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Thâbit ibn Qurra's Generalization of the Pythagorean Theorem

By Aydın Sayılı *

IN AN Istanbul manuscript collection of scientific and philosophical articles, there are several monographs by Thâbit ibn Qurra, ninth-century mathematician and scientist. One of them is on the "Socratic proof" of the Pythagorean theorem.¹ This article is a letter written by the author to a friend who was somewhat disappointed with the "Socratic proof" of the Pythagorean theorem, as this proof deals with a special case only, namely, that of an isosceles right triangle, and he wished Thâbit to communicate to him the general proof. Thâbit's answer constitutes an elaborate treatment and an important contribution to the subject, although the request itself apparently did not envisage such a comprehensive answer.

The so-called Socratic proof refers to the well-known passage in Plato's *Menon*, but this work is not specifically mentioned in Thâbit's article. Thâbit conceives the question as one requiring the reconstruction of the "Socratic general proof," or as necessitating the establishment of a new proof which is in the spirit of the "Socratic special proof." Thus the fact that Euclid's *Elements* contains a general proof of the theorem does not make the question a superfluous one.

Thâbit ibn Qurra calls his method, which is in accord with the Socratic one, the method of reduction and composition, or the method of reduction to triangles and rearrangement by juxtaposition. He gives two different proofs for the general case, i.e., for the relation between the squares constructed on the sides of any right triangle, and both these proofs constitute examples of the method of reduction and composition.

The first proof is shown in Figure 1. The triangle is ABC; AA'BB' and DFB'D' are the squares of the two right sides, and ACDE is the square of the hypotenuse. The first two squares are obtained by adding triangles 1 and 2 to the shaded polygon, and the square of the hypotenuse is obtained by adding the same polygon to triangles 3 and 4. But, as all these triangles are equal by construction, the sum of the first two squares must be equal to the latter square. That Thâbit had given such a proof was known to us from another source, i.e., from Al Nayrizî's (ca. 900 A.D.) commentary of Euclid's *Elements*,² but apparently the rest of the information contained in the present article is new.

* Ankara University, Turkey. This paper was read at the December 1956 meeting of the History of Science Society.

¹ The volume is in the Library of Ayasofya Museum and is registered under the number 4832. Ritter gives a description of this manuscript collection which is apparently over nine hundred years old (H. Ritter, *Schriften Ja'*

qûb ibn Ishâq al-Kindî's in Stambuler Bibliotheken, Archiv Orientalni, 1932, 4: 363); the article in question occupies pp. 39a-41a. Another copy of this article exists in Cairo (H. Suter, *Die Mathematiker und Astronomen der Araber*, 1900, p. 37).

² Codex Leidensis, 399, 1: *Euclid's Elementa ex interpretatione Al Hadschdschadsch*

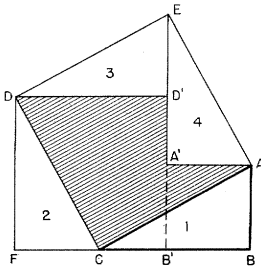


FIG. 1

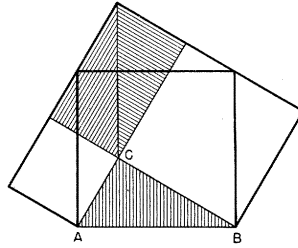


FIG. 2

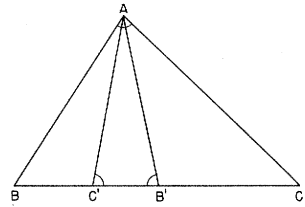


FIG. 3

The second proof, shown in Figure 2, is based on the fact that when, from the whole figure given, the three shaded triangles are deducted, the squares on the right sides are obtained, while the square of the hypotenuse results when from the same total figure the three triangles on the corners with heavily drawn sides are subtracted. As, however, all these triangles are equal to each other by construction, the sum of the two former squares is thus seen to be equal to the latter square.

Such proofs are familiar today and are found in certain text books. Their origin is sometimes traced to late ninth-century Indian mathematicians.³ Such proofs may therefore have been found independently by other mathematicians also. The more important contributions of Thâbit to the subject concern the further generalization of the theorem to any triangle.

After giving the above-mentioned two proofs, Thâbit compliments his friend for seeking a comprehensive knowledge of things and adds that the generalization achieved by the proofs given above may not be considered sufficient. One could wish, for example, he says, not to restrict oneself either to squares or to right triangles. In the first case, one would show, as Euclid has done, that the sum of any similar figures similarly located on the right sides of any right triangle is equal to a similar figure similarly placed upon the hypotenuse.⁴ In the latter case, on the other hand, one would wish to generalize the theorem to any triangle whatsoever.

