

Mathematics and Music Theory: From Pythagoras to Rameau

*M*usic theory is a wide and beautiful subject, and some basic mathematical ideas are inherent in it. Some of these ideas were introduced in music theory by mathematicians, and others by musicians with no special mathematical skill. This paper describes some

of the connections between music theory and mathematics. The examples are chosen mainly from the works of Pythagoras and of J. Ph. Rameau, who both were important music theorists, although the former is usually known as a mathematician, and the latter as a composer.

Before going into the works of Pythagoras and Rameau, I present, in the next section, a summary history of the relation between music and mathematics.

A Few Historical Markers

I start with Greek antiquity.

It is well known that the schools of Pythagoras, Plato, and Aristotle considered music as part of mathematics, and a

Greek mathematical treatise from the beginning of our era would usually contain four sections: Number Theory, Geometry, Music, and Astronomy. This division of mathematics, which has been called the *quadrivium*¹ (the “four ways”), lasted in European culture until the end of the middle ages (ca. 1500). One can see bas-reliefs and paintings representing the four branches of the *quadrivium* on the walls or pillars of cathedrals in several places in Europe (see for instance the pictures in [1]). The situation changed with the Renaissance, when theoretical music became an independent field, but strong links with mathematics were maintained.²

Several important mathematicians of the seventeenth and eighteenth centuries were also music theorists. For in-

¹This terminology is due to Boëthius (ca. 480–524 AD), who worked on the translation and the diffusion of Greek science and philosophy in the Latin world. He is responsible in particular for a Latin translation and a commentary of the mathematical treatise of Nichomachus. Boëthius considered the study of the *quadrivium* to be a prerequisite for philosophy, and this idea was at the basis of Western European curricula for almost ten centuries.

²The AMS subject classification has a section called Astronomy, but none called Music.

stance, the first book that René Descartes wrote is on music (*Compendium Musicae*, 1618). Marin Mersenne wrote several treatises on music, among them the *Harmonicorum Libri* (1635) and the *Traité de l'harmonie universelle* (1636), and he had an important correspondence on that subject with Descartes, Isaac Beekman, Constantijn Huygens, and others. John Wallis published critical editions of the *Harmonics* of Ptolemy (2d c. AD), of Porphyrius (3d c. AD), and of Bryennius (a Byzantine musicologist of the fourteenth century). Leonhard Euler published in 1731 his *Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae*. Jean d'Alembert wrote in 1752 his *Eléments de musique théorique et pratique suivant les principes de M. Rameau* and in 1754 his *Réflexions sur la musique*; and there are many other examples.

Until well into the Renaissance, the term “musician” referred to music theorists rather than to musical performers. Research and teaching in music theory were much more prestigious occupations than musical composition or performance. Some famous mathematicians were also composers or performers, but this is another subject.³

J. Ph. Rameau, who is certainly the greatest French musicologist of the eighteenth century, wrote in his *Traité de l'harmonie réduite à ses principes naturels* (1722):

La musique est une science qui doit avoir des règles certaines; ces règles doivent être tirées d'un principe évident, et ce principe ne peut guère nous être connu sans le secours des mathématiques. Aussi dois-je avouer que, nonobstant toute l'expérience que je pouvais m'être acquise dans la musique pour l'avoir pratiquée pendant une assez longue suite de temps, ce n'est cependant que par le secours des mathématiques que mes idées se sont débrouillées, et que la lumière y a succédé à une certaine obscurité dont je ne m'apercevais pas auparavant.

Music is a science which must have determined rules. These rules must be drawn from a principle which should be evident, and this principle cannot be known without the help of mathematics. I must confess that in spite of all the experience which I have acquired in music by practising it for a fairly long period, it is nevertheless only with the help of mathematics that my ideas became disentangled and that light has succeeded to a certain darkness of which I was not aware before.

³For instance, Pythagoras, according to his biographers, besides being a geometer, a number-theorist, and a musicologist, was a composer, and he also played several instruments; see e.g., [7], Chapter XV, p. 32.

Music theory as well as musical composition requires a certain abstract way of thinking and contemplation which are very close to mathematical pure thought. Music makes use of a symbolic language, together with a rich system of notation, including diagrams which, starting from the eleventh century (in the case of Western European music), are similar to mathematical graphs of discrete functions in two-dimensional cartesian coordinates (the x -coordinate representing time and the y -coordinate representing pitch). Music theorists used these “cartesian” diagrams long before they were introduced in geometry. Musical scores from the twentieth century have a variety of forms which are close to all sorts of diagrams used in mathematics. Besides abstract language and notation, mathematical notions like symmetry, periodicity, proportion, discreteness, and continuity, among others, are omnipresent in music. Lengths of

