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Considers the distinction between history and heritage in mathematics and for mathematics education.

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History or Heritage? An Important Distinction in Mathematics and for Mathematics Education

Ivor Grattan-Guinness

To the fond memory of John Fauvel (1947–2001)

1. INTEREST AND DISAGREEMENTS. During recent decades there has been a remarkable increase in work in the history of mathematics, including its relevance to mathematics education. But at times considerable differences of opinion arise, not only about its significance but even concerning *legitimacy*—that is, whether or not an historical interpretation counts as history at all. In this paper I consider the latter issue, and also note some consequences for education.

The disagreements are general, in that they may arise for any branch of mathematics in any period or culture; so they need a general resolution. I offer one in the form of a distinction in the ways of interpreting a piece of mathematics of the past. Take such a mathematical notion N ; it could be anything from one notation through a definition, proof, proof-method or algorithm to a theorem, a wide-ranging theory, a whole branch of mathematics, and ways of teaching it. By its ‘history’, which becomes a technical term, one considers the development of N during a particular period: its launch and early forms, its impact, and applications in and/or outside mathematics, and so on. It addresses the question ‘What happened in the past?’ by offering descriptions. Maybe some kinds of explanation will also be attempted, to answer the companion question ‘Why did it happen?’.

History should also regard as important two companion questions, namely ‘What did not happen in the past?’ and ‘Why not?’. The reasons may involve the other side of this distinction, which I call ‘heritage’. There one is largely concerned with the effect of N upon later work, during any relevant period including that of its launch. Some modernised versions of N are likely to be taken, for heritage is largely concerned with the question ‘How did we get here?’, that is, to some current version of the context in question.

The distinction between history and heritage is often sensed by people who study some mathematics of the past, and feel that there are fundamentally different ways of doing so. Hence the disagreements can arise; one man’s reading is another man’s anachronism, and his reading is the first one’s irrelevance. The discords often exhibit the differences between the approaches to history usually adopted by historians and those often taken by mathematicians.

The claim put forward here is that *both history and heritage are legitimate ways of handling the mathematics of the past; but muddling the two together, or asserting that one is subordinate to the other, is not*. Many consequences flow from this stance, which will be treated in sections 3 and 4; first let us take a simple and well-known example, from the distant past.

2. PYTHAGORAS’S THEOREM, EUCLID STYLE. One of the best-known theorems in Euclid’s *Elements* (fourth century B.C.E.) concerns the sides of a right-angled triangle ABC in Figure 1.

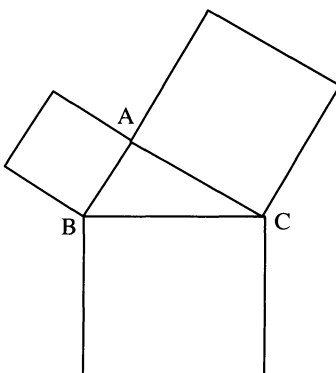


Figure 1.

We recognise it as saying of the sides AB, AC, and BC that

$$AB^2 + AC^2 = BC^2; \quad (1)$$

but Euclid actually says something quite different [11, Book 1, Proposition 47]: ‘In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle’. There is an attached diagram, of which Figure 1 is part, and the differences between it and (1) are basic. Not only is (1) algebraic whereas the figure is geometric; the diagram shows the squares outside the triangle, which (1) does not convey. Were any of the squares to lie over the triangle, then both (1) and the theorem would still be true; but the complicated proof, not shown in the figure, could not be effected. The algebraic character of (1) emerges further when, as was and is commonly done, the letters ‘a’, ‘b’, and ‘c’ are used for the sides: for algebra is the branch of mathematics in which special words and especially symbols are used to a significant extent to represent constants, unknowns, variables, and operations.

Another important difference concerns the word ‘on’. Euclid never used the phrase ‘side squared’, for in his geometrical Books he never multiplied geometric magnitudes together, either in the statement of theorems or (more importantly) in any proof. For example, he did not draw upon side-squaring when proving Pythagoras’s theorem, either in the complicated proof just mentioned, which relies upon congruence, or in a more elegant one for the more general theorem about rectangles with the same ratio of sides set upon the sides of the triangle, where the proof deploys similar triangles and ratio theory [11, Book 6, Proposition 31]. Thus ‘BC²’ is already a transgression from his geometry (and the frequent use in diagrams of small letters such as ‘a²’ even more so). Instead Euclid constructed a square *on* a given line—indeed, in the proposition immediately preceding Pythagoras’s theorem [11, Book 1, Proposition 46].

