

(3.13)—and therefore without consulting a monochord, polychord, or conducting any empirical test at all.¹⁴

Heilmann's critique of Barbera has merit, but she still shares with him one fundamental premise: namely, that Boethius must have known that his conclusions were true by some means other than the proofs that he offers his readers. In short, both Barbera and Heilmann think that Boethius's *demonstratio per numeros* is somehow disingenuous, and they do not take it seriously. Barbera goes so far as to claim that we cannot take the arithmetical *demonstratio* at face value, for if we do, "then we are treated here to one of the most fantastic coincidences in the history of science and musical theory. For there is no reason other than accident that would explain why the mere manipulation of numbers would depict three sophisticated acoustical relationships."¹⁵

I submit that we can and should take the proofs seriously: the method is not the chicanery it may seem, nor are the correct conclusions a "fantastic" accident. This is borne out when we take a closer look at the proof's inner workings (and here again I restrict discussion to only the first of the three). Example 1 works out the math, which amounts to this: to claim that M is greater than 3K but less than 4K is to claim that the minor semitone is greater than a numerical approximation of three commas and less than a numerical approximation of four commas. These approximations are calculated by substituting an arithmetic series (which maintains a common *difference* between terms) for a geometric series (which maintains a common *ratio* between terms).

Example 1: Functional equivalences within Boethius's method

A geometric series of conjunct 'descending' commas is properly expressed:

$$Y, Y \times (Y:X)^1, Y \times (Y:X)^2, Y \times (Y:X)^3, Y \times (Y:X)^4, \dots Y \times (Y:X)^n \text{ (for } n = 0, 1, 2, \dots),$$

for which Boethius substitutes the arithmetic approximation:

$$Y, Y - K, Y - 2K, Y - 3K, Y - 4K, \dots Y - nK \text{ (for } n = 0, 1, 2, \dots),$$

where $K = X - Y$

$$\therefore 3K < M < 4K = Y : (Y - 3K) < Y : (Y - M) < Y : (Y - 4K),$$

where $Y : (Y - 3K)$ is an arithmetic approximation of $Y : (Y \times (Y:X)^3)$,

$Y : (Y - 4K)$ is an arithmetic approximation of $Y : (Y \times (Y:X)^4)$,

and $Y : (Y - M) = Y : Z$, the ratio of the minor semitone (256 : 243).

14 Heilmann points out (ibid., 227) that it is a simple matter of straightforward arithmetic to conclude that 74 is greater than $3\frac{1}{2} \times 19$ but less than 4×19 .

15 Barbera, "Interpreting an Arithmetical Error," 33.

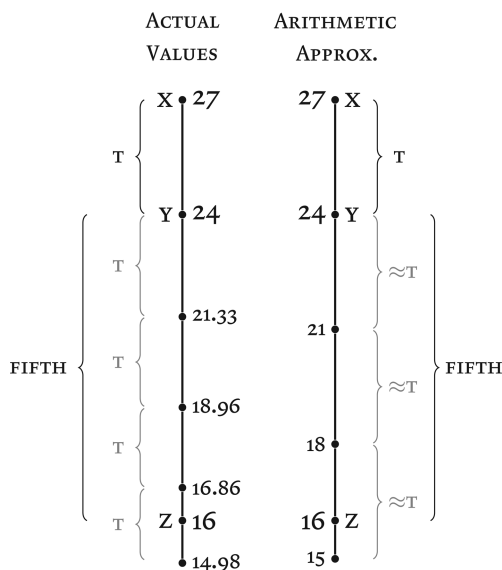
Let me clarify this with two simple examples (Exx. 2a and 2b). If we calculate—as in Example 2a—a sequence of four conjunct fifths (3 : 2, here calculated with 48 : 32 as the base ratio), we produce the geometric series 32, 48, 72, 108, 162. If we replace this series with a sequence of terms generated by the “distance” contained within the first ratio (48 - 32 = 16), we obtain the arithmetic series 32, 48, 64, 80, 96. Clearly, the arithmetic series quickly falls short of its geometric counterpart: four conjunct fifths sum to two octaves and a major third (or 2,808 cents), but their arithmetic approximation sums only to (roughly) an octave and a fifth (or 1,902 cents). This same shortcoming will befall any arithmetic approximation of a geometric series that is generated by a relatively large ratio, such as an octave, a fifth, a fourth, or even a tone. And the same holds true for descending series as well, with the necessary inversion that a descending arithmetic series exceeds its geometric counterpart at about the same rate. But consider the case of 74 : 73 (Ex. 4b). Four such ratios—correctly multiplied according to the geometric series 73, 74, 75.01, 76.04, 77.08—sum to 77.08 : 73, or 94.2 cents. The arithmetic approximation of the same—73, 74, 75, 76, 77—sums to 77 : 73, or 92.4 cents. The correct calculation and its arithmetic approximation differ by only 1.8 cents, a difference that lies well below the threshold of human hearing (which hovers around 5 cents).

Example 2: Geometric and arithmetic series: large vs. small ratios

GEOMETRIC $a_n = a_0 (X:Y)^n \rightarrow 32, 48, 72, 108, 162$ ARITHMETIC $a_n = a_0 + (nD) \rightarrow 32, 48, 64, 80, 96$ 162 : 32 > 96 : 32 2808 cents > 1902 cents	GEOMETRIC $a_n = a_0 (X:Y)^n \rightarrow 73, 74, 75.01, 76.04, 77.08$ ARITHMETIC $a_n = a_0 + (nD) \rightarrow 73, 74, 75, 76, 77$ 77.08 : 73 ≈ 77 : 73 94.2 cents ≈ 92.4 cents
(a) X : Y = 3 : 2, $a_0 = 32$, $D = a_1 - a_0 = 16$	(b) X : Y = 74 : 73, $a_0 = 73$, $D = a_1 - a_0 = 1$

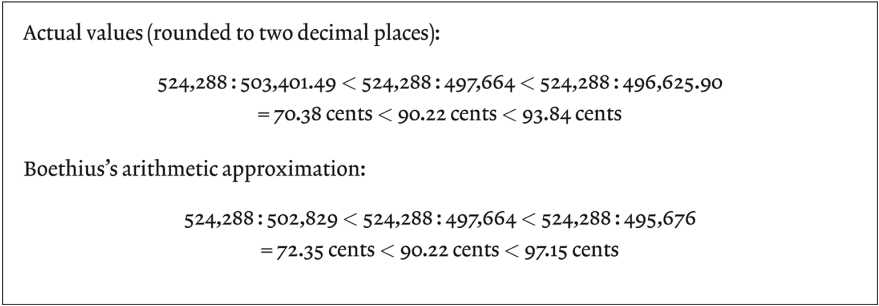
The discrepancy in the accuracy of the arithmetic approximation in Example 2a (a large ratio) versus 2b (a small ratio) is easily explained. As the common ratio in a geometric progression shrinks, that is, as $x : y$ approaches 1, the corresponding arithmetic series becomes, at least for its initial terms, an increasingly accurate approximation; and in the case of small, microtonal ratios, it succeeds with surprising precision. Since Barbera’s counter-proof is based on an arithmetic approximation of a relatively large ratio (9 : 8), it is bound to produce a nonsensical result (as is demonstrated in Fig. 3).

