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The Origin of Mathematics 3

C H A P T E R

Pythagoras of Samos

Pythagoras lived about 570–490 B.C.E. The only roughly determined date in his life is ≈ 530 , when he left Samos to settle in Crotona, in southern Italy. At Crotona he founded a religious and philosophical society that soon came to exert considerable political influence in the Greek cities of southern Italy. He was forced to leave Crotona about 500 and retired to Metapontum, where he died (see Fig. 6.1).

The Pythagoreans, as his followers were called, continued to exert political power until sometime in the middle or late fifth century, when a democratic revolution occurred and they were forced to leave the Greek cities of southern Italy. Some of them went to Sicily and others to the Greek mainland, where they found new centers for their activities. The last of the Pythagoreans were known in about 350 B.C.E. as poor vegetarian wandering pilgrims.

The city (and island) of Samos together with its close neighbors Miletus and Ephesus on what is now the Turkish coast were booming economic and intellectual centers in the sixth century B.C.E. Thales and his student Anaximandros taught in Miletus in the first half

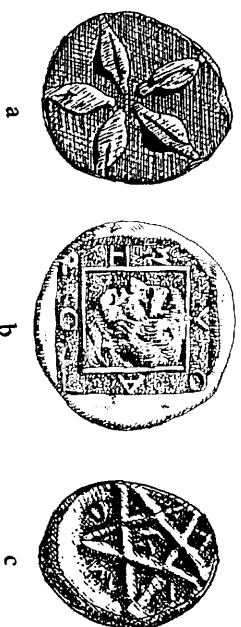


FIGURE 6.1 (a) Coin from Metapontum with a “pentagram” composed of grains of barley (about 440 B.C.E.?). (b) Coin from Abdera (430 B.C.E.?) showing an idealized portrait of (Π) YΘAIOPHΣ, that is, (P)YTHAGORES. (c) Coin from Melos (before 420 B.C.E.) with pentagram

of the sixth century. The philosopher Heraclitus of Ephesus was a contemporary of Pythagoras.

Two examples of practical geometry will illustrate the atmosphere of the city of Pythagoras’s youth. Outside of Samos was the ancient sanctuary of the goddess Hera. In 570–560 the city commissioned the architects Rhoikos and Theodoros with the construction of a new temple of dimensions hitherto unheard of. Its ground plan was 100×200 cubits (52.5×105 meters); its 104 columns were 18 meters high. The bases of the columns had a diameter of up to 1.80 meters and weighed about 1500 kg each. In spite of these extraordinary dimensions, they were turned on lathes! This temple was to be the prototype of the Ionian style in architecture. It was constructed during Pythagoras’s youth. Just completed, it was destroyed in a revolution in the 530s, which brought the tyrant Polykrates into power. Polykrates immediately ordered the construction of a new and even bigger temple.

The second example of high technology in the second half of the sixth century in Samos is the tunnel of Eupalinus. This tunnel, for a new water main, was about 1 km long and, this is the noteworthy point, was built from both sides of the mountain! The diggers met in the middle of the mountain with a deviation of about 10 meters. The people who gave the money for the construction did indeed trust geometry and geodesics.

Such was the background of Pythagoras’s youth. He may have traveled to Egypt and Babylon for studies, but Samos was where the action was in his time.

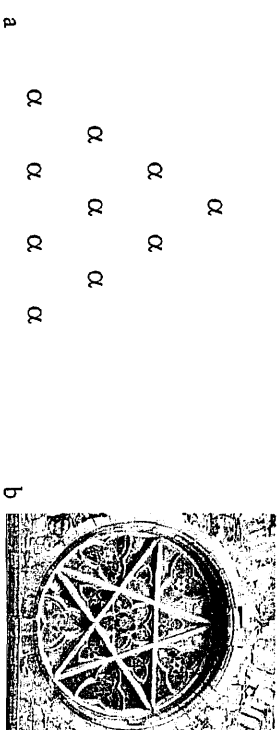


FIGURE 6.2 (a) Tetrakys after Iamblichus: (α stands for the unit one.) (b) Pentagram from St. Mary’s Church, in Lemgo, Germany (1300 C.E.).

The Mathematics of the Pythagoreans

Almost nothing is known for sure about Pythagoras’s own mathematical achievements. The oral tradition of his followers was written down rather late, some of it by the students of Aristotle. The best available information about Pythagoras is collected in the book by Burkert [1972]. It seems that the students of Pythagoras in southern Italy were split into two groups: the Akusmatikoi, who followed the master’s sayings verbatim, and the Matematikoi, who tried to develop and expand his teachings. The Pythagoreans used two symbols: the so-called tetrakys, the arrangement of ten dots representing $1 + 2 + 3 + 4 = 10$, and the pentagram (Fig. 6.2).