The issue is more profound than it may seem. Both here and everywhere else in the *Elements* Euclid works with *lines* rather than *lengths*, the latter being lines upon which some arithmetical measure has been imposed. Euclid presented geometry without *arithmetic* in the sense just explained; numbers are also present, but for other purposes, such as saying that this line is twice that line, or that the ratio of two lines is the same ratio as 5:7. In the same way he worked with planar regions but not (measured) areas, with solids but not volumes, with angles but not in degrees. By contrast, which is sometimes overlooked, in the arithmetical Books 7–9 multiplication of integers themselves occurs as usual [15].

These remarks concern the history of Euclid. When one moves to its heritage, then a quite different situation arises, in which (1) and many other such equations are prominent. For the *Elements* played a major role in the development of common algebra among some of its Arabic initiators, and a still greater one when Europe at last woke up during the twelfth century and began to elaborate that algebra with symbols introduced both for unknown quantities and for operations. *Both* (1) and Pythagoras's theorem as shown by the figure are legitimate readings of Euclid, but are quite different from each other.

The *Elements* is a particularly interesting historical example, because common algebra as in (1) became the dominating reading of Euclid (including in mathematics education) to such an extent that during the nineteenth century it also became the normal historical interpretation; apparently Euclid had been a 'geometric algebraist', talking geometry but really practising common algebra. A supporter of this reading was T. L. Heath, whose English edition and translation, first published in 1908, is still the most widely used, usually now in the second edition [11]. Greek specialists tell me that his translation is very reliable both to the language and to the mathematics; in particular, for Pythagoras's theorem and all other contexts he says there 'square on the side', not 'square of the side' as many earlier translations had rendered (the word 'apo' can admit both 'on' and 'of' as translations) but which can easily lead to the algebraic 'side squared'. Nevertheless, Heath added to his translation many algebraic versions of the propositions without seeming to notice the differences entailed.

While some historians of that time did not follow the algebraic interpretation of Euclid—for example, the Dutchman E. J. Dijksterhuis [26, chap. 5]—the standard view came under severe challenge only from the 1960s onwards. In particular, in the mid 1970s the historian Sabatei Unguru attacked it strongly, to the opposition of some mathematicians interested in history. Unguru's charges of anachronism and ahistory are largely vindicated; his mathematician opponents were inheritors [20].

We shall take another Euclid example in section 8. First, though, let us explore some general consequences of the distinction.

3. SOME PRINCIPAL DIFFERENCES BETWEEN HISTORY AND HERITAGE. The distinction between the history of a notion N and its heritage obviously involves its respective pre- and post-histories; but much more is at hand, for history has to use post-history also. To see this, let us consider the advice, which is quite often put forward for history of all kinds, about a way of being 'history-minded' about N (say, Pythagoras's theorem in Euclid); namely, forget everything that has happened since N was formed, and read Euclid with the eyes with which he wrote it. But this advice *begs the question at hand*. For in order to forget everything E that has happened since N , then one has to know E already; however, to do that one needs to be able to distinguish E from the history and pre-history of N ; but this is the task to be attempted.

Thus the distinction between history and heritage rests in part upon the *ways* in which notions later than N are to be used. When they are determined to be later notions, the view urged here is this: by all means bring them to bear, and deploy them to understand the heritage from N , but *avoid* feeding them back to appraise its history (such-and-such did not happen). Further, when considering periods intermediate between that of N and some later ones such as now, apply the distinction carefully. Thus, in our example the equation (1) is not only part of the history of René Descartes and the heritage of Euclid but also belongs to the heritage from, among others, the algebraist François Viète in the sixteenth century, whose work also belongs to the history of Descartes. Note also that history is usually *a history of heritages*; it is a tale

of mathematicians taking and modifying notions from the past (often pretty recent) without enquiring about the history of those notions.

Various other matters can be explored; a more detailed discussion, largely focussed upon history, is given in a companion paper [16]. The following table summarises the main features of handling past notions N in the two different ways suggested. An apparent contradiction between the third and fourth rows needs to be addressed. When the historian reconstructs past muddles, he will conflate notions that we now know to be different, a feature that the inheritor will stress. But the difference that the reconstruction exposes is that between past ignorance of the distinction, which is different from our (and the inheritor's) present knowledge of it.