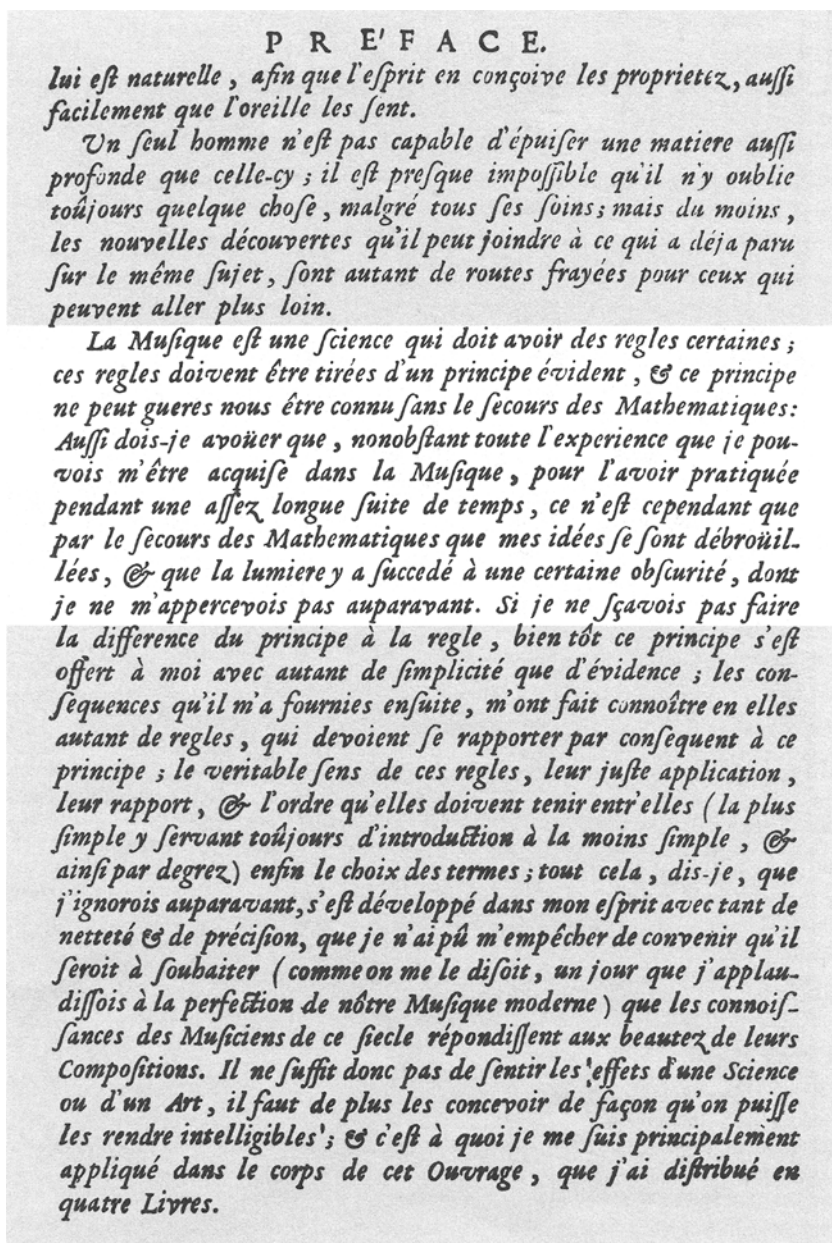


Figure 1. Glowing words from Rameau's *Traité de l'harmonie réduite à ses principes naturels*.

musical intervals, rhythm, duration, tempi, and several other musical notions are naturally expressed by numbers. The mathematical use of the word “harmonic” (for instance in “harmonic series” or “harmonic analysis”) has its origin in music theory. The composer Milton Babbitt, who taught mathematics and music theory at Princeton University, writes in [2] that a musical theory should be “statable as a connected set of axioms, definitions and theorems, the proofs of which are derived by means of an appropriate logic.”

It is important to realize that there are contributions in both directions. On the one hand, mathematical language and mathematical ideas have shaped the language and the concepts of music theory. This is illustrated in the work of Rameau discussed below, but there are several other instances. For example, Milton Babbitt uses group theory and set theory in his theoretical musical teaching and in his compositions. Olivier Messiaen speaks of “symmetric permutations.” Some pieces of Iannis Xenakis are based on game theory, others on probability theory; and so on.⁴ On the other hand, questions and problems arising in music theory have constituted, at several points in history, strong motivation for investigations in mathematics (and of course in physics). For example, phenomena like the production of beats or the production of the harmonic frequencies were noticed and discussed by music theorists several decades before they were explained by mathematical and physical theories. Some of the theories developed in the seventeenth century by Wallis, J. Sauveur, and others were essentially motivated by these phenomena. I shall discuss the question of the harmonic frequencies in the last part of this article. It is also fair to acknowledge that there are instances where music theorists have used mathematical notions in an intuitive manner, before these notions had been shaped and refined by mathematicians. One such example is the use of logarithms, also discussed below.

Now let us start from the beginning, that is, with Pythagoras.