To achieve this latter generalization, Thâbit draws from the vertex A of any triangle ABC two lines forming with the base the angles $AB'B$ and $AC'C$, both equal to A. (See Fig. 3.) He then states without proof that the sum of the squares of the sides AB and AC is equal to the rectangle $(BB' + CC') \times BC$; he adds that the proof can easily be obtained with the help of the *Elements*.

This theorem seems to have been rediscovered only recently.⁵ The theorem is reducible to or derivable from the Euclidean propositions II, 12 and II, 13,⁶

cum commentariis Al Narisii, ed. R. O. Besthorn and J. L. Heiberg, part 1, 1893, pp. 184-188; *Anarithi in decem libros priores Elementorem Euclidis commentarii*, ed. Maximilianus Curtze, *Euclidis Opera Omnia*, ed. I. L. Heiberg and H. Menge, *Supplement*, Leipzig 1899, pp. 84-86. See also J. Tropicke, *Geschichte der Elementar Mathematik*, 1923, 4: 143, 149. The Arabic text and Turkish translation of this article of Thâbit ibn Qurra, together with its explanation and analysis, have appeared in print in *Belleten*, quarterly

journal of the Turkish Historical Society: see, Aydın Sayılı, "Sâbit ibn Kurra'nın Pitagor Teoremini Tamimi," *Belleten*, 1958, 22: 527-549.

³ W. Lietzmann, *Der Pythagoreische Lehrsatz*, Stuttgart, 1953, p. 24. Earlier editions of this book appeared in 1930 and 1937.

⁴ Euclid, Book VI, proposition 31.

⁵ Lietzmann, *op. cit.*, 1937, pp. 38-40, 1953, pp. 46-47. The earlier edition of 1930 of his book does not contain this theorem.

⁶ See, A. Sayılı, *Belleten*, 1958, 22: 528-530.

and it is probable therefore that it was on these that Thâbit based his proof. Thâbit's article contains references to various special cases, e.g., that in which A is a right angle (B' and C' coinciding); those in which B or C is a right angle; the case when $A < 60^\circ$ (C' and/or B' being situated outside the segment BC); and the case in which $A > 90^\circ$ and $BB' + CC' < BC$.

Thâbit goes on to say that, should one wish to be in possession of a still more general theorem, i.e., one in which neither the type of triangle nor the shape of the figures drawn on its sides is restricted, then one may say that the sum of any similar figures similarly drawn upon two sides of any triangle is equal to a figure whose ratio to the similar figure similarly drawn upon the third side is the same as the ratio of $BB' + CC'$ to BC .⁷

In concluding, Thâbit remarks that our knowledge is perfect when it combines the most general and comprehensive with the special and the particular; for, he says, in our purely general knowledge the knowledge of the particular cases exists only potentially. He also states that in the course of instruction one has to follow a procedure in which there is a gradual increase in generalization and comprehensiveness, and adds that the reason why Socrates mentioned only the proof of a special case was that the person he was teaching was a beginner in the subject and not an advanced student.

Proof of the Theorem

$$\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2 + 2BA \cdot AC \cdot \cos A$$

$$= \overline{BC}^2 + BA \cdot AC (\cos C' + \cos B')$$

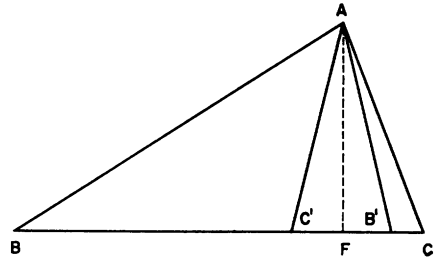
$$= \overline{BC}^2 + BA \cdot AC \cdot \frac{FC' + FB'}{AB'}; \text{ and from the}$$

similarity of triangles $AB'B$ and ABC

$$\text{we have } \frac{BA}{AB'} = \frac{BC}{CA}.$$

$$\therefore \overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2 + \frac{BC}{CA} \cdot AC (FC' + FB')$$

$$= BC(BC + C'B') = BC(BB' + CC').$$



In the case where $A > 90^\circ$, the same proof would naturally hold with an alteration made for the sign of $\cos A$. Thus the proof is quite general.

⁷ Thâbit makes no reference to Pappus' generalized Pythagorean theorem (see, e.g., Euclid, *Elements* [3 vol., 2nd edn., republished in 1956], I, 349-368, especially p. 366; or *Selections Illustrating the History of*

Greek Mathematics [translated by Ivor Thomas, 2 vol., 1939-1941], II, 575-579. Cf. Dr. E. Smith, *History of Mathematics* [2 vol., 1923-1925], II, 289, or Lietzmann, 1937, p. 31).