Table.

Feature	History	Heritage
Motivation(s) to N	Important issue; maybe hard to find (for example, for Euclid's <i>Elements</i>)	Probably only of minor interest
Types of influence	Can be negative as well as positive; both should be noted	Likely to draw only upon the positive cases
Relationships of N to earlier and to later notions	Major issue; differences stressed as much as similarities, maybe more	Important issue; similarities stressed more than differences
Handling unclarities evident in N	Reconstruct them, and as clearly as possible	Recognise them, but clean them up
Successful developments	Very important; but also study failures, delays, missed opportunities, and late arrivals	Likely to be the main concern
Role of chronology	Usually important; can be hard to establish	Beyond broad details, not likely to matter so much
Historical consequences	May try to reconstruct the <i>foresight</i> (hopes, and so on) for N held by the historical figures	May try to construct <i>hindsight</i> and historical perspective of the developments after N
Determinism?	Preferably not claimed: the actual developments were so-and-so, but not necessarily so-and-so	May carry a determinist flavour; we <i>had</i> to get here (but see the history column!)
Foundations of a theory	Dig down to them, and build upon a swamp	Lay them down and build up from them, like on solid ground
Level of importance or popularity of N	Can vary over time, independently of content; should be noted (and maybe explained)	Not normally considered; current importance assigned

4. CHANGING HABITS. A further type of issue, which is not susceptible to tabular expression, concerns the use of notions that have become standard and therefore are now used habitually. Such habits may well help in determining the heritage from N: but historical anachronism can easily arise, which needs to be controlled. I note three important examples.

First, after an interesting history of its own from the 1870s [9], Georg Cantor's set theory has been part of our mathematical furniture for just over a century; so for the mathematics of this period its use may well be faithful. Now collections of things have been handled in mathematics since at least Greek antiquity; but the earlier theory of so doing was part-whole theory, where (say) British women form part of the class of women, membership is not distinguished from inclusion, and an object is not distinguished from its unit class. The differences between part-whole and set theories are considerable, both technically and philosophically, and the historian needs to mark them carefully. By contrast, the inheritor can deploy set theory with little chance of deception.

Second, while the influence of Euclid was great in Western mathematics, his stress on axioms and common notions was rarely imitated (though to some extent Newton's *Principia* is an example). The axiomatisation of mathematical theories became more prominent only during the late nineteenth century, especially in connection with the axioms of Euclidian and non-Euclidian geometries, and the emergence of abstract algebras [8]. Both developments attracted the attention of David Hilbert, and led him to launch the wide-ranging use of axiomatisation during the first half of the twentieth century, an attitude that has now become pretty standard: a clear path of heritage can be traced up to present-day practises. But the historian should be careful when looking at the structure of earlier mathematical theories, for axiomatisation may well not be prominent beyond specifying basic principles or laws. Cantor's set theory is a good example: while it too was developed during the late nineteenth century, he showed little interest in the axioms that it may require.

Third, vector and matrix theory have become standard fare in mathematics, though (especially in the second case) only from the 1930s onwards and after rather scrappy historical developments in various contexts during the nineteenth century. Once again, care should be exercised in applying them to earlier work. For example, much of the mechanics developed by figures such as L. Euler, J. L. Lagrange, and P. S. Laplace can be rewritten in vectorial and matricial forms, but historical understanding will not profit. For none of these figures knew that their theories could be developed in terms of strings or arrays of scalar elements; they worked instead in terms of collections of simultaneous linear or differential equations, or quadratic and bilinear forms [14, chaps. 5–6]. The introduction of vectors or matrices is *not* merely a matter of changing notation; new theories are involved. It is of course nice to save such space, for one thing; but if the historian does deploy these theories, then a chronological health warning should be appended.

By contrast to all these cautions to the historians, the inheritors can execute all these reformulations of theory quite legitimately; indeed, much nice heritage mathematics may emerge. Further, some history of mathematics produced after the initial period under study might be created; for, as was mentioned in section 1, mathematicians normally read the past in a heritage spirit.