Pythagoras and the Theory of Musical Intervals

Historians of science usually agree that Pythagoras (sixth c. BC) is at the origin of mathematics as a purely theoretical science.⁵ At the same time, Pythagoras is regarded as the first music theorist (from the point of view of European music). The major musical discovery of Pythagoras is the relation of musical intervals with ratios of integers. This is described by Jamblichus ([7], Chap. XXVI, p. 62) in these terms: Pythagoras was “reasoning with himself, whether it

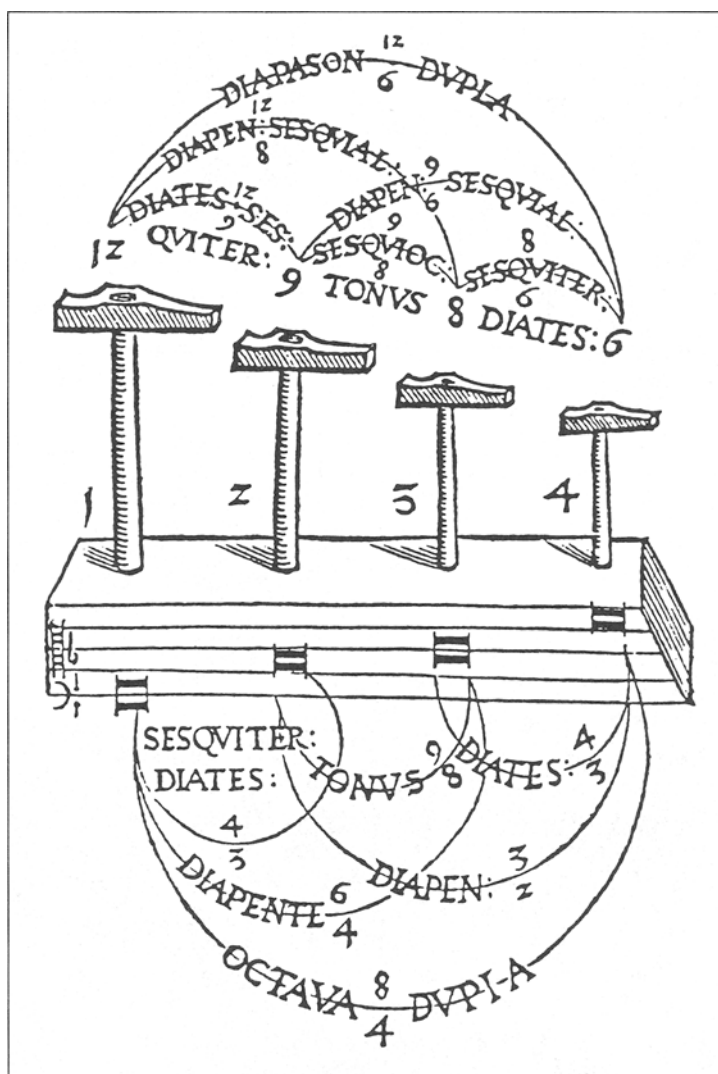


Figure 2. The hammers of Pythagoras, according to Gafurius (1492).

would be possible to devise instrumental assistance to the hearing, which could be firm and unerring, such as the sight obtains through the compass and the rule.” Walking through a brazier’s shop, Pythagoras heard the different sounds produced by hammers beating an anvil. He realized that the pitch, that is, the musical note, that was produced by a particular hammer, depended only on the weight of the hammer and not on the particular place where the hammer hit the anvil, or on the magnitude of the stroke. Pythagoras realized also that the compass of a musical interval between two notes produced by two different hammers depended only on the relative weights of the hammers, and in particular that the *consonant* musical intervals, which in classical Greek music were the intervals of octave, of fifth and of fourth, correspond, in terms of

⁴The idea of using mathematical theories in musical composition is not new. Athanasius Kircher, a seventeenth-century mathematician at the Court of Vienna, wrote a treatise on musicology, *Misurgia Universalis* (1622) in which he described a machine, *Arca Musicarithma*, which produces musical compositions based on mathematical structures.

⁵The theories and results which Pythagoras and his school developed were not intended for practical use or for applications, and it was even forbidden for the members of the Pythagorean school to earn money by teaching mathematics, and the exceptions confirm the rule: Jamblichus (see [7], Chap. XXIV, p. 48) relates that “the Pythagoreans say that geometry was divulged from the following circumstances: A certain Pythagorean happened to lose the wealth that he possessed; and in consequence of this misfortune, he was permitted to enrich himself from geometry.”

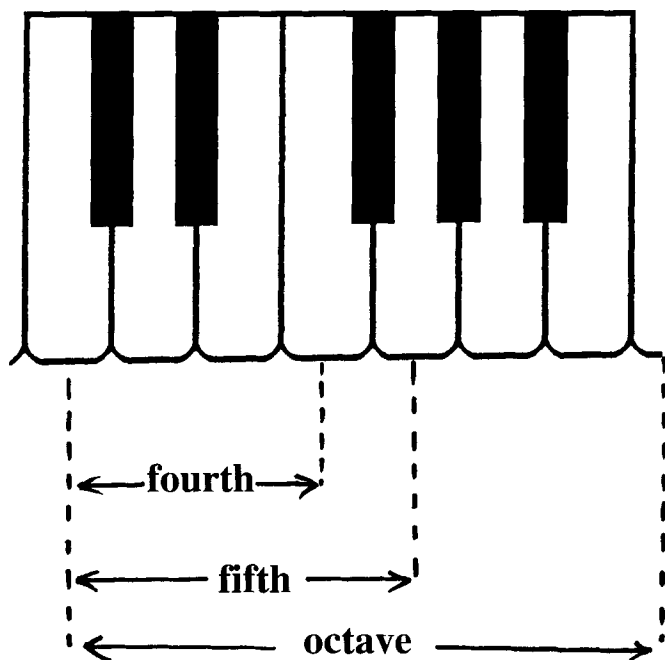


Figure 3. The classically “consonant” intervals.