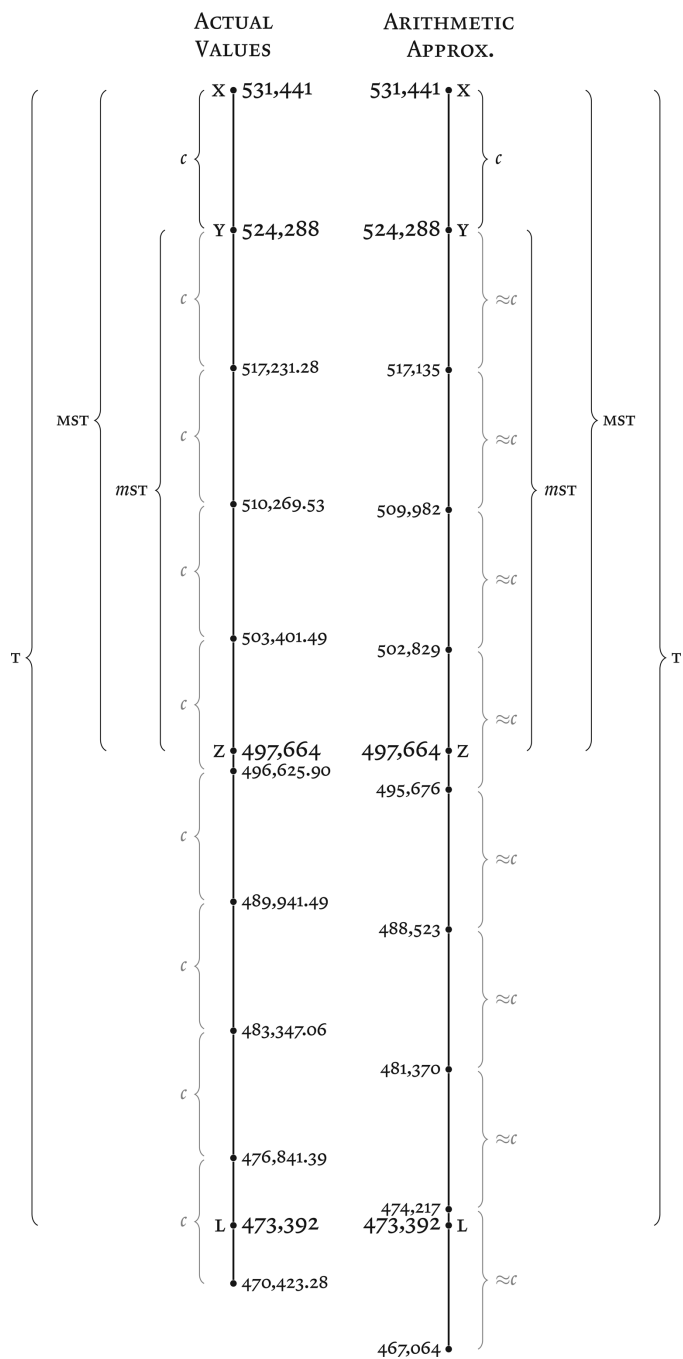
Figure 3: Comparison of actual values and approximate values in the computation of sequential tones (T) against a fifth (Barbera's counter-proof)



The arithmetic series 27, 24, 21, 18, 15 (on the right of Fig. 3), which maintains the common difference of 3, is simply a poor approximation of the geometric series 27, 24, 21.33, 18.96, 16.86, 14.98 (on the left), which maintains the common ratio of 9 : 8. Barbera's counter-proof fails to land a decisive blow against Boethius's arithmetic method for one simple reason: approximate methods are rarely generalizable.¹⁶ Indeed, Boethius's arithmetic method fails miserably for large ratios, but it succeeds with surprising precision when dealing with small ones. Because Boethius's construction of three and four commas is based on a ratio even smaller than 74 : 73, his approximations are as accurate as that calculated in Example 2b, as is clear from Figure 4. The approximation of three commas is off by a mere 1.97 cents and the approximation of four, by only 3.31 cents. Of course, the margin of error increases with every multiplication of the arithmetical difference; but the approximation still holds true for the calculation of eight and nine commas, between which falls the whole tone. This is clear in Figure 5, which summarizes all three proofs.

16 For instance, the shorthand conversion of Celsius to Fahrenheit (double and add thirty) or Fahrenheit to Celsius (subtract thirty and halve) works well enough for a normal range of temperatures (and it works out exactly at 50°F/10°C) but fails miserably at either extreme; it is not recommended for baking or Canadian winters.





The payoff for these calculations is now, I hope, clear. Barbera claims in no uncertain terms that Boethius could not have determined these relationships with his *demonstratio per numeros*, and that he must have relied upon his ear to determine them through empirical investigation. But in fact, the arithmetic demonstration calculates the relationships with greater accuracy than empirical investigation could reliably achieve. Why? Because Boethius's method ingeniously exploits the fact that geometrical progressions of very small ratios can be reasonably approximated with their arithmetic counterparts. And the method amounts to something very much like Ptolemy's observation (made in the course of his tetrachordal division in *Harmonics* 1.15) that when dividing microtonal ratios, "the differences [are] kept equal, and the ratios almost equal, as equal ratios are impossible."¹⁷ Within the context of Boethius's proofs, it's not so much that the ratios could not be equal (they are all rational, whole-number ratios), but rather that it's unreasonable to expect him to calculate them. The exact numerical demonstration, employing only whole-number ratios, requires numbers that reach into the quadrillions (10^{15}) in only a few short calculations and quickly reach into the nonillions (10^{30})—and that's just for the first proof.¹⁸ Although Boethius is capable of astonishing feats of staggering tedium,¹⁹ we can hardly fault him for not attempting to work out, in Roman numerals no less, whether the cube and fourth power of three to the twelfth over two to the nineteenth was less or greater than two to the eighth over three to the fifth, as the first proof alone would demand. Even with Arabic numerals, it proved too much to keep the calculations in line when, nearly a millennium after Boethius, Jacques Lefèvre d'Étaples mounted a (partial) exact proof in his 1496 *Elementa musicalia* (II.35–36; see Appendix II for an annotated translation of Lefèvre's proof). Lorenzo Gaizo, at the prompting of Pietro Aaron, attempted to rectify Lefèvre's errors (which he charitably attributed to printer's errors),²⁰ but he inadvertently introduced new errors of his own.²¹ Aaron, in reply, claimed to have begun

17 Ingemar Düring, *Die Harmonielehre des Klaudios Ptolemaios*, Göteborgs Högskolas Årsskrift 36 (Göteborg: Elanders Boktryckeri, 1930), 34.13–14: "τῶν μὲν ὑπεροχῶν τηρουμένων ἴσων, τῶν δὲ λόγων παρίσων, ἐπεὶ μὴ δυνατόν ἴσων."

18 Both Barbera ("Interpreting an Arithmetical Error," 30–31) and Heilmann (*Boethius' Musiktheorie und das Quadrivium*, 228) acknowledge this fact.

19 See, e.g., *De hypotheticis syllogismis*, books 2 and 3.

20 "There is a printer's error in Faber's statement where he says that the tone is more than seven and less than eight commas [in fact Faber never makes such a claim]; compare my figures with Faber's. Also, the figure on his lines o, r, and p are wrong, and in proposition 35, where he says *I multiply d by h and c by k and d by l*, it should be 'I multiply e by h and f by k and e by l,' and there are many other errors that I believe are typographical" (Bonnie J. Blackburn, Edward E. Lowinsky, and Clement A. Miller, *A Correspondence of Renaissance Musicians* [Oxford: Oxford University Press, 1991], 947 [Sparato Corr. 102]). Gaizo's exact calculations are missing—the letter refers to "la presente figura," not included in the surviving manuscript (Vaticanus latinus 5318, fol. 173r)—but he concludes that a tone "has more than three quarters of the ninth comma"; in fact, it has less than three-quarters of the ninth comma (but more than five-eighths).