As an example, take Lagrange and others in mechanics. A major problem, which he formulated in the 1770s, was to prove mathematically that the planetary system was stable. (Previous figures such as Newton and Euler had relied on God to watch out for danger; that is, a religion influenced mathematics.) In terms of matrix theory, Lagrange's brilliant theory sought proof of the reality of all the eigenvalues and

eigenvectors (to use modern terms) of a certain matrix. But he had no such theory, and worked with the corresponding quadratic forms; so did Laplace, who adapted his results to some extent; neither man found a watertight proof. The next major contribution came in 1829 from (surprisingly) A. L. Cauchy, and in 1829 he *did* formulate ‘tableaux’ of scalar entries in his own work on this problem [17]. Thus matrix theory may—indeed, should—be used to describe Cauchy’s contribution, and thus to help us to grasp an important part of his heritage from his predecessors. And we also have a nice example of the ‘What did not happen?’ question; for Cauchy never realised the significance of his achievement and rarely used it later, so that unfortunately he was not an influential founder of the spectral theory of matrices.

5. SOME PHILOSOPHICAL BACKGROUND. It is obvious that this talk of earlier and later notions, the development of theories, and so on, is not confined to mathematics: such features occur also in the histories of other sciences (including technology, engineering, and medicine), and indeed elsewhere (for example and a nice one, practices to be adopted and avoided in the so-called authentic performance of older music). The main general principles that underlie the foregoing discussion are as follows.

First, history is *unavoidable*, whether one likes it or not. A mathematician who presents his theory without concern with history is not thereby immune from it. For example, an enthusiast for axiomatics mentioned in section 3 will lay out his theory in a very formal way without reference to predecessors or precedents; but they will be there, including previous formal theories laid out by preceding axiomatists without reference to their own predecessors or precedents. Thus the question of whether or not one can use history in mathematics is *miscast*: it is rather the question of whether it is done consciously or not. Indeed, independent of the content of this paper, it is useful to have some general historical idea of a topic of interest, whatever it may be.

Second, knowledge and ignorance go together. This symbiosis has not received the general philosophical attention that it deserves. In particular and of special significance for mathematics, there is *knowledge of ignorance*, especially when one formulates a problem. When, for example, J. P. G. Dirichlet studied the convergence problem of Fourier series in the late 1820s, he knew that he did not know sufficient conditions on a function to establish convergence to it: finding some was precisely his problem. Having done so, he knew that he did not know whether or not they could be weakened, thereby setting the next problem in this chain (to which the first answer was the Lipschitz condition, by the way). One can also have ignorance of ignorance, or unawareness, where people do not know that they do not know something because the required connections between notions have not yet been laid down. Thus Dirichlet did not know that he did not know how his proof bore upon the specification of function spaces, because that notion did not emerge until the late nineteenth century [21].

Third, and following from the preceding line of thought, knowledge of all kinds is stratified into theory, metatheory, . . . For mathematics this means not only metamathematics of the technical kind that Hilbert launched, but also informal kinds. In particular, the history of notion N is one kind, its heritage is another, manners of its possible teaching a third, heuristic strategies to explain its significance a fourth, and there may well be others. The relationship between knowledge and ignorance just outlined lie in the metatheory of the notions involved. Similarly, metatheory requires metametatheory as its own forum for discussion, and so on upwards as far as is needed. An example of metametatheory is the history of the history of mathematics, an interesting story recently recorded in detail in [10]; the comments on Heath in section 2 form an example of it; and this paper itself is a self-referring example, with its heritage (if any) awaited!

The recognition of history and heritage as metatheoretic also releases both historians and inheritors from the need to like what they find in the past that they study. Why should they? After all, they were not there (as a rule). The point seems obvious enough; after all, one can be a good historian (or inheritor) of, say, military history without being a militarist. Yet not infrequently historians and inheritors become overly attached to their objects and figures of study, in any kind of history, and feel that they have to defend what they find. While of course such attachment can be felt if it arises naturally, no compunction to it should even be encouraged.