weights, to the numerical fraction $2/1$, $3/2$, and $4/3$, respectively. Thus, Pythagoras thought that the relative weights of two hammers producing an octave is $2/1$, and so on. As soon as this idea occurred to him, Pythagoras went home and performed several experiments using different kinds of instruments, which confirmed the relationship between musical intervals and numerical fractions. Some of these experiments consisted of listening to the pitch produced by the vibrations of strings that have the same length; he had suspended the strings from one end and attached different weights to the other end. Other experiments involved strings of different lengths, which he had stretched end-to-end, as in musical instruments. He also did experiments on pipes and other wind instruments, and all these experiments confirmed him in his idea that musical intervals correspond in an immutable way to definite ratios of integers, whether these are ratios of lengths of pipes, lengths of strings, weights, etc.⁶

Theon of Smyrna, in Part 2, Chapter XIII of his mathematics treatise [12], describes other experiments which illustrate this relation between musical intervals and quotients of integers. He relates, for instance, that the Pythagoreans considered a collection of vases, filled partially with different quantities of the same liquid, and observed on them the “rapidity and the slowness of the movements of air vibrations.” By hitting these vases in pairs and

listening to the harmonies produced, they were able to associate numbers to consonances. The result is again that the octaves, fifths, and fourths correspond respectively to the fractions $2/1$, $3/2$ and $4/3$, in terms of the quotients of levels of the liquid.

These experiments were repeated and reinterpreted by the acousticians of the seventeenth century. The ideas and observations of Pythagoras and his school established the relation between musical intervals and ratios of integers.

Logarithms

The arithmetic of musical intervals involves in a very natural way the theory of logarithms. For an example, we return for a moment to Jamblichus, who relates in Section XV of [7] that Pythagoras defined the *tone* as the *difference* between the intervals of fifth and of fourth. (The definition may seem circuitous, but it becomes natural if we recall that the definitions of musical intervals had to be based on those of consonant intervals, which are naturally recognisable by the ear.) The point now is that the fraction associated to the tone interval is not the difference $3/2 - 4/3$, but the *quotient* $(3/2)/(4/3) = 9/8$.

It is natural to define the compass of a musical interval as the number (or the fractions of) octaves it contains. Thus, when we say that two notes are n octaves apart, the fraction associated to the interval that they define is 2^n . The definition of the compass can be made in terms of frequency, and in fact one usually defines the *pitch* as the logarithm in base 2 of the frequency. (Of course, the notion of frequency did not exist as such in antiquity, but it is clear that the ancient Greek musicologists were aware that the lowness or the highness of pitch depends on the slowness or rapidity of the air vibration that produces it, as explained in Theon’s treatise [12], Chapter XIII.) The relation of musical intervals with logarithms can also be seen by considering the lengths of strings (which in fact are inversely proportional to the frequency). For instance, if a violinist (or a lyre player in antiquity) wants to produce a note which is an octave higher than the note produced by a certain string, he must divide the length of the string by two.

Thus, music theorists dealt intuitively with logarithms long before these were defined as an abstract mathematical notion. (It was only in the seventeenth century that logarithms were formally introduced in music theory, by Isaac Newton, and then by Leonhard Euler and Jacques Lambert.) The theory of musical intervals is a natural example of the practical use of logarithms, an example easily explained to children, provided they have some acquaintance with musical intervals.

⁶We must note that the experiment with the hanging weights is considered to be a mistake of Pythagoras, or an extrapolation due to Pythagoras’s disciples, or a misinterpretation of what Pythagoras really said. This mistake was noticed by Vincenzo Galilei (the father of Galileo Galilei). Vincenzo was a most cultivated person, in particular a music theorist and a music composer. He did the experiment with the hanging weights and realized that to produce the intervals of octave, fifth, and fourth, the ratios of the pairs of weights should be respectively $4/1$, $9/4$, and $16/9$, which are the squares of the numbers which occur in the experiments involving the lengths of strings. Galilei was proud of that discovery (and of the discovery of a mistake in the theory of Pythagoras), and he published it in his famous musical treatise, the *Discorso intorno alle opere de Gioseffo Zarlino*. The physical reason behind this fact is that the frequency of a vibrating string, while it is proportional to the length of the string, is proportional to the square root of the tension. Nonetheless, the relation between musical intervals and ratios of integers is still there, even though it is not so direct in all cases. We note too that the same experience with the hanging weights is described by Vincenzo’s son, Galileo (see [5], p. 98 to 110).

Music in the Mathematical Treatise of Theon of Smyrna

It is interesting to go through the music theory part of a mathematics treatise of the classical Greek era. I consider here the section on Music (Part 2) of Theon's treatise [12]. This section deals with the definition and the combinations of musical intervals, with proportions, musical units, and so on. It involves non-trivial arithmetic, and Theon, in this section, often refers to the discoveries made by Pythagoras and the Pythagoreans.

