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ARITHMETIC AND GEOMETRIC DIVISIONS OF THE TETRACHORD

C. André Barbera

Modern investigations into ancient Greek conceptions of numbers have often confused the meanings of tetrad (*τετράς*) and tetractys (*τετρακτύς*). The following distinction provided by Delatte will be of functional importance here.¹ “Tetrad” signifies the number 4 as well as the first four positive integers, whereas “tetractys” is defined as an ensemble of four things, a quaternary. Three well-known examples of the tetractys are the four elements, the quadrivium, and the set of four numbers that can be arranged proportionally to define the consonant and structural intervals of the Pythagoreans ($\{12, 9, 8, 6\}$).² By the fourth century B.C. the tetractys was dually manifested in music as the intervals of $\{12, 9, 8, 6\}$ and as the tetrachord, four strings or notes spanning a fourth.³

The distinction between multitude and magnitude is as ubiquitous as the tetractys in Pythagorean writings; multitude is associated with the study of numbers in and of themselves (arithmetic) while magnitude is linked to the material display of numbers perceivable by the sense of sight (geometry). This distinction stands at the nexus of Pythagorean cosmologic theory.⁴

By number the Pythagoreans meant integer, and in music they found a material working out, a representation perceivable by the sense of hearing, of the relationship of multitude to multitude. This relationship was expressed in terms of a series of six kinds of proportions ordered according to a conceptual departure from unity: equal, multiple, epimore, multiple-epimore, epimere, multiple-epimere.⁵ The Pythagoreans defined as consonant those intervals that could be represented by multiple and epimore (or superparticular) proportions such that the individual terms involved in the proportions were elements of the tetrad (2:1, 3:1, 4:1, 3:2, 4:3).

The emphasis on superparticular proportions runs throughout the history of tetrachord divisions, but this emphasis is counterbalanced by a geometric conception of music, initiated by Aristoxenus and reflective of the larger mathematical issue regarding incommensurable or irrational magnitudes. Discovery of incommensurability is attributed to Hippasus of Metapontum (early fifth century B.C.),⁶ although the earliest known specific reference to incommensurability occurs in Plato's *Theaetetus* (147b).⁷

Thus the general case was one of conflict between the truths of arithmetic and the truths of geometry, an irreconcilability of magnitude to multitude. This conflict was mirrored by two divergent conceptions of music theory; the tetrachord divisions outlined below serve as instances of the general case.

The theorists from whom we have tetrachord divisions can be arranged into three unequal categories, the first of which represents a period before Aristoxenus, a period before emerging geometry influenced and penetrated the realm of music theory. The second category begins with Aristoxenus (late fourth century B.C.) and includes those music theorists who continue to expound the Aristoxenian divisions of the tetrachord. The third category contains the music theory after Aristoxenus that does not conform with the geometric conception and ordering of the Aristoxenians. This third group includes theorists who may be called Pythagoreans, neo-Pythagoreans, or proportionalists—i.e., music theorists

who represent their tetrachord divisions in terms of numerical proportions.

The core of the first category is represented by Archytas of Tarentum (early fourth century B.C.), a friend of Plato, possibly a student of Philolaus, and the first Greek music theorist from whom we have tetrachord divisions. His extant writings contain definitions of arithmetic, geometric, and harmonic means.⁸ Evidence for his tetrachord divisions, however, is contained in Ptolemy's *Harmonics* (I 13, 31), written in the second century A.D.⁹ Archytas divides the tetrachord into three genera using numerical proportions to characterize the intervals.¹⁰

	enharmonic	chromatic	diatonic
Mese			
Lichanos	5:4	32:27	9:8
Parhypate	36:35	243:224	8:7
Hypate	28:27	28:27	28:27

Unique to Archytas's system is his use of the pure major third (5:4) in the enharmonic genus and his retention of 28:27 as the interval of parhypate to hypate in all three genera. Subsequent theorists tend to use the ditone (81:64) as the highest interval in the enharmonic and to change the parhypate when moving from the enharmonic to the chromatic.