The generality of stratification is an insight forged in connection with symbolic logic in the early 1930s, thanks principally to Kurt Gödel and Alfred Tarski. In logic the distinction of (object-level) logic itself from metalogic is especially tricky but thereby all the more important; as was known already in Greek times, failure to make a distinction of some kind admits nasty paradoxes. Gradually stratification spread into other disciplines, especially mathematics and some types of philosophy. One follower, inspired by Tarski in the mid 1930s, was Karl Popper. Several parts of his philosophy of fallibilism are metaphilosophical; for example, his preference for indeterminism over determinism [19]. Of particular relevance to this paper is his essay 'On the Sources of Knowledge and Ignorance' [18, introduction], for it contains an insight largely missing from other kinds of philosophy; that *ignorance is nice*, for it is the site (in metatheory) of our problems when construed as knowledge of ignorance. In most other philosophies ignorance is a disease to be cured by the acquisition of knowledge however that acquisition is claimed to occur (see [25, chaps. 1–6] for the various forms of this view maintained within the sceptical tradition of philosophy). So far explicit use of stratification has not been widely canvassed among prevalent philosophies of history (which are well surveyed in [23]); but it seems worthy of further elaboration.

6. GENERAL REMARKS ABOUT HISTORY IN MATHEMATICS EDUCATION. In recent decades a considerable and international increase has developed in the use of history in mathematics education, in order to temper and challenge the normal picture of mathematics as a human-free zone, all answers but no questions, all solutions but no problems. Several edited or authored books and special issues of journals have appeared containing material of various kinds: textbooks significantly informed by the relevant history; summary histories of particular developments; surveys of the lives and works of important historical figures; international and/or multicultural comparisons of the development of (more or less) the same theories; translations of original texts with commentary; and suggested strategies for using history in teaching practice, both in specific contexts and in general. The emphasis often falls upon motivation and context, on showing that mathematics is after all human activity despite appearances, and moreover that much of it is not Western in origin. The range of concerns is well captured in a recent volume [12].

Most attention seems to have fallen on teaching at school and college level, but the university level has also been addressed. Much more work has been done on pure mathematics than on applied or applicable mathematics, or on probability and statistics; a redress of balance would be most welcome. I do not attempt to review this literature here, but I consider the place and utility of the distinction between history and heritage in mathematics education in general.

As with researchers in history mentioned in section 1, there is an evident sense of the distinction in this kind of educational literature, or at least an intuition that the mathematics of the past can be used in different ways. Where is mathematics education to be found between history and heritage? My answer is that *that is exactly where it should be found*, so that it can profit from *both* sides. In particular, if notion N is to be

taught, then both its history and its heritage can be used. Euclid's *Elements* is a good example, where the inherited use of algebra has been well used quite frequently. In addition, the historical Euclid deserves attention, with its geometry presented without *arithmetic* with lines rather than lengths, and the beautiful theory of ratios used in both his geometry and his arithmetic.

7. HISTORY-SATIRE AND THE CALCULUS. In the paper [13] I introduced long ago the term 'history-satire' to characterise a way in which history and also heritage can be used in mathematics education. Under it the broad features of the historical record are respected and used; but usually many detours and complications occur that, while they attract the historian, will impede teaching and so should be set aside or at most treated only in passing. The 'genetic method' of Otto Toeplitz, which he introduced initially in the late 1920s in connection with teaching the differential and integral calculus, is similar in sentiment [24]. More recently the Mathematical Association of America published a novel and important textbook in real-variable mathematical analysis by David Bressoud, in which he gives prominent places to the main developments, especially of the nineteenth century, such as Fourier series [5].

As Bressoud duly notes, a major innovation of the century was the founding of analysis in the 1820s by Cauchy. His approach was based upon a newly sophisticated *theory* of limits, not with limit left as an intuitive notion. Undoubtedly it was much superior to the preceding versions in the organisation of the subject and statements and proofs of the theorems; however the loss in heuristics was heavy, and both his colleagues and students objected forcefully to it [14, chaps. 10–11 *passim*, and 20.8].

For an explicit example, here is a use of history-satire that I found helpful in my own teaching. In a remarkable analogy, Cauchy adapted his real-variable analysis to complex variables and their functions and thereby introduced a major new subject into mathematics. But it seems a strange subject when first learnt: it uses the corresponding expressions as in real-variable analysis, but there are no curves, tangents to them, or areas underneath them to think about or look at. Among the many theorems that Cauchy proved, a main one is now named after him: namely, that the integral of a single-valued and differentiable function with a continuous derivative around and inside a closed contour C is zero. To students, including me long ago, it seemed to be a peculiar result; and a quick and doubtless valid proof using the Cauchy-Riemann equations and Green's theorem did not assuage the perplexity.