The title of Part 2 of Theon's mathematical treatise is "A book containing the numeric laws of music." In the introduction, he says, "Harmony is spread in the world, and offers itself to those who seek it only if it is revealed by numbers." The first part of this sentence, that "Harmony is spread in the world," has been repeated throughout the ages, and it was at the basis of a strong feeling of cosmic structure and order. There are important philosophical and esoteric traditions behind this idea, which led eventually to explanations of physical phenomena, like the motion of planets. Famous adepts and advocates of such traditions include, after Pythagoras himself, Plato, Boëthius, Copernicus, and Kepler (see for instance [8], Book V, where Kepler gives a relation between the eccentricities of the orbits of the planets and musical intervals). The second part of Theon's sentence, that "harmony is revealed by numbers," has also been repeated throughout the ages, for instance in the citation of Rameau mentioned earlier and in the following citation of Gottfried Wilhelm Leibniz, from his *Principles of nature and of grace* (1712): "*Musica est exercitium arithmeticae occultum . . .*" (Music is a secret exercise in arithmetic).

Let us look at the treatment of *semitones* in Theon's treatise. There are several kinds of semitones used in ancient Greek music, two of which are the "diatonic semitone" and the "chromatic semitone," the values of which are, respectively, $16/15$ and $25/24$. One could expect that there is a semitone whose value is equal to half of the value of a tone, in the sense that if we concatenate two such semitones, we obtain a tone. This is not the case for any of the semitones used by the Pythagoreans, however. Indeed, by the discussion on logarithms above, we know that if the semitone were half of the tone, then its numerical value should have been $\sqrt{9/8}$, which is an irrational number. For the Pythagoreans, dealing with irrational numbers would have been incompatible with their philosophy. Theon writes in §VIII of Part 2 that "one can prove that" the tone, the value of which is $9/8$, cannot be divided into two equal parts, "because 9 is not divisible by 2." Of course, this is nonsense: the point is not to divide 9 by 2, but to take the square root of $9/8$. Although Pythagoras and his school were aware of the existence of irrational numbers, they consid-

ered them unnatural and a threat to their philosophical system, based on positive integers. The adjective "irrational" which they introduced clearly indicates this. It is also well known that the Pythagoreans wanted to keep the existence of irrational numbers (the discovery of which is attributed to Pythagoras himself) a secret. Jamblichus relates in [7] Chapter XXIX (p. 126) that "he who first divulged the theory of commensurable and incommensurable quantities, to those who were unworthy to receive it, was so hated by the Pythagoreans that they not only expelled him from their common association, and from living with them, but also constructed a tomb for him."

The reasons why ancient Greek music used semitones of $16/15$ or $25/24$ are certainly related to the fact that these intervals are acceptable by the ear. But it is also a fact that the ancient Greek musicologists liked to deal with *superparticular ratios* de-

derived from 2, 3, and 5, that is, fractions of the form $(n + 1)/n$ with numerator and denominator having only 2, 3, and 5 as prime factors. Pythagorean number symbolism is involved here, but that subject is

beyond the scope of this paper. The following is a list of "useful" musical intervals, which was known to Gioseffo Zarlino and Descartes:

- $2/1$ octave
- $3/2$ fifth
- $4/3$ fourth
- $5/4$ major third
- $6/5$ minor third
- $9/8$ major tone
- $10/9$ minor tone
- $16/15$ diatonic semitone
- $25/24$ chromatic semitone
- $81/80$ comma of Didymus.

There is a discussion of this list in both [6] and [9]. Many years after this list was known to music theorists, C. Størmer proved that this is a complete list of the superparticular ratios derived from the prime numbers 2, 3, and 5 [11].

Scales

Scales are building blocks for musical compositions. (This is true at least in tonal music, that is, in almost all pre-twentieth-century European music.) I shall talk in this section about the arithmetic of scales, and I remark by the way that in addition to this arithmetic, there is a more abstract relation between scales and mathematics, namely in the context of formal languages. Classical musical compositions are based on scales, fragments of which appear within a piece in various forms, constituting a family of privileged sequences of musical motives. This fact has been exploited

and systematically generalized in certain twentieth-century compositional techniques (for instance, serial music), which are related to mathematics, but which are beyond the subject matter of this paper.

The major part of post-Renaissance Western European classical music uses a very limited number of scales; in fact, since the general acceptance of the tempered scale in the eighteenth century, there are basically two scales, the major and the minor scale. The tempered scale (the one we play on a piano keyboard), is based on the division of the octave into 12 equal intervals, the unit being the *tempered semitone*, the value of which is equal therefore to $^{12}\sqrt{2}$. Any two major (respectively, minor) tempered scales are translations of each other on the set of pitches. (In musical terms, these translations are called transpositions.) This was not the case in pre-Renaissance music.

In contrast, the theory of harmony in classical Greece included a complicated and very subtle system of scales. Greek mathematical treatises usually contain a description of scales in terms of fractions, with a discussion of the logic behind the definitions. For instance, the scale which is known today as the “scale of Pythagoras” is defined by the following sequence of numbers:

1, $9/8$, $81/64$, $4/3$, $3/2$, $27/16$, $243/128$, 2.