According to Ptolemy, Archytas derived the chromatic lichanos by referring to the diatonic lichanos because the second highest tone in the chromatic genus is to the corresponding tone in the diatonic genus as 256:243. Archytas probably used the remainder of two whole tones subtracted from a fourth to derive the chromatic lichanos, as follows:

mese:diatonic lichanos::9:8

chromatic lichanos:hypate:: $(28:27 + 243:224) = 9:8$

mese:hypate::4:3

$(4:3 - (9:8 + 9:8))::256:243$, thus

chromatic lichanos = diatonic lichanos + 256:243, i.e.,
32:27:: $(9:8 + 256:243)$

The following is a seemingly less complex method of conceiving and achieving the same results:

Let the chromatic pyknon¹¹ be 9:8
 mese:chromatic lichanos::(4:3 – 9:8) = 32:27
 since chromatic parhypate:hypate::28:27,
 chromatic lichanos:chromatic parhypate::(9:8 – 28:27)
 = 243:224

Contemporary with Archytas, Plato's *Timaeus* (35b–36b) is a fascinating source for Pythagorean cosmologic theory as well as for the utilization of arithmetic and harmonic means.¹² There Plato defines a diatonic scale from which the following tetrachord easily can be extracted:

Mese	9:8
Lichanos	9:8
Parhypate	256:243
Hypate	

This division of the tetrachord defined the diatonic genus for over a millennium. Furthermore, the *Timaeus* probably represents a Pythagorean conception of the numerical characterization of the tetrachord prior to the divisions of Archytas.¹³

A second category of music theorists is represented by Aristoxenus, perhaps the most important figure in the history of tetrachord divisions or for that matter in the entire history of Greek music theory. Born at Tarentum c. 375–365 B.C. and in part educated there by his father, Spintharus, he became the author of voluminous works including biographies of Pythagoras and Archytas. He joined the Lyceum opened by Aristotle at Athens in 336;¹⁴ the inductive logic and empiricism of Aristotle is manifested clearly in Aristoxenus's *Elements of Harmony*.¹⁵

For Aristoxenus, as for his Pythagorean predecessors, the fourth is the smallest consonant interval (*Elements* I 20). Of particular interest here are his divisions of the tetrachord into three genera and his subsequent divisions of the chromatic and diatonic genera into shades (*Elements* I 22–27 and II 44–52), which divisions depend both upon his assumption that two and a half tones equal a fourth and upon his division of the tone.¹⁶ The tone is defined as the difference between the fourth and the fifth and can be divided equally in half, in thirds, and in quarters (*Elements* I 21). Table 1 shows Aristoxenus's divisions of the tetrachord.

Table 1. Aristoxenus's divisions of the tetrachord

	enharmonic	soft chromatic	hemiotic chromatic	tonic chromatic	soft diatonic	syntonic diatonic
2½ tones	Mese └─					
	Lichanos └─	2	1⅓	1½	1¼	1
	Parhypate └─	¼	⅓	½	¾	1
Hypate └─	¼	⅓	⅜	½	½	½

These six genera and shades are derived from infinitely many possible divisions of the tetrachord, for Aristoxenus defines the boundaries or loci of the movable notes (lichanos and parhypate) in the enharmonic and syntonic diatonic genera and admits a continuous gradation of pitch within these limits.

Particularly representative of Aristoxenus's geometric approach to dividing the tetrachord is the hemiolic chromatic genus. In establishing the lichanoi for the enharmonic and soft chromatic, he proceeds by letting the pyknon of the enharmonic be the two smallest dieses ($\frac{1}{4} + \frac{1}{4}$) while the pyknon of the soft chromatic is the two smallest chromatic dieses ($\frac{1}{3} + \frac{1}{3}$). The pyknon of the hemiolic chromatic is $\frac{3}{4}$ tone (*Elements* I 24–25), and Aristoxenus divides this interval in half (*Elements* II 52).