21 *Ibid.*, 945–46 (Sparato Corr. 102).

computing the proof anew but abandoned the project, not because the numbers were too large, but (he said) because he had better things to do.²² Realistically speaking, the exact numerical proof was incalculable, and the approximate method provided an easily calculable, methodologically sound, and surprisingly accurate alternative. It remained the only kind of proof even attempted for the first millennium after Boethius's treatise.

PART TWO: NICOMACHUS OR BOETHIUS?

Both Barbera and Heilmann are sensitive to the fact that another source, namely Nicomachus's lost music treatise, may lurk behind Boethius's proofs, but neither seriously considers the implications of this (possible) debt.²³ If the arithmetic procedure that Boethius employs were one that he had devised himself, my reconstruction of the method's pre-Boethian origins would be seriously undermined. Thus, a review of the evidence is in order so as to determine whether, in fact, Boethius has translated his proof from Nicomachus.

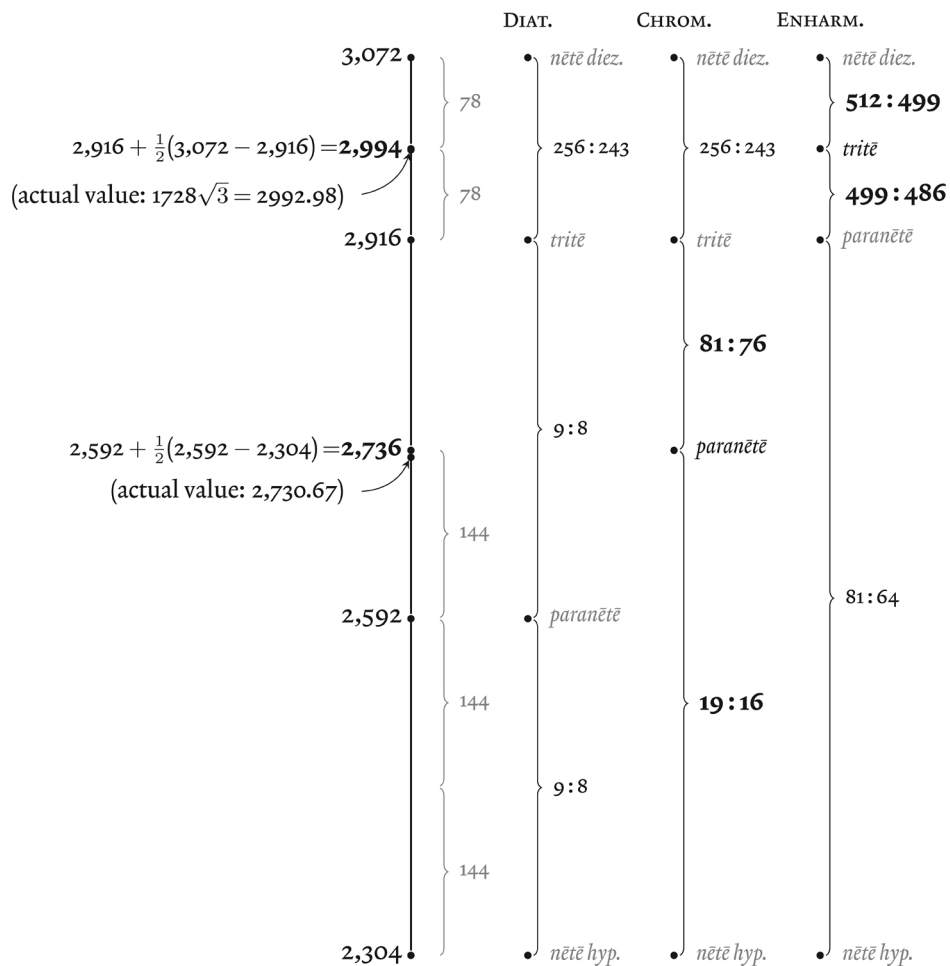
Internal cross-references within the *De institutione musica* strongly suggest that book 3 is drawn, as a whole, from the same source as the first two books, namely Nicomachus's lost *Εἰσαγωγή μουσική*.²⁴ But this is merely circumstantial evidence. More conclusive, and more relevant to my argument, is the presence of similar arithmetic approximations in the course of the fourth book's monochord division, a division that Bower and Stefan Hagel persuasively demonstrate to have come likewise from Nicomachus.²⁵ An examination of the first stage of that division, which begins with the *hyperbolaion* tetrachord at 4.6 (*Monochordi nectarum hyperboleon per tria genera partitio*), will suffice to illustrate the point (see Fig. 6).

22 Ibid., 949 (Sparato Corr. 103): "non tanto per gli grandi numeri che gli occorreano, quanto per essere in altre cose più necessarie occupato."

23 Barbera, "Interpreting an Arithmetical Error," 27; Heilmann, *Boethius' Musiktheorie und das Quadrivium*, 228. Heilmann even suggests at one point that it might result from Boethius's misunderstanding of his source: "[es] ist nicht auszuschliessen, dass sich die Inkonsistenz schon in Boethius' Quelle befand oder dass er sie fehlerhaft übertragen hat" (ibid.).

24 See Calvin Bower, "Boethius and Nicomachus: An Essay Concerning the Sources of *De institutione musica*," *Vivarium* 16 (1978), 9–11, doi: 10.1163/156853478X00012; Ubaldo Pizzani, "Studi sulle fonti del 'De institutione musica' di Boezio," *Sacris Erudiri* 16 (1965), 83–87, doi: 10.1484/J.SE.2.303339.

25 Bower, "Boethius and Nicomachus," 19–26; Stefan Hagel, *Ancient Greek Music: A New Technical History* (New York: Cambridge University Press, 2010), 160–66, doi: 10.1017/CBO9780511691591.



The chromatic *paranētē* (2,736) differs from its ‘correct’ value (2,730.67) by only 3.38 cents:

$$2,736 : 2,304 = 297.51 \text{ cents and } 2,730.67 : 2,304 = 294.13 \text{ cents.}$$

Likewise, the ‘Aristoxenian’ quarter-tones,

$$512 : 499 \text{ and } 499 : 486 = (\text{respectively}) 44.52 \text{ cents and } 45.70 \text{ cents,}$$

are nearly identical, differing by only 1.18 cents, well below the threshold of perception.

number ratios throughout the double-octave division. The partitions of the chromatic and enharmonic tetrachords are both derived from this diatonic framework. Two calculations are important here. First, the chromatic *paranete*, the second-highest pitch in the tetrachord, is determined through an arithmetic approximation of three semitones ($144 + 144 + 144$); second, the two closely packed quarter tones on either side of the enharmonic *trite* at the bottom of the tetrachord are derived by simply dividing the bounding minor semitone “space” in half, a method that generates intervals with equal distances of 78. Once again Boethius has substituted an arithmetic series (or mean) for a geometric series (or mean) that is far more difficult, if not even impossible, to calculate. And again, the resultant intervals are, acoustically speaking, indistinguishable from their correctly computed counterparts.