These numbers can be regarded as representing ratios of lengths of strings, the n th number being the ratio of a pair of strings having the same section and stretched at the same tension, producing the interval between the first and the n th note. Thus, for instance, the interval between the first and the last note in the list is an octave, the interval between the first and the fourth note is a fourth and the interval between the first and the fifth note is a fifth, as expected, since the Pythagorean scale needs to contain these three consonant intervals. The intervals between consecutive notes, except those between the third and the fourth and the seventh and the eighth, have the value $9/8$. The intervals which we have excluded have the common value $256/243$, which corresponds to another semitone. The scale of Pythagoras sounds approximately, but not exactly, like our tempered major scale. The semitone which is used in our tempered scale, $^{12}\sqrt{2}$, is closer to the diatonic semitone, $16/15$, than to the other two which we encountered.

There is a logic behind the defini-

tion of the scale of Pythagoras. One starts by assigning the values 2, $3/2$, and $4/3$, respectively, to the eighth, fifth, and fourth notes in the list. The rest of the values are obtained by an iterative process involving fifths whose values are $3/2$ (such fifths are called *pure* fifths). Thus, for example, if we start from the first note (with value 1) and concatenate two pure fifths, we obtain an interval of ninth, with value $3/2 \times 3/2 = 9/4$, which is greater than 2 (as expected, since this interval is larger than an octave). To come back inside our octave, we divide by two, obtaining the value $9/8$. In the same way, the value $27/16$ is found as $(3/2)^3$ divided by 2, and so on. Unfortunately the process gives an infinite number of notes, but it is reasonable to stop after the octave has been divided into these seven intervals.

The scale of Pythagoras has beautiful properties. One is that all fifths and all fourths are pure, their common values being $3/2$ and $4/3$. For instance, the value of the interval between the second and the fifth note is $(3/2)/(9/8) = 4/3$. This is a remarkable property which does not follow obviously from the construction.

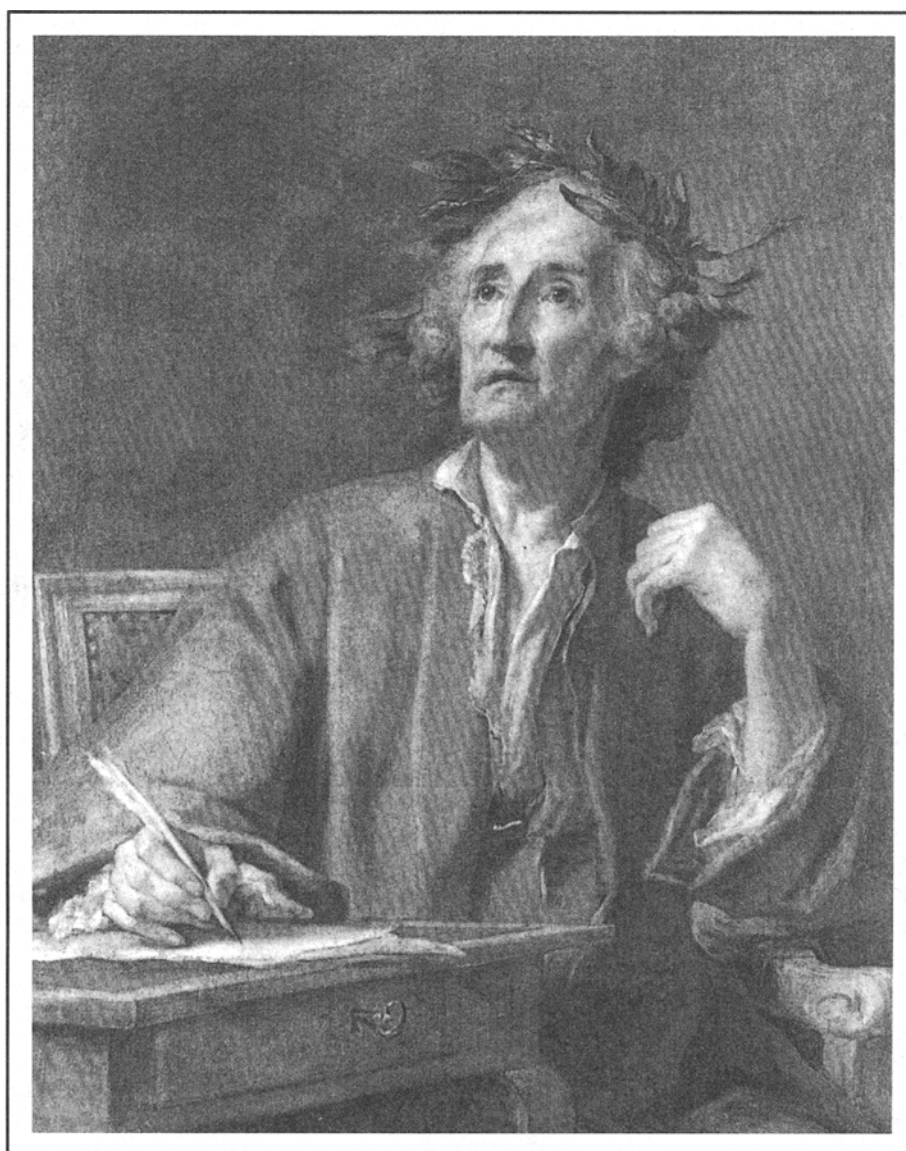


Figure 4. Jean-Philippe Rameau. Portrait by Jean Bernard Restout, titled “The inspired poet.”