They used to call the pentagram “Health.” (There is indeed a very old tradition of the pentagram as a medical symbol, cf. Notes.) Undoubtedly, musical harmonics was central to Pythagoras’s teachings. The Middle Ages knew him as “inventor musicae,” not as a mathematician. Aristotle writes about the Pythagoreans:

Contemporaneously with these philosophers and before them, the Pythagoreans, as they are called, devoted themselves to mathematics; they were the first to advance this study, and having been brought up in it they thought its principles were the principles of all things. Since of these principles numbers are by nature the first, and in numbers they seemed to see many resem-

balances to the things that exist and come into being—more than in fire and earth and water (such and such a modification of numbers being justice, another being soul and reason, another being opportunity—and similarly almost all other things being numerically expressible); since, again, they saw that the attributes and the ratios of the musical scales were expressible in numbers, since, then, all other things seemed in their whole nature to be modelled after numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number. And all the properties of numbers and scales which they could show to agree with the attributes and parts and the whole arrangement of the heavens, they collected and fitted into their scheme; and if there was a gap anywhere, they readily made additions so as to make their whole theory coherent. (*Metaphysics* 985 b 23–986 a 7)

In the first part of this quotation, Aristotle praises the Pythagoreans for their mathematical studies. In the second part, he criticizes them for unfounded speculations. It seems that there were four “mathematical” parts of the Pythagorean teachings: arithmetic, geometry, harmonics (music), and astronomy. This is the classical “quadrivium”, part of the seven liberal arts. In the Middle Ages, the three “trivial” parts, the “trivium,” grammar, rhetoric, dialectics, were the introductory ones of the seven undergraduate courses. Plato was the first to call the subjects of the quadrivium “mathemata,” things to be learned. (*Republic* 527 c 10)

Arithmetic

Aristotle points out the prominent role of numbers in Pythagorean mathematics. There are considerable traces of Pythagorean arithmetic in the writings of Nicomachus, which we will present and contrast to Euclid’s style in an appendix to the arithmetical Book VII of Euclid.

The Application of Areas

We have seen the first example of the application of areas in I.44, the simple application of an area to a line. The next, and more substantial, examples would be II.5,6 when supplemented by the theorem of Pythagoras so as to give the solution of a quadratic problem. (Readers should compare the more detailed statements and comments in the discussion of Book II.) It reads like this. Let an area c^2 and a line b be given. Find a line x such that

$$bx - x^2 = c^2 \quad (\text{application with square defect})$$

or

$$bx + x^2 = c^2 \quad (\text{application with square excess}).$$

The geometric equivalent proceeds like this. (We take the case with quadratic excess.) By II.14, the given area might be a square. By II.6, we are able to transform the problem into the geometric version of

$$\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2 = c^2 + \left(\frac{b}{2}\right)^2.$$

Pythagoras’s theorem then provides us with z such that

$$c^2 + \left(\frac{b}{2}\right)^2 = z^2,$$

and from this we find x as in Fig. 6.3. Proclus ascribes the discovery of this technique to the Pythagoreans in his comment on *Elements* I.46:

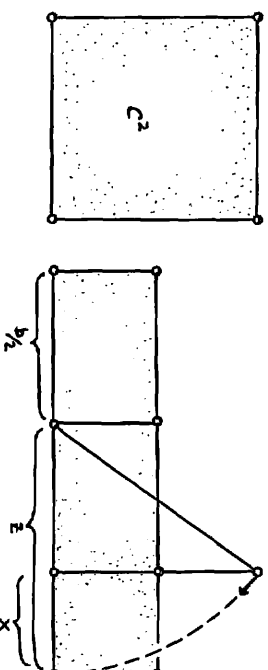


FIGURE 6.3

Eudemus and his school tell us that these things—the application of areas, their exceeding and their falling short—are ancient discoveries of the Pythagorean muse.... Those godlike men of old saw the significance of these terms in the describing of plane areas along a finite straight line. (Proclus–Morrow p. 332)

Incommensurable Segments

Incommensurable segments, or, as we might say today, segments of irrational length with respect to a given unit segment, are one of the most important discoveries of the Greek mathematicians. Pappus tells us that the theory of incommensurable lines “had its origin in the school of Pythagoras” (Heath [1921], 154). The history and significance of this subject will be discussed in an appendix to Book X.

The Dodecahedron

One of the “mathematikoi” students of Pythagoras was Hippasus of Metapontum. Sources from late antiquity report about him that he

was a Pythagorean but, owing to his being the first to publish and write down the (construction of the) sphere with the twelve pentagons, perished by shipwreck for his impiety, but received credit for the discovery. (Heath [1921], 160 after Iamblichus)

Again there will be a special appendix dealing with the regular polyhedra and their history following Book XIII.