Cauchy developed his theory fitfully from the 1810s to the 1840s [22], and this version of his theorem is the last one, with the complex plane available as the site for C . I found that an earlier stage of his theory helped in understanding the theorem. In his treatment of the real-variable integral of $f(x)$ over the range $x_0 \leq x \leq X$ (I use his symbols) he formed the area sum S for a partition of values of x over the range, took successive subpartitions and formed the corresponding sums, and defined the integral as the limiting value of the sequence *if* it existed at all. This manner of defining the integral has long been standard, and his version is still worth reading and teaching [6, lecture 21].

Soon afterwards Cauchy deployed his analogy. He defined the integral of a finite-valued and continuous complex-variable function ' $f(x + y\sqrt{-1})$ ' by forming the expression corresponding to S for $f(x)$ but with $x + y\sqrt{-1}$ taking a sequence of values between the limits $A = 'x_0 + y_0\sqrt{-1}'$ and $B = 'X + Y\sqrt{-1}'$ for which both x and y were continuous functions of a parametric variable t . Then, drawing upon integration by parts and the calculus of variations, he proved that the value of this integral between A and B '*is independent of the nature of the functions*' involved [7, sec. 3]. The closed contour theorem then follows by taking the integral along one sequence of

values between A and B and back along another sequence under which the required conditions obtain; the two integrals for the two sequences cancel out, so that the value of the integral over C is zero.

Working through the theorem this way certainly took more than a few lines; but the understanding increased substantially, especially as the definition of the real-variable integral had already been taught elsewhere. My account follows Cauchy historically to the extent of deliberately avoiding diagrams, for both types of integral: at that stage in his career he regarded geometric notions as unrigorous and so wished to avoid them. The status of geometry makes a nice point to debate in the classroom, and in fact I increased the measure of satire by using diagrams myself. I also ignored several special cases and other details of the theory as Cauchy was then developing it. But I raised questions such as whether or not Cauchy assumed the derivative of f to be continuous (yes, but implicitly); and I also taught his 1825 version of the residue theorem, noting that, contrary to most later practice, he allowed $x + y\sqrt{-1}$ to go *through*, and not just *round*, a pole of $f(x + y\sqrt{-1})$ [6, sec. 8].

The considerations of this section have used the calculus and mathematical analysis because these case studies happen to come from it. But genetic approaches and history-satire can be applied to any mathematical notion or level of teaching.

8. THE PROPOSALS OF BASHMAKOVA. The relationships between knowledge and ignorance outlined in section 5 deserve serious consideration, including the niceness of ignorance as the source of problems (big or small) to tackle. One important area of education where these relationships are prominent is the design of a course syllabus and the manner and order of teaching the topics proposed, when in effect the designer is considering the stage at which the pupils or students should cease to be ignorant of some specific notions.

Let us take an example, examining the historiography proposed in recent years by the Russian historian I. G. Bashmakova, for two of her books have recently been translated into American and published by the Mathematical Association of America for their utility in mathematics education. While dealing with the history of common algebra, her position is put forward in a general way, most explicitly in a joint paper with I. M. Vandaloukis [4]. For them, there are two main stages in handling an historical text. ‘First the text should be “translated” into the [*sic*] contemporary mathematical language, i.e. an adequate model for it should be constructed. This is absolutely necessary in order to *understand* the text, to reveal its mathematical content’ (p. 251). In the next stage ‘it is necessary to *embed* the considered work in the context of science of its day’ (p. 252).

The authors state that the second stage is ‘more difficult’ than the first: in my view it might well be impossible, since the first stage will have put so much heritage in place that the historical context could be masked. They state the aim of heritage very clearly: ‘the mathematicians of every new age reconsider the previous material and restate it in new terms, thus making it readily available and applicable for the contemporary scientist’ (p. 250).

The examples given in Bashmakova’s writings seem to exhibit the conflation of history and heritage, without the stress on the distinction between them that was argued in section 7. For example, she takes Proposition 4 of Book 2 of Euclid’s *Elements* to express the quadratic identity

$$(a + b)^2 = a^2 + 2ab + b^2 \quad (2)$$

as a legitimate prime reading [1, p. 88], [3, p. 165].

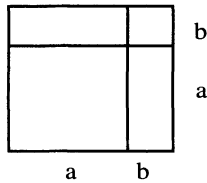


Figure 2.