Providing a scale with the maximum number of pure intervals was a domain of research of early music theory. In sixteenth-century Western European music, the intervals of minor and major third began to be considered as consonant, and the scale of Pythagoras was less suitable for new harmonies that involved many of the new intervals. (The value of a pure major third interval is $5/4$, whereas in the scale of Pythagoras the value of the interval between the first and the third notes is $81/64$, which is a little bit greater than $5/4$). A scale which was useful in that respect is the one named after Gioseffo Zarlino, a famous sixteenth-century Venetian musicologist. Zarlino's scale makes a compromise between pure thirds, pure fourths, and pure fifths. The sequence of numbers is

$$1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8, 2.$$

Some of the fifths in this scale are pure, but not all of them. For instance, the value of the interval between the second and the sixth note is $40/27$, which is strictly less than $3/2$. The value of the difference is $(3/2)/(40/27) = 81/80$, the Didymus comma, which is an audible interval.

It is impossible to have only pure intervals in a scale, unless the scale is short. Aristoxenus (fourth c. BC) made a systematic theory of scales based on "tetrachords," scales consisting of four notes corresponding to different divisions of the fourth by tones and semitones. A long scale would be obtained by concatenating tetrachords.

Let us return for a moment to the scale of Pythagoras. Problems are encountered as soon as one needs to concatenate several such scales, for instance in order to play musical instruments whose ranges cover several octaves. For example, one would expect that the concatenation of 12 fifths gives 7 octaves (as is the case for instruments like the guitar or the harpsichord). This cannot be the case if one uses the scale of Pythagoras, since $(3/2)^{12}$ is not equal to $(2/1)^7$. The interval with value $(3/2)^{12}$ is larger than the one with value $(2/1)^7$. The difference is a small (but nevertheless audible) interval, $(3/2)^{12}/2^7$. This small interval is called a "Pythagorean comma."

Similar problems occur in all the other scales based on pure intervals. For instance, we would expect that the concatenation of 4 fifths gives an interval of 2 octaves and one major third. If we do the computation in Zarlino's scale, we find that this is not the case, and the difference is the Didymus comma ($81/80$).

It is worthwhile to mention here that music theorists in ancient China encountered similar arithmetical problems in their theory of scales.

It should be clear now that the definition of a scale involves some arbitrariness and depends strongly on which intervals we insist be pure. One solution to the problem was, instead of making a restricted choice, to keep different possibilities. This is one of the reasons why there are so many scales in antique Greek music. In this music, different scales were adapted to different melodies and different types of instruments. The choice of scale for a musical piece determined much of the character of the piece and of its psychological effects on the listener. (This is also

related in Jamblichus [7].) This subtle dependence of the piece upon the scale lasted in European music until the adoption of the tempered scale. For instance, Rameau gives a list of characteristics of different *tonalities* in his *Traité de l'harmonie réduite à ses principes naturels*, Book II, Chapter 24 (Vol. 1 of [10]).

Rameau and the Harmonic Sequence

Like Pythagoras 2000 years before him, the composer and theoretician Jean-Philippe Rameau made a real synthesis between music as an art whose aim is to express and to create emotions, and music as a mathematical science with a deductive approach and rigorous rules. Pythagoras established the important relation between musical intervals and pairs of integers, Rameau went a step further and gave a musical content to the whole sequence of positive integers.

One of the main ideas for Rameau is that the infinite sequence of integers is contained, in a beautiful way, in nature, as a sequence of frequencies.

When a sonorous body (Rameau's terminology: "*corps sonore*") vibrates, it creates a local periodic variation of the pressure of air. This vibration propagates as an acoustic wave. It hits our ear drums, and we hear a musical note. The musical note produced by a vibrating string (bowed or plucked), consists usually in a superposition of a fundamental tone and overtones. The frequencies of the overtones, which are called the *harmonic frequencies*, are integral multiples of the frequency of the fundamental tone. The sequence of harmonic frequencies is naturally parametrized by the positive integers. For instance, the frequency of the note C_1 (which corresponds to the lowest C key on a piano keyboard) is (approximately) $f = 33$ Hz (cycles per second). The frequencies of the corresponding overtones are therefore

$$f, 2f, 3f, 4f, 5f, 6f, \dots$$

whose values in Hz are

$$33, 66, 99, 132, 165, 198, \dots$$

The corresponding sequence of notes is

$$C_1, C_2, G_2, C_3, E_3, G_3, \dots$$

In principle, one can hear the first four or five overtones on an instrument like an organ. (Mersenne, in his *Harmonie Universelle*, says that he can hear the first nine overtones.)