Barbera, in a 1977 study of geometric and arithmetic tetrachord divisions, asserts that Boethius’s method of dividing the enharmonic *pyknon* (that is, the minor semitone between the *paranete* and the *nete*) “produces two different intervals, $499 : 486$ and $512 : 499$.”²⁶ He is, of course, absolutely correct. But such a claim in this context amounts to acoustic hair-splitting, for these “two different intervals” differ acoustically by a mere and nearly imperceptible 1.18 cents. Barbera further claims that the “curious looking” ratios that the method generates (printed in boldface in Fig. 6) “necessarily must have been considered before determining that 2,304 is the smallest integer such that all of the intervals of the Greater and Lesser Perfect Systems could be characterized by integers.”²⁷ But this is only partly true, and it makes the determination of 2,304 appear more difficult than it actually need be. Boethius did not have to concern himself with whether any given number in the partition had a 499th part or a 486th part. All that must be ensured for the division to succeed is that a number be chosen such that all generated values of the diatonic double octave are even numbers. That is to say, the “curious looking” ratios are better explained as artifacts of the partition method and not as *a priori* generative principles determined in advance.

Bower tries to account for these odd ratios by appealing to the primacy of the arithmetic mean, but in so doing he slightly mischaracterizes Nicomachus’s arithmetic method. Bower dismisses the notion that computational expediency was the driving force behind the arithmetic derivation of the chromatic *paranetai* and enharmonic *tritai*, arguing instead that “Nicomachus, the *arithmetician* [Bower’s emphasis], considered arithmetic proportionality prior by nature to the other types of proportionality.”²⁸ This may be true, but Boethius’s language, assuming it’s at least marginally representative of Nicomachus’s original Greek, does not fully bear this out. Boethius explicitly describes the computation of the chromatic

26 André Barbera, “Arithmetic and Geometric Divisions of the Tetrachord,” *Journal of Music Theory* 21 (1977), 309, doi:10.2307/843492.

27 Ibid.

28 Bower, “Boethius and Nicomachus,” 26.

paranete as a means through which “we can obtain a number distant by three semitones from the *nete hyperboleon*.”²⁹ Moreover, the number generated by Boethius’s method (2,736) is not the arithmetic mean of any two numbers in the tetrachord. If the arithmetic mean had ultimate priority, Boethius could have derived an (admittedly much less accurate) chromatic *paranete* by calculating the arithmetic mean of 2,916 : 2,592 and placing the *paranete* at 2,754. But he doesn’t. Boethius also explicitly describes the two enharmonic quarter tones as “half the interval [*spatium*] of the minor semitone,”³⁰ and he claims, without corrective or qualification, that these quarter tones can be obtained by calculating half the distance between the terms in the minor semitone ratio, which again maintains equal distances, not equal ratios.³¹ The most straightforward interpretation of Boethius’s and, by extension, Nicomachus’s language is precisely what Bower wants to deny: Boethius describes geometric relationships but expedites the math by using arithmetic calculations.

Boethius’s manner of dividing the chromatic and enharmonic tetrachords is strikingly similar to the manner in which he proves that the minor semitone is larger than three commas and smaller than four. In both cases, Boethius reasonably calculates an arithmetic approximation of a geometric proportion that either (a) cannot be expressed as a rational or whole number (in the case of quarter tones) or (b) exceeds the limits of reasonable computation. These two instances of the same basic principle strongly suggest the presence of a single mind solving different problems in similar ways, and evidence independent of this methodological congruency points to Nicomachus as the likely, if not certain, source.

PART THREE: INTERVALS AND RATIOS IN THE TIMAEAN TRADITION

The final piece of the puzzle, then, is to consider how and where this approximate method might have arisen. The answer may lie in the letter of Plato’s “harmonic” division of the world soul (*Timaeus* 35b4–36b5):

ἤρχετο δὲ διαιρεῖν ὥδε. μίαν ἀφεῖλεν τὸ πρῶτον ἀπὸ παντὸς μοῖραν, μετὰ δὲ ταύτην ἀφήρει διπλασίαν ταύτης, τὴν δ’ αὖ τρίτην ἡμιολίαν μὲν τῆς δευτέρας, τριπλασίαν δὲ τῆς πρώτης, τετάρτην δὲ τῆς δευτέρας διπλὴν, πέμπτην

29 *Inst. mus.* 4.6 (320.27–321.4): “si distantiam paranetes hyperboleon et netes hyperboleon diatonici generis sumpserimus eiusque dimidium paranete hyperboleon, quae est diatonici generis, apponamus, habebimus numerum tribus semitoniiis ab hyperboleon nete distantem; et erit haec in chromatico genere paranete hyperboleon.”

30 *Inst. mus.* 4.6 (321.17–19): “constat autem tetrachordum enarmonii generis ex duobus integris tonis et diesi ac diesi, quae sunt dimidia spatia semitonii minoris.”

31 *Inst. mus.* 4.6 (321.19–322.2): “distantiam eam, quae est inter neten diezeugmenon et paraneten hyperboleon enarmonion sumo. Sed quoniam nete diezeugmenon est III·LXXII-paranete autem hyperboleon enarmonios II·DCCCCSVI-horum distantia erit CLVI. Horum sumo dimidiam partem, qui sunt LXXVIII. Hos adicio II·DCCCCXVI, fient II·DCCCCXCIII. Haec erit EE trite hyperboleon enarmonios.”

δὲ τριπλὴν τῆς τρίτης, τὴν δ' ἕκτην τῆς πρώτης ὀκταπλασίαν, ἐβδόμην δ' ἑπτακαικεκοσιπλασίαν τῆς πρώτης· μετὰ δὲ ταῦτα συνεπληροῦτο τὰ τε διπλάσια καὶ τριπλάσια **διαστήματα**, μοίρας ἔτι ἐκεῖθεν ἀποτέμνων καὶ τιθεὶς εἰς τὸ μεταξὺ τούτων, ὥστε ἐν ἑκάστῳ **διαστήματι** δύο εἶναι μεσότηας, τὴν μὲν ταυτῷ μέρει τῶν ἄκρων αὐτῶν ὑπερέχουσαν καὶ ὑπερεχομένην, τὴν δὲ ἴσῳ μὲν κατ' ἀριθμὸν ὑπερέχουσαν, ἴσῳ δὲ ὑπερεχομένην. ἡμιολίων δὲ **διαστάσεων** καὶ ἐπιτρίτων καὶ ἐπογδῶν γενομένων ἐκ τούτων τῶν δεσμῶν ἐν ταῖς πρόσθεν **διαστάσεσιν**, τῷ τοῦ ἐπογδόου **διαστήματι** τὰ ἐπίτριτα πάντα συνεπληροῦτο, λείπων αὐτῶν ἑκάστου μόριον, τῆς τοῦ μορίου ταύτης **διαστάσεως** λειφθείσης ἀριθμοῦ πρὸς ἀριθμὸν ἐχούσης τοὺς ὄρους ἕξ καὶ πεντήκοντα καὶ διακοσίῳ πρὸς τρία καὶ τετταράκοντα καὶ διακόσια.