In a more recent short history of algebra, coauthored with G. Smirnova, (2) is held to be ‘equivalent’ to the diagram [3, chap. 2]; and throughout this book the modern notations dominate, although the older terms and symbols are also presented in some detail [3, chaps. 4–5]. The dominance of heritage is clear in a general historiographical appendix, where in specifying the term ‘geometric algebra’ the authors characterise algebra as an historical category drawn from ‘the class of problems associated with algebra today’ (p. 164). For them, therefore, Euclid’s Book 2 is concerned with algebraic identities such as (2) (see especially Bashmakova in [2]): indeed, her most recent stance is to impose algebraic readings onto ancient arithmetic and geometry for *all* cultures (see [3, pp. 163–172], where Bashmakova and Smirnova vote for the mathematicians and against Unguru in the disagreement noted earlier in section 2).

The preference for modern notations in the book fits its primarily educational purpose well, exposing an important chain of heritage influences. But the quoted general statements of *historical interpretation* seem to involve heritage mistaken as history. For me, in Book 2 Euclid presents theorems relating subregions of planar rectilinear constructions involving rectangles, squares, and triangles (as in the cited example, where the relative locations of the subsquares and subrectangles are lost in the ubiquitous sign ‘+’); the algebraic content is empty, as also in all his other geometry Books. By contrast, algebra looms very large in the post-Grecian heritage from Euclid’s geometry. Both readings are valuable to mathematics education, though better presented as distinct sources. Indeed, like Euclid himself the history of the theory of polynomial equations is especially suitable for historical satire.

9. CONCLUDING REMARK. In this paper, and in more detail in its companion [16], I assert that the history of mathematics differs fundamentally from heritage studies in the use of the mathematics of the past, and that both are beneficial in mathematics education when informed by the mathematics of the past. The majority of the examples presented come from fairly modern periods. This is no accident, for they constitute my specialist areas; thus the examples as such have no particular significance. Indeed, since the distinction between history and heritage is held to be general, then indefinitely many more examples could be presented; the reader is invited to construct some of his or her own. A rich resource comes from considering the many ways in which notions are changed, especially when they are (major) theorems or theories. These include the alteration of known results by extension, generalisation, and/or abstraction; reaction to counterexample; the exposure as axioms or as procedures of assumptions previously taken for granted; the adaptation of algorithms; the introduction, or maybe removal, of connections between branches (such as geometry with or without arithmetic); classifications into kinds of objects in a theory; switches between axiom, theorem, and definition; and new applications, both within mathematics and to other disciplines.

More attention has been paid in this paper to issues concerning history and historiography than to heritage and heritage studies; but no value judgement is involved, for,

as stated in section 1, neither activity is subordinate to the other one. A companion paper concentrating on good and bad practices in heritage work could be written. The two activities are distinct but they interact in fruitful ways, each posing questions for the other to address.

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The Frobenius endomorphism is an endomorphism: A new proof

Let $\mathbb{F}$ be a field of characteristic p . We consider the *Frobenius endomorphism*, the map $\phi : \mathbb{F} \rightarrow \mathbb{F}$ given by $\phi(x) = x^p$. The usual way to show that $\phi(x + y) = \phi(x) + \phi(y)$ is to use Newton's binomial formula and to notice that p divides $\binom{p}{k}$ whenever $0 < k < p$.

Here is an alternative proof that uses formal derivatives of polynomials, avoiding binomial coefficients. We fix y in $\mathbb{F}$ and consider the polynomial $P = (X + y)^p$ in $\mathbb{F}[X]$. Then P is monic of degree p and can be written $P = X^p + \sum_{i=0}^{p-1} a_i X^i$. The formal derivative $D(P)$ of this polynomial gives at the same time $D(P) = p(X + y)^{p-1} = 0$ and $D(P) = \sum_{i=1}^{p-1} i a_i X^{i-1}$, so $a_i = 0$ for $i = 1, 2, \dots, p - 1$. Thus P has the form $X^p + a_0$. Now we substitute 0 for X , showing that $a_0 = y^p$, i.e., $(X + y)^p = X^p + y^p$. Since we can substitute any element x of $\mathbb{F}$ for X and since y in $\mathbb{F}$ was arbitrary, the relation $\phi(x + y) = \phi(x) + \phi(y)$ for all x and y follows.

—Submitted by Marc Bernot, ENS, Cachan, France