The Pythagorean Theorem as a Paradigm for Mathematics

The most frequent answer to the question of what people who are disconnected from mathematics in their adult life remember of school mathematics is “Pythagoras.” And I think that this is as it should be. I would rank the Pythagorean theorem as a cultural asset of the first order, the knowledge of which should be bestowed upon every student as a standard for life. It belongs to a basis of common intellectual possessions of mankind. In Greek antiquity, cultural coherence rested on the general widespread intensive knowledge of Homer. In the Middle Ages the Bible (besides the Latin language) had a similar function in Western Europe. More recently, in the English-speaking world one recognizes cultural coherence in that an allusion to Shakespeare’s *Hamlet* is immediately understood. (In German, Goethe’s *Faust* plays an analogous role.) Mathematics, however, is independent of specific languages and transcends cultural borders. The theorem of Pythagoras is taught and understood all over the world. And it is more important than pop-music.

An approach to the theorem of Pythagoras through a work of art will help us to understand its significance for mathematics as a whole.

When I saw from afar a sculpture by the German sculptor Helmut Lander at an exhibition my spontaneous reaction was, “Pythagoras at work on the square on the side.” (See Fig. 6.4.)

The right-angled triangle and the square block lead a mathematician immediately to the association “Pythagoras,” but the artist himself chose “Sisyphus” as the title of his work. Can we adhere to “Pythagoras,” beyond the external similarity, and argue with the sculptor about his title?

A few remarks to this purpose about the Pythagorean theorem. One indeed knows the figure, but the assertion of the theorem does remain invisible. The equality of the areas expressed in $a^2 + b^2 = c^2$ cannot be seen from the figure. Entirely different from the case of the intersection of the three altitudes of a triangle, knowledge here eludes the direct view. Only the proof gives a reason for believing

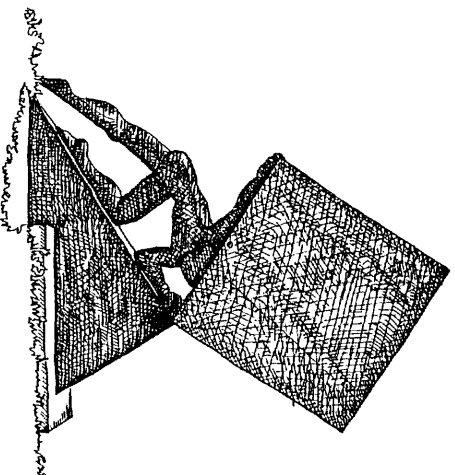


FIGURE 6.4

and insight into the truth of the theorem. And that is exactly what one calls “deep” in mathematics.

Here something is not just verified, or something that in any case could be read off from a good drawing: it is placed within the framework of logical deduction. The theorem of Pythagoras is one of the few important examples of substantial mathematics that is usually treated in school. Quite rightly does it often stand as a symbol for all of mathematics. It is a good sign for the intellectual perception of many nonmathematicians if it is exactly this theorem that they remember from their schooldays. Part of the symbol is also the laborious way to knowledge and the drudgery with formulas and proofs that on the boundaries of research is the same as initiation in high school.

Though to be sure, with some real distinctions—which brings us back to the sculpture: In research there is no longer a teacher who knows the way and has the solution ready. One works hard and often, again and again, in vain. One believes that one has just mastered the problem when the whole structure breaks down again at some decisive point. And if one finally manages to put everything together successfully, every answer simply generates new questions. The game starts all over again.

Thereby we would be with Sisyphus, who in never-ending effort pushes his block up the mountain. Our modern image of this ancient figure is really stamped with the existential meaning that Albert Camus has given him: the heroic man who consciously takes upon himself the contradiction of life. To him corresponds the formal figuration of the sculpture. Exerting all its power, the organic form asserts itself between the overpowering abstract figures of the triangle and the square. It is lost if the blocks slam together: In the geometrically precise conception there is no place for human beings.

And so it is with the Pythagorean theorem: When the assertion is ready, it stands there in cool precision. The human element enters in the proof where the squares on the sides must be transformed and reformed until they come together again on the hypotenuse.

So now should one say “Pythagoras” or “Sisyphus”? Lander himself says it doesn’t matter. The figure forms a bridge between the proverbial two cultures of science and the humanities (where mathematics, in the sense of this classification, belongs to the sciences, even if she should protest against it). I plead for Pythagoras. Naming it so would be more unconventional and would provoke more association and further thought. The starting point becomes more specific, and the viewer is spoken to more personally than with the rather philosophical Sisyphus, who then appears already alone.

IN THE MOUNTAIN 'THE CREATION'
OF MATHEMATICS