This is how he began to divide. First he took away one part from the whole, then another, double the size of the first, then a third, one and a half times the second and three times the first, then a fourth, double the second, then a fifth, three times the third, then a sixth, eight times the first, then a seventh, twenty-seven times the first. Next he filled out the double and triple **intervals**, once again cutting off parts from the material and placing them in the intervening gaps, so that in each **interval** there were two means, the one [the harmonic mean] exceeding and exceeded by the same part of the extremes themselves, the other [the arithmetic mean] exceeding and exceeded by an equal number. The hemiolic, epitritic, and epogdoic **spaces** arose from these links within the previous **spaces**; and he filled up all the epitritics with the epogdoic kind of **interval**, leaving a part of each of them, where the **space** of the remaining part had as its boundaries, number to number, 256 : 243.³²

There are several musicological problems (noted by commentators ancient and modern) with this passage that cannot detain us here.³³ As Barker notes, Plato does not set himself the “task of making his construction correspond at every point to the shape of a system that could be used in practice. [. . .] There are no notes or pitches in his *harmonia*; there are only numbers.”³⁴ But this is not quite the whole truth, for there are more than just numbers, that is, discrete quantities: there are also parts (μοῖραι), as well

32 I follow the translation, with minor modifications, of Barker, *The Science of Harmonics*, 319.

33 The literature is extensive. Classic and recent studies include Jacques Handschin, “The ‘Timaeus’ Scale,” *Musica Disciplina* 4 (1950), 3–42; Francis MacDonald Cornford, *Plato’s Cosmology: The Timaeus of Plato, Translated with a Running Commentary* (London: Routledge & Kegan Paul, 1937), 66–72; Barker, *The Science of Harmonics*, 318–26; Dirk Baltzly, trans., *Proclus: Commentary on Plato’s Timaeus*, vol. 4, Book 3, Part II: *Proclus on the World Soul* (Cambridge: Cambridge University Press, 2009).

34 Barker, *The Science of Harmonics*, 322.

as intervals and spaces (διαστήματα and διαστάσεις)—that is, continuous magnitudes. In fact, it could be argued that there are no discrete numbers at all, just magnitudes related to one another in determinate, numerically specifiable ways. A quick scan of the Greek terms printed in boldface above, corresponding to boldface text in the English translation, reveals a striking feature of this famous passage: spatial terms (διάστημα and διάστασις) appear where we might expect Plato to use λόγος, or “ratio,” a term not used anywhere in the division. Although λόγος does appear in the conclusion of his description of the world soul, where Plato observes that the circles within it move with speeds ἐν λόγῳ (“according to ratio”: 36d6), in the division of the world soul it is spaces that are explicitly deemed duple, triple, hemiolic, and so on. Even the λείμμα, the minor semitone, is not a ratio but a space bounded, number to number, by 256 and 243. Of course, διάστημα does not necessarily have the musical meaning of “interval” here, for the term’s primary signification is geometrical, where it means simply the distance between geometrical points, lines, and surfaces.³⁵ On this point, it is worth recalling that Aristotle (*De anima* 407a) had criticized Plato for presenting the soul as a spatial magnitude (μέγεθος),³⁶ and in the Academy, perhaps as early as Speusippus, there arose a debate, which likely sprung from the *Timaeus*, about whether or not the soul, the world soul in particular, is a geometrical magnitude.³⁷ In addition to Aristotle’s criticism, we have the report of Iamblichus, who, at the beginning of his own *De anima*, gives us a doxographical history of his predecessors’ beliefs. There were, he writes, three kinds of mathematical conceptions of the soul: the geometrical, the arithmetical, and the harmonic. To the first category he assigns both Speusippus and Severus.³⁸ Iamblichus’s testimony to Severus’s view has independent confirmation via Proclus, who also criticizes Severus for holding the position that the soul has a “geometric hypostasis” (γεωμετρικὴ ὑπόστασις) that results from the point and from extension (διάστασις).³⁹

35 See Nathan Sidoli, “On the Use of the Term *Diastema* in Ancient Greek Constructions,” *Historia Mathematica* 31 (2004), 2–10, doi: 10.1016/j.hm.2003.08.003.

36 407a2: “πρῶτον μὲν οὐκ οὐ καλῶς τὸ λέγειν τὴν ψυχὴν μέγεθος εἶναι.”

37 On which see Leonardo Tarán, *Speusippus of Athens: A Critical Study with a Collection of the Related Texts and Commentary*, *Philosophia antiqua* 39 (Leiden: Brill, 1981), 367–70; cf. John Dillon, *The Middle Platonists: 80 B.C. to A.D. 220*, rev. ed. (Ithaca, NY: Cornell University Press, 1996), 263–64.

38 John F. Finamore and John M. Dillon, *Iamblichus De anima: Text, Translation, and Commentary*, *Philosophia antiqua* 92 (Leiden: Brill, 2002), 28–29 (§4), commentary at 79–85; cf. Tarán, *Speusippus of Athens*, 154 (F54ab), commentary at 365–71. There is debate between Dillon and Tarán over whether the Old Academic Speusippus actually held the position that the world soul is a geometrical magnitude, but that in and of itself is immaterial for my purposes. It is enough to note that there was a tradition that posited the world soul as a quasi-geometrical entity.

39 Ernst Diehls, ed., *Procli Diadochi in Platonis Timaeum commentaria*, 3 vols., *Bibliotheca scriptorum Graecorum et Romanorum Teubneriana* (Liepzig: B. G. Teubner, 1903–6), 2.153.21–25.

Plato's diffuse language seems, then, to have occasioned—or at least encouraged—some confusion between λόγος and διάστημα, and Barker has highlighted the marked overlap between these domains within the commentary tradition.⁴⁰ Theon of Smyrna's *Expositio rerum mathematicarum ad legendum Platonem utilium* preserves Thrasyllus's definition of διάστημα as “a specifically qualified relation [σχέσις] that notes have to one another, such as the fourth, the fifth, and the octave” (διάστημα δέ φησιν εἶναι φθόγγων τὴν πρὸς ἀλλήλοθς σχέσιν, οἷον διὰ τεσσάρων, διὰ πέντε, διὰ πασῶν), which echoes the Euclidian definition of λόγος as ἡ κατὰ πηλικότητά ποια σχέσις (“a sort of relation in respect of size”).⁴¹ Nicomachus himself is not exempt from criticism: in his *Ἀρμονικὸν ἐγχειρίδιον*, he initially toes the Aristoxenian line and defines διάστημα as what lies between (μεταξύτης) two discrete pitches. But he, too, relates διάστημα to λόγος: “a relation [σχέσις] is the ratio that *measures the distance* within each interval” (διάστημα δ' ἔστι δυοῖν φθόγγων μεταξύτης. σχέσις δὲ λόγος ἐν ἐκάστῳ διαστήματι μετρητικός τῆς ἀποστάσεως).⁴² Plutarch, Adrastus, and Theon of Smyrna all write themselves into hopeless muddles, and Barker wryly observes that the “confusion is endemic.”⁴³

Though Barker was concerned only with the Greek tradition, the conflation spills into Latin as well. The fourth-century (partial) translation and commentary on the *Timaeus* by the enigmatic Calcidius continues to deploy the Latin terms *interuallum* (διάστημα), *spatium* (διάστασις), and *ratio* (λόγος) interchangeably in his remarks on this same passage, presumably reflecting the terminological mix in his Greek source for this section of the commentary.⁴⁴ For instance, 6 and 12 differ “according to the *ratio* [*ratio*] of duple quantity,”⁴⁵ 32 and 24 “produce the *interval* [*interuallum*] of the epitritic quantity,”⁴⁶ 9 measured against 8 “produces the epogdoic *space* [*spatium*],”⁴⁷ and so on. There is no need

40 Barker, “Early *Timaeus* Commentaries.”

41 Edward Hiller, ed., *Theonis Smyrnaei philosophi Platonici Expositio rerum mathematicarum ad legendum Platonem utilium*, Bibliotheca scriptorum Graecorum et Romanorum Teubneriana (Leipzig: B. G. Teubner, 1878), 48.8–10, echoing Euclid *El.* V, def. 3.