The result of such a division is that the interval from the hypate to the parhypate ($\frac{3}{8}$ tone), while being larger than the smallest diesis and larger than the smallest chromatic diesis, is incomposite. Specifically, $\frac{3}{8}$ tone is greater than $\frac{1}{4}$ tone by $\frac{1}{8}$ tone, and $\frac{1}{8}$ tone cannot be practically measured according to Aristoxenus; similarly, $\frac{3}{8}$ tone is greater than $\frac{1}{3}$ tone by $\frac{1}{24}$ tone. Thus $\frac{3}{8}$ tone is a structural unit in and of itself, without relation to the dieses, and therefore represents Aristoxenus's sensory, geometric perception of music.

By associating Aristoxenus with a geometric approach to music theory I do not mean to imply that he actually deals with physical space. However, his conception and description of musical space is analogous to a geometric treatment of physical space. This analogy extends to a common use of terms by Aristoxenus and by Euclid in his *Elements*.¹⁷ In noting that there are infinitely many lichanoi (*Elements* I 26) Aristoxenus uses *ἄπειρος* to indicate this magnitude; Euclid uses the same word for the same concept in the first book of his *Elements*, 23rd definition.

As is the case with very large magnitudes, there is also a conceptual correspondence between hearing and sight with respect to very small magnitudes. A basic principle of geometry is that there is no smallest magnitude. Similarly, Aristoxenus notes that there is no smallest musical interval

(*Elements* II 46). In doing so he uses *διάστημα* to mean interval. This is the same word that Euclid uses to mean distance in the first book, third postulate, of his *Elements*. There is, then, a conceptual and a terminological correspondence between Aristoxenus's description of musical space and Euclid's treatment of physical space.

Furthermore, by breaking with the Pythagorean tradition of representing intervals by numerical proportions, Aristoxenus's tetrachord divisions reflect the contemporary mathematical crisis over incommensurables in general, and in particular over $\sqrt{2}$. A faulty discussion regarding the representation of $\sqrt{2}$ as a proportion of two integers is contained in Plato's(?) *Epinomis* (990d-991a),¹⁸ and a proof that this representation is impossible is interpolated into the end of Book X of Euclid's *Elements*.¹⁹

The incompatibility of Aristoxenus's tetrachord divisions with the Pythagorean method can be shown by representing Aristoxenus's enharmonic genus in terms of numerical proportions such that mese:hypate::4:3, an assumption that Aristoxenus never would have made.²⁰

Mese	$\sqrt[10]{65,536}:\sqrt[10]{6561}$	i.e., $\sqrt[10]{4^8}:\sqrt[10]{3^8}$
Lichanos	$\sqrt[10]{4}:\sqrt[10]{3}$	
Parhypate	$\sqrt[10]{4}:\sqrt[10]{3}$	
Hypate	$\sqrt[10]{4}:\sqrt[10]{3}$	(approximately 11,486,984:11,161,229)

The approach of Aristoxenus, germinated by Aristotle, became a tradition that spanned nearly the entire history of Greek music theory. As late as the fourth century A.D. we find music theorists continuing to set forth tetrachord divisions identical to those of Aristoxenus. Peculiarly, however, there is at least a four-hundred-year gap between Aristoxenus and the next known music theorist of this tradition, Cleonides (second century A.D.).

In his *Harmonic Introduction* Cleonides names the notes of the three genera—e.g., “diatonic lichanos,” “chromatic lichanos,” etc.—and then blends the genera in forming the Greater and Lesser Perfect Systems.²¹ Within each tetrachord, however, Cleonides defines three lichanoi but only one parhypate, e.g.:

Proslambanomenos
 Hypate hypaton
 Parhypate hypaton
 Enharmonic Lichanos hypaton
 Chromatic Lichanos hypaton
 Diatonic Lichanos hypaton
 Hypate meson

Following this blending Cleonides lists as fixed notes the proslambanomenos, hypate hypaton, hypate meson, mese, nete synemmenon, paramese, nete diezeugmenon, and nete hyperboleon. All the notes that lie between these are movable (*JanS* 185).

Of course the parhypate hypaton lies between the hypate hypaton and the hypate meson and therefore would be movable although it appears fixed in the blended genera. Judging from the fact that Cleonides stems from the Aristoxenian tradition—he equates six tones with the diapason (*JanS* 194)—there is reason to believe that more than one parhypate hypaton exists in Cleonides's system.