Rameau's theoretical work is based on scientific discoveries in acoustics which were made in the seventeenth century, in particular by the mathematician Joseph Sauveur. The phenomenon of "harmonics" in music had been noticed long before Rameau, but Rameau was the one who used it as the basis of a coherent theoretical teaching of music, in particular in his *Traité de l'harmonie réduite à ses principes naturels*.

Rameau's textbooks on music theory (about 2000 pages) include the basics of figured bass, accompaniment, chords, modulation, and composition techniques. All the theories he developed are based on simple rules

TRAITE' DE L'HARMONIE, DE'MONSTRATION.

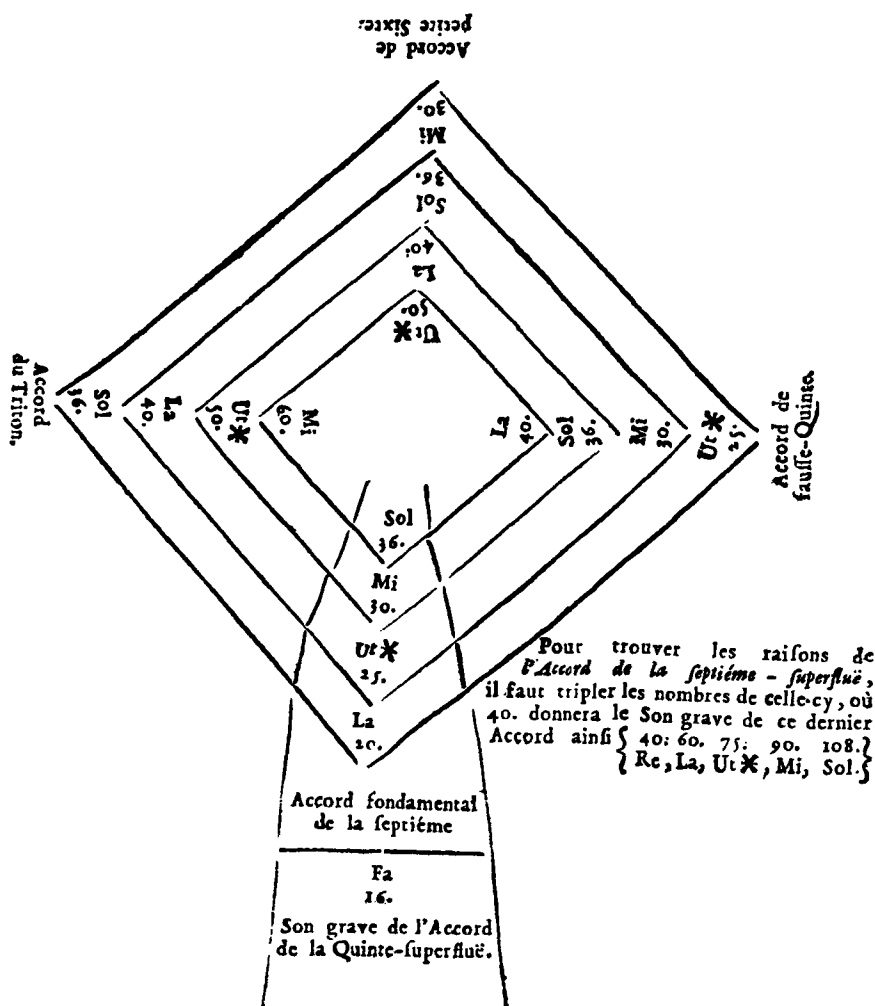


Figure 5. A diagram in Rameau's *Traité*, discussing ratios of frequencies of a dissonant chord (containing a minor seventh).

Rameau liked to consider the harmonic sequence of frequencies emitted by a sonorous body as a proof that the principles of music theory are contained in nature. Later on (starting from the year 1750), and especially in his *Nouvelles réflexions sur le principe sonore*, Rameau argued that since the fundamental objects of mathematics are derived from the sequence of positive integers, and since this sequence is contained in music, then mathematics itself is part of music. These reflections provoked a dispute between Rameau and eighteenth-century French mathematicians, like L. B. Castel and J. d'Alembert, and with the encyclopaedists, like Denis Diderot, Jean-Jacques Rousseau, and Friedrich von Grimm. The details of the controversy are worth studying, but they cannot be included in this short report. A very strong hostility followed several years of friendship and mutual praise between Rameau and d'Alembert; do not fall into the facile conclusion that the interaction between music theorists and mathematicians was always friendly. Still the interaction was there.

In this report I have concentrated on examples, starting with Pythagoras and ending with Rameau. To support the choice of Pythagoras and Rameau, let me conclude by citing Jacques Chailley [4]⁷

En 2500 ans d'histoire écrite, la musique n'a peut-être connu que deux véritables théoriciens, dont les autres n'ont guère fait qu'aménager ou rapetasser les propositions. L'un, au VI^e siècle avant notre ère, fut le fabuleux Pythagore. L'autre mourut à Paris en 1764: ce fut Jean-Philippe Rameau.

In 2500 years of written history, music has perhaps known only two genuine theoreticians, and what the others did was only to repackage or patch up their propositions. The first one, in the VIth century before our era, was the fabulous Pythagoras. The other one died in Paris in 1764: this was Jean-Philippe Rameau.

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