42 Karl von Jan, ed., *Musici scriptores graeci: Aristoteles, Euclides, Nicomachus, Bacchius, Gaudentius, Alypius, et melodiarum ueterum quidquid exstat* (Leipzig: B. G. Teubner, 1895), 261.8–10.

43 Barker, “Early *Timaeus* Commentaries,” 82.

44 Probably Adrastus the Peripatetic. Calcidius's arguments often overlap with the citations from Adrastus's (lost) *In Timaeum* given by Theon of Smyrna. The commentary on the world soul division, however, has no exact parallel in Theon, and thus we cannot know precisely what his Greek text would have looked like. Throughout the passage, there are occasional hints that Calcidius is thinking in Greek, e.g.: “Rursum enim decem et octo numerus aduersum decem et sex [only this once, everywhere else *sedecim*] numerum epogdoi rationem obtinet” (J. H. Waszink, ed., *Timaeus a Calcidio translatus commentarioque instructus*, 2nd ed., Plato Latinus 4 [London: Warburg Institute, 1975], 44 [91.24–25]).

45 *In Tim.* 41 (89.22–90.1): “Quia sex numerus facit unum limitem et item duodecim secundum efficit limitem iuxta rationem duplicis quantitatis et a se distant, interponuntur duae medietates, una octonarii numeri, altera nouenarii.”

46 *In Tim.* 42 (91.13–15): “Triginta duo medietas aduersum uiginti quattuor limitem comparata facit interuallum epitritae quantitatis.”

47 *In Tim.* 43 (91.21–23): “Quia igitur nouem et octo epogdoum faciunt spatium et ex his duo limites dupli ratione distantes, id est sex et duodecim conplentur, recte dixit epogdoi spatio epitritorum omnium interualla conpleri.”

to belabor the point with more examples, which could indeed be multiplied; nor do I want unduly to fault Calcidius. I wish only to highlight the fact that even though he gives us no definition of λόγος or διάστημα such as we find in Thrasyllus, Theon, and Nicomachus, Calcidius's free use of the terms *ratio*, *intervallum*, and *spatium* tellingly reveals the fluid employ of λόγος, διάστημα, and διάστασις in his source: both writings follow the long tradition, which has deep Platonic roots, of using the terms interchangeably.

This usage of λόγος and διάστημα was not lost on at least one of the ancient commentators, the third-century Neoplatonic philosopher Porphyry. In his commentary on Ptolemy's *Harmonics*, Porphyry prefaces his discussion of 1.5 (on the concords and their Pythagorean ratios) with a substantial doxographical account of the ancient views on these terms, noting (*in nuce*) that their usage varied considerably.⁴⁸ Some writers use διάστημα in the sense of λόγος (the common usage among the ancients, according to Porphyry), some define διάστημα as the difference (διαφορά, elsewhere ὑπεροχή) between terms, and the Aristoxenians make the διάστημα a spatial designation.⁴⁹ In the initial category, that of those who equate the terms, Porphyry's first example is none other than the Timaeon division of the world soul: "for instance, in the *Timaeus* by the Divine Plato, who says 'the hemiolic, epitritic and epogdoic spaces arose from these links, and he filled up all the epitritics with the epogdoic kind of interval.'" ⁵⁰ In the second category, encompassing those who define the διάστημα as the ὑπεροχή and thus as different from the λόγος, his first example is the Hellenistic mathematician Eratosthenes. Although Porphyry adduces the testimony of Eratosthenes to distinguish the terms, he still faults Eratosthenes for failing in his usage to "establish either what is meant by διάστημα or in what respect it differs from λόγος."⁵¹

What occasioned this scolding? As Barker and Hagel both note (though to different conclusions), Eratosthenes seems to have attempted to reconcile Pythagorean ratios with Aristoxenian pitch space,⁵² a reconciliation that amounts to the same basic principle that we have seen at work in Nicomachus and Boethius: the preservation of equal distances as an approximation of equal intervals and thereby of equal ratios. According to Ptolemy's *Harmonics*, Eratosthenes, in his computation of the enharmonic tetrachord (diagrammed

48 See Andrew Barker, *Porphyry's Commentary on Ptolemy's Harmonics: A Greek Text and Annotated Translation* (Cambridge: Cambridge University Press, 2015), 279–93 (which appeared after the completion of this article).

49 Düring, *Porphyrios Kommentar*, 94.31–95.13: "Οἱ μὲν γὰρ τὸν λόγον καὶ τὴν σχέσιν τῶν πρὸς ἀλλήλους συμβλητῶν ὄρων τὸ διάστημα καλοῦσι. [...] Οἱ δὲ τὴν διαφορὰν τῶν ὁμογενῶν τε καὶ πρὸς ἀλλήλους συμβλητῶν ὄρων τὸ διάστημα λέγουσι. [...] Οἱ δ' Ἀριστοξένειοι τοπικὸν τίθενται τὸ διάστημα." On Porphyry's fuller treatment of the question, see Massimo Raffa, "The Debate on *logos* and *diastema* in Porphyry's *Commentary on Ptolemy's Harmonics*," *Greek and Roman Musical Studies* 1 (2013), 243–52, doi: 22129758-12341245.

50 Düring, *Porphyrios Kommentar*, 92.12–18.

51 *Ibid.*, 91.9–10: "Ερατοσθένης μὲν οὖν φησιν ἕτερον εἶναι διάστημα λόγου. [...] ἐκ δὲ τοιούτων οὔτε τί καλεῖται διάστημα, οὔτε καθ' ὃ διαφέρει τοῦ λόγου παρέστησεν."

52 Barker, "Early *Timaeus* Commentaries," 86; Hagel, *Ancient Greek Music*, 182–87.

in Fig. 7), generated superparticular ratios by employing the Aristoxenian principle that the tone is equally divisible into twelve hypothetical units, three of which comprise the enharmonic quarter tone.⁵³ Correspondingly, the quarter tones in Eratosthenes' partition share the common difference of 3 (120 : 117 and 117 : 114), but not a common ratio.

Figure 7: Eratosthenes's enharmonic tetrachord (according to Ptolemy, *Harmonics* 2.14)



Eratosthenes's 'Aristoxenian' quarter-tones,

40 : 39 and 39 : 38 = (respectively) 43.83 cents and 44.97 cents,

are nearly identical, differing by only 1.14 cents, well below the threshold of perception.

This is not an isolated example, and other fragments, such as *Excerpta Neapolitana* 19 (wherein Eratosthenes is credited with dividing the tone into the four quarter tones: 36 : 35, 35 : 34, 34 : 33, and 33 : 32),⁵⁴ demonstrate this same principle. It is difficult to pinpoint precisely what Eratosthenes was actually up to.⁵⁵ It is not at all clear, nor is it at all likely, that his Pythagorean translation of Aristoxenian pitch space was as naïve as Ptolemy and Porphyry might have us believe. Nor is it at all clear that the transmitted ratios properly belong to the science of canonic.⁵⁶ Nonetheless, it does seem likely that the original (or at least a plausible) context for Eratosthenes' tetrachordal ratios was his

53 Düring, *Die Harmonielehre des Klaudios Ptolemaios*, 2.14.

54 von Jan, *Musici scriptores graeci*, 416.19–417.11.