In discussing the shades of the genera Cleonides uses a combination of Aristoxenian and Ptolemaic terminology. In terms of enharmonic dieses, whole and half tones, Cleonides defines the soft diatonic, syntonic diatonic, etc., but he also provides an artificial, numerical definition (*JanS* 192–93). The number three is equated with the smallest diesis, the $\frac{1}{4}$ tone, and thus the division of the tetrachord is defined in terms of a triple division of 30, e.g., $3 + 3 + 24$ defines the enharmonic genus. This symbolization does not appear in Aristoxenus's *Elements*, although the equation of the fourth with 30 is similar to Ptolemy's treatment of Aristoxenus (*Harmonics* II 14, 71).²²

Other theorists in the Aristoxenian tradition include Aristides Quintilianus (third or fourth century A.D.), who presents Aristoxenus's tetrachord divisions in his *De musica*,²³ and Gaudentius (probably fourth century A.D.), who does the same in his *Introduction to Harmonics* (*JanS* 331).²⁴ At a later point in his treatise, however, Gaudentius provides numerical proportions for the intervals of the Greater Perfect System (see below, p. 306).

Spanning a period of 800 years, the third and largest cate-

gory of music theorists who deal with tetrachord divisions is comprised of those post-Aristoxenians who represent their divisions in terms of numerical proportions. Although these tetrachord divisions are not identical from theorist to theorist, as is the case with the followers of Aristoxenus, there is a commonality among the divisions, namely the Pythagorean, arithmetic conception of music theory.

The earliest known complete formation of the Greater Perfect System is found in Euclid's(?) *Sectio canonis* (c. 300 B.C.), wherein the arithmetic, Pythagorean approach to music theory is clearly manifested.²⁵ The author divides the monochord into four diatonic tetrachords, each one containing the intervallic relationships expounded in Plato's *Timaeus*. He makes clear reference, however, to the enharmonic genus such that the interval from the mese to the lichanos is a ditone (81:64), leaving the pyknon to be divided unequally by the parhypate (*JanS* 162).

In the third century B.C. more evidence of the arithmetic approach is found in the divisions of the tetrachord by Eratosthenes, for which Ptolemy is our source (*Harmonics* II 14, 71-73):

	enharmonic	chromatic	diatonic
Mese	19:15	6:5	9:8
Lichanos	39:38	19:18	9:8
Parhypate	40:39	20:19	256:243
Hypate			

Again the diatonic genus is that of the *Timaeus*, and Eratosthenes's emphasis on superparticular proportions is obvious. As for the division of the enharmonic genus, in which Eratosthenes uses neither the major third of Archytas (5:4) nor the ditone (81:64), Winnington-Ingram postulates that the determining factors are the choice of 6:5 as the highest interval in the chromatic genus and the assumption (also made by Didymus and Boethius) that the pyknon of the enharmonic should equal the lowest interval of the chromatic.²⁶

(4:3 - 6:5)::10:9, the chromatic pyknon

10:9::20:18

20:18:: (20:19 + 19:18), the arithmetic mean

Thus 20:19 is the enharmonic pyknon

(4:3 – 20:19)::19:15, the highest interval of the enharmonic
20:19::40:38

40:38:: $(40:39 + 39:38)$, the arithmetic mean

Even greater emphasis on superparticular proportions can be found in the tetrachord divisions of Didymus, first century B.C. (Ptolemy, *Harmonics* II 14, 71–73). In fact, all of Didymus's intervals are related by superparticular proportions, which means the abandoning of the diatonic genus found in the *Timaues*:

	enharmonic	chromatic	diatonic
Mese			
Lichanos	5:4	6:5	9:8
Parhypate	31:30	25:24	10:9
Hypate	32:31	16:15	16:15

Of particular interest are the intervals in the chromatic genus from lichanos to parhypate and from parhypate to hypate. In this instance the latter interval is greater than the former, which violates the Aristoxenian rule regarding the ordering of sizes of intervals making up the pyknon.²⁷