55 Hagel, *Ancient Greek Music*, 185.

56 David E. Creese, *The Monochord in Ancient Greek Harmonic Science* (Cambridge: Cambridge University Press, 2010), 179–80.

lost *Πλατωνικός*, which must have included a discussion of the Timaeon world soul (and indeed, Eratosthenes' diatonic division is the Timaeon division). Thus Eratosthenes' method is brought into the same Platonic and Timaeon realm as that of later authors, such as Theon of Smyrna and Nicomachus of Gerasa,⁵⁷ who preserve the few surviving *testimonia* to the work of the Hellenistic mathematician.⁵⁸

In the conclusion to his interpretation of Boethius's "arithmetical error," Barbera maintained that, "although [Boethius's] method of representing the stacking of commas differed in no way from the Aristoxenian method of counting up quarter tones, Boethius seems to have been satisfied by the apparent numerical verification of what he could hear."⁵⁹ I think it unlikely that Boethius heard anything. But Barbera's instincts were fundamentally sound: as I have argued, one possible origin of Boethius's arithmetic method lies precisely in a Pythagorean re-interpretation of Aristoxenian quarter-tone stacking—ironically, a very non-Pythagorean method that developed in the service of Platonic harmonics. If we convict Boethius of an "arch arithmetical crime," then we must convict Nicomachus, Ptolemy, and Eratosthenes as well. Admittedly, the case can easily be made, and all have been weighed and found wanting at some point. But each of them, Nicomachus in particular, could and did exploit the possibilities of arithmetic approximation with sound results. Although Boethius's *De institutione musica* stands at a great distance from the cosmological realm of Timaeus's "likely story," it may still harbor within the nitty-gritty minutiae of its mathematical methods the remnants of a lost, Hellenistic, specifically Timaeon musicology.

APPENDIX I: SUMMARY OF *DE INSTITUTIONE MUSICA* 3.14–16⁶⁰

3.14 "The minor semitone is larger than three commas, but smaller than four"

Let A be 262,144. Let B be 472,392, five tones above A. Let C be 524,288, a diapason above A. Let D be 531,411, six tones above A and a tone above B. Let K be 7,153, the difference between D and C, the comma. Construct a minor semitone below C by descending two tones from B to E, 373,248, and ascending a diatesseron from E to F, 497,664. (To confirm the construction, ascend another diatesseron from F to G, 663,552, and descend two tones from G to P, 524,288, the same value as C.) Let M be 26,624, the difference between C and F, the minor semitone. Let 3K be L, 21,459,

⁵⁷ Notably, Nicomachus claims to know Eratosthenes' division of the *kanon* (see von Jan, *Musici scriptores graeci*, 260.12–17).

⁵⁸ Barker, "Early *Timaeus* Commentaries," 84; Creese, *The Monochord*, 207–8.

⁵⁹ Barbera, "Interpreting an Arithmetical Error," 41.

⁶⁰ Ed. Godfrey Friedlein, *De institutione musica, libri quinque*, Bibliotheca scriptorum Graecorum et Romanorum Teubneriana [Leipzig: B. G. Teubner, 1867], 293–99.

and 4K be N, 28,612. M is greater than L but less than N. “Therefore it is correctly stated that the minor semitone is smaller than four commas but larger than three.”

3.15 “The apotome is larger than four commas but smaller than five; the tone is larger than eight commas but smaller than nine”

Let A be 262,144. Ascend five tones to B, 472,392, and another sixth tone to D, 531,441. Let C, 497,664, be a minor semitone above B. Since BD is a tone, and BC is a minor semitone, CD is an apotome. Let the difference between D and C be E, 33,777, and the difference of a comma (calculated above) be F, 7,153. Let 5F be G, 35,765, and 4F be K, 28,612. G is more than E, but K less than E.⁶¹ “Therefore it is correctly stated that the apotome is smaller than five commas but larger than four.” Summing the relationships determined thus far proves that the tone is larger than eight commas but smaller than nine.⁶²

3.16 “Proof through numbers of the things discussed above”

Let A be 262,144. Let B, 472,392, be five tones above A, and let C, 524,288, be a diapason above A. Let D, 531,441, be six tones above A and a tone above B. Since CD is a comma, let D minus C be E, 7,153. Since BD is a tone, let D minus B be F, 59,049. Let 9E be H, 64,377, and 8E be G, 57,224. H is greater than F, but G less than F. “Thus it has been demonstrated that the tone is smaller than nine commas but larger than eight of the same commas.”

APPENDIX II: TEXT AND ANNOTATED TRANSLATION OF JACQUES LEFÈVRE D’ÉTAPLES, *ELEMENTA MUSICALIA* II.35–36⁶³

II.35. Semitonium minus tribus commatibus maius est, minus uero quatuor. Vnde manifestum est apotomen plura quatuor et pauciora quinque continere commata. Non est graeca curiositas calculi labore deterrita quo minus quot commata in diesi, quot in apotome, quot denique in tono sint, peruestigare. quod nisi a prioribus tentatum cognouissem, cum id quoque plus laboris quam (ut michi uisum est) in musicis modulationibus usus utilitatisque afferat, missum fecissem. qui tamen id cognoscere desiderauerint, hoc pacto deprehendent. sint *a b* minimi numeri semitonii minoris et *c d* minimi commatis: per decimamseptimam et decimamoctauam [decimamoctauam et

61 “G igitur ab eo, quod est E, maius est, K minus.” Bower translates this as: “G is larger than E but smaller than K” (110)—doubtless a printer’s error.

62 Boethius is wrong on this point. He has only proved that the whole tone is greater than *seven* (three plus four) and smaller than nine (four plus five) commas. Since he has not specified that the minor semitone is greater than three and a half commas (or the apotome four and a half commas), the lower boundary cannot be fixed at eight.

63 Jacques Lefèvre d’Étaples, *Elementa musicalia* (Paris: Johannes Higman and Wolfgang Hopyl, 1496), g2r–g2v; 2nd ed. (Paris: Guilemus Cauellat, 1551), 20v–21r.

uicesimamoctauam *in 1551 editione*] huius reperti: duco b in c et d et proueniant ef et a in c et veniat g : per septimam secundi Arithmetices f ad e est commatis habitudo, et per octauam eiusdem g ad e habitudo dieseos, semitonique minoris. Deinde duco e in e et f in f et e in g et nascentur $h k l$: per sextam quarti Arithmetices perquam facile cognosci potest h continere duo commata. et per septimam secundi eiusdem l ad h esse semitonium minus. Deinde duco d in h , et c in k , et d in l : et eo ordine veniant $m n o$: per eandem sextam quarti cognitu facillimum est $n m$ continere tria commata. et per septimam secundi $o m$ continere semitonium minus. at n numerus cognoscitur esse minor o . ergo o ad m semitonium minus tria uincit exsuperatque commata. Deinde duco h in h et k in k et h in l , et suo ordine exsurgant orianturque $p q r$. manifestum est per idem quod prius $q p$ continere quattuor commata, et $r p$ continere semitonium minus. at numerus r minor est numero q . igitur quattuor commata amplius sunt semitono minore. Correlarium autem hinc notum est quod semitonium maius solo commate superat semitonium minus. atqui semitonium minus plura tribus et pauciora quattuor ut modo uisum est, continet commata. igitur unico superadiecto commate semitonium maius quod uocant apotomen, plura quattuor et pauciora quinque continere est necesse.

II.36. Tonum plura septem continere commata necesse est. Nam tonus ex semitono minore et apotome coalescit atque constituitur. at semitonium minus per perultimam tria continent commata et amplius, et per praecedetem apotome quatuor et amplius. tria autem et quatuor et amplius septem sunt et amplius. igitur tonus plura quam septem continet commata.

II.35. The minor semitone is greater than three but smaller than four commas. Whence, it is clear that the apotome contains more than four and fewer than five commas. Laborious calculation did not deter the curiosity of the Greeks from thoroughly investigating how many commas are in a *diesis* [minor semitone], how many are in an *apotome* [major semitone], and finally how many are in a whole tone. If I had not known that it had been attempted by my predecessors, since it entails more labor than (as it seemed to me) usefulness and utility in musical melody, I would have left it aside. Nevertheless, those who desire to understand this can figure it out in the following way: Let a [256] and b [243] be the least numerical expression for the minor semitone, and c [524,288]⁶⁴ and d [531,441] the comma, as found in the 18th and 28th [propositions] of this [book].⁶⁵ I multiply b by c and by d and obtain

64 In Faber's printed table, the values of c (531,441) and d (524,288) must be flipped for the calculations to succeed.

65 *Elementa musicalia* II.18 ("Semitonii minoris minimos numeros reperire") and II.28 ("Comma in minimis numeris constituere"). This translates the corrected text of the 1551 edition (Paris: Guilemus Cauellat, 1551). The original 1496

e [127,401,984] and

f [129,140,163],

and I multiply a by c to produce

g [134,217,728].

According to the seventh [proposition] of the second book of the *Arithmetic*,⁶⁶ f to d [*recte*: e] is the relation [*habitudō*] of a comma. Likewise, according to the eighth [proposition] of the same,⁶⁷ g to e is the relation of a diesis, i.e., a minor semitone. Next, I multiply e by e and f by f and e by g to obtain

h [16,231,265,527,136,256],

k [16,677,181,699,666,569],⁶⁸ and

l [17,099,604,835,172,352]⁶⁹.

According to the sixth [proposition] of the fourth book of the *Arithmetic*,⁷⁰ it can be easily understood that k h contain two commas and, by the seventh [proposition] of the second book of the same, that l to h is the minor semitone. Then I multiply d [*recte*: e] by h and c [*recte*: f] by k and d [*recte*: e] by l and obtain in order

m [2,067,895,430,987,964,852,731,904],

n [2,153,693,963,075,557,766,310,747], and

o [2,178,523,581,616,950,626,746,368].⁷¹

Through the same sixth [proposition] of the fourth book it is simple to know that n m contain three commas and, through the seventh [proposition] of the second book, that o m contain the minor semitone. But the number n is recognized to be less than o ; therefore, o to m , the minor semitone, surpasses and exceeds three commas. Next, I multiply h by h and k by k and h by l and in order there arise

edition reads “per decimamseptimam et decimamoctauam huius reperti” (as found in the seventeenth and eighteenth [propositions] of this [book]).

66 Jordanus of Nemore's *De elementis arismetice artis*, printed (as the *Elementa arithmetices*) with a commentary by Faber in the same volume as the *Elementa musicalia* (Paris: Johannes Higman & Wolfgang Hopyl, 1496). The seventh proposition of the second book reads: “Si unus numerus duos multiplicet, erit productorum et multiplicatorum eadem proportio.” H. L. L. Busard, ed., *Jordanus de Nemore, De elementis arithmetice artis: A Medieval Treatise on Number Theory*, Texte und Abhandlungen zur Geschichte der Mathematik und der Naturwissenschaften 22, 2 vols. (Stuttgart: Franz Steiner Verlag, 1991), I.76–77.

67 *De elementis arismetice artis* II.viii (Busard I.77): “Si duo numeri unum multiplicent, erit que multiplicantium eadem productorum proportio.”

68 Numbers printed in bold indicate my emendations to Faber. Faber's value for k is 1,667,718,169,966,569, a probable printing error as subsequent calculations with k utilize the correct value.

69 Faber's value for l is 17,098,604,835,172,352, a probable error in Faber's original calculation, as subsequent calculations with l compound the error.

70 *De elementis arismetice artis* IV.vi (Busard I.97): “Numeros quotlibet secundum datam proportionem minimos investigare.”

71 Faber's value for o is 2,178,396,179,632,950,626,746,368, an error propagated from l .

$$p [263,453,980,612,401,802,360,312,389,697,536],^{72}$$

$$q [278,128,389,443,693,511,257,285,776,231,761],^{73} \text{ and}$$

$$r [277,548,226,488,785,437,877,530,747,994,112].^{74}$$

It is clear through the same reasoning as before: q p contain four commas and r p contain the minor semitone. But the number r is less than the number q ; therefore, four commas are greater than the minor semitone. But a corollary, hence, is known: that the greater semitone surpasses the minor semitone by a single comma, but the minor semitone contains more than three and fewer than four commas, as was just now seen; therefore, with the addition of a single comma, it follows that the greater semitone, which they call the apotome, contains more than four and fewer than five commas.

II.36. It is necessary that the whole tone contain more than seven commas. For the tone is comprised of the minor semitone and the apotome. The minor semitone, as in the above [proof], contains three commas and a bit more, and the apotome, as above, contains four commas and a bit more. Three plus four plus a bit more is more than seven. Therefore the whole tone contains more than seven commas.

Abstract

In an influential 1981 article, “Interpreting an Arithmetical Error in Boethius’s *De institutione musica* (iii.14–16),” André Barbera drew attention to a problematic set of arithmetical proofs. At *Ins. mus.* 3.14ff. Boethius purports to prove that (1) the minor semitone is larger than three commas but small than four, (2) the major semitone is larger than four commas but smaller than five, and (3) the tone is larger than seven commas but smaller than eight. All of these conclusions are correct, but the mathematical procedure seems inherently flawed, for Boethius manipulates the numerical difference between the terms of a ratio as if it were an accurate quantification of the resultant interval. Barbera maintained that the rationale motivating the erroneous mathematics “lies at the heart of Pythagorean cosmogony.” Boethius (Barbera argued) set out to find the numerical truth underlying acoustic phenomena and “seems to have been satisfied by the apparent numerical verification of what he could hear.” The origin of this “arithmetical error” can be specified with greater precision than the vague invocation of “Pythagorean cosmogony” allows. First, I demonstrate that Boethius’s arithmetical error is not nearly as erroneous as Barbera and others would have us believe; rather, it represents an approximate method of calculation used to prove relationships otherwise incalculable. Secondly, I argue that this

72 Faber’s value for p is 263,600,061,952,401,802,360,312,389,697,536, too far off to be merely a printing error.

73 Faber’s value for q is 328,128,389,443,693,511,257,285,776,231,761, a printing error?

74 Faber’s value for r is 277,531,995,223,258,301,621,530,747,994,112, another error propagated from l .

approximate method was not developed by Boethius but was faithfully translated from his immediate Greek source, Nicomachus of Gerasa's (now lost) *Eisagoge musike*. Thirdly, I suggest that this method, or at least its basic principle, was not independently developed by Nicomachus either, for similar arithmetical methods arose within the early stages of the Greek commentary tradition on Plato's *Timaeus*.

About the author

Andrew Hicks is an assistant professor of music and medieval studies at Cornell University, cross-appointed as the Medieval Latinist for Cornell's Graduate Program in Medieval Studies, and a member of the graduate fields of Classics and Near Eastern studies. His research focuses on the history of music theory and the ways in which the harmonic sciences were mobilized within philosophical and cosmological discourses in late antiquity and the early Middle Ages, with emphasis on the reception of Platonic cosmology, which forms the subject of his forthcoming book, *Composing the World: Harmony in the Medieval Platonic Cosmos*.