In Nicomachus's *Manual of Harmonics* we find a tantalizingly sketchy presentation of many aspects of Pythagorean doctrine regarding music.²⁸ Chapter 12 contains the division of the three genera, but here Nicomachus uses terminology characteristic of Aristoxenus rather than the numerical proportions of the Pythagoreans (*JanS* 262):

	enharmonic	chromatic	diatonic
Mese			
Lichanos	ditone	tri-semitone	tone
Parhypate	$\frac{1}{2}$ semitone	semitone	tone
Hypate	$\frac{1}{2}$ semitone	semitone	semitone

Nicomachus refers to the interval from mese to lichanos in the enharmonic genus as a true ditone but divides the remaining semitone in half (*JanS* 262). If Nicomachus is referring here to the division by two of the musical interval of a semitone, then the intervals resulting from his division of the enharmonic genus can be expressed in numerical proportions only by using irrational numbers:

$$(4:3 - 81:64)::256:243$$

lichanos:parhypate::parhypate:hypate

Thus, lichanos:parhypate:: $16:9\sqrt{3}$, i.e., $\sqrt{256}:\sqrt{243}$

Ptolemy's *Harmonics* is a source not only for his own tetrachord divisions as well as those of Archytas, Aristoxenus, Eratosthenes, and Didymus, but also for a thorough discussion of the conceptual bases and inconsistencies of Pythagorean and Aristoxenian music theory. In dividing the tetrachord Ptolemy specifically states that the interval between the mese and the lichanos should be defined by a superparticular proportion. He then goes one step further than Aristoxenus in decreeing that for all genera the deepest interval should be smaller than those remaining (*Harmonics* I 15).²⁹

With these general guidelines set down, Ptolemy then takes the first three superparticular proportions less than 4:3—i.e., 5:4, 6:5, 7:6—and uses them to define the interval from mese to lichanos.³⁰ Subtracting 5:4, 6:5, and 7:6, respectively, from 4:3, the following composite intervals (pykna) remain: 16:15, 10:9, and 8:7. In each of these three cases he divides the pyknon by first multiplying each term of each proportion by three—e.g., $(3 \cdot 16):(3 \cdot 15)::48:45$ —and then assigning to the lowest interval of the tetrachord the superparticular proportion determined by the second term of the multiplied proportion, e.g., 45; thus 46:45 represents the lowest interval. The middle interval in each case is determined by subtracting the lowest interval from the pyknon, e.g., $(48:45 - 46:45)::24:23$.

In this way Ptolemy divides the tetrachord into one enharmonic and two chromatic genera. Of the five diatonic genera that he proposes, one is determined by this method. The arithmetic for these four divisions follows.

Enharmonic

$$(4:3 - 5:4)::16:15$$

$$(16 \cdot 3):(15 \cdot 3)::48:45$$

Thus, 46:45 is the lowest interval

$$(48:45 - 46:45)::24:23$$

$$\text{and } (5:4 + 24:23 + 46:45)::4:3$$

Soft Chromatic

$$(4:3 - 6:5)::10:9$$

$$(10 \cdot 3):(9 \cdot 3)::30:27$$

Thus, 28:27 is the lowest interval

$(30:27 - 28:27)::15:14$

and $(6:5 + 15:14 + 28:27)::4:3$

Syntonic Chromatic

$(4:3 - 7:6)::8:7$

$(8 \cdot 3):(7 \cdot 3)::24:21$

Thus, 22:21 is the lowest interval

$(24:21 - 22:21)::12:11$

and $(7:6 + 12:11 + 22:21)::4:3$

Syntonic Diatonic

$(4:3 - 10:9)::6:5$

$(6 \cdot 3):(5 \cdot 3)::18:15$

Thus, 16:15 is the lowest interval

$(18:15 - 16:15)::9:8$

and $(10:9 + 9:8 + 16:15)::4:3$

The formation of the tonic diatonic genus (equivalent to Archytas's diatonic) uses almost the same arithmetic method; in this instance, however, Ptolemy does not multiply the two terms of the apyknon by 3:

Tonic Diatonic

$(4:3 - 9:8)::32:27$

Thus, 28:27 is the lowest interval

$(32:27 - 28:27)::8:7$

and $(9:8 + 8:7 + 28:27)::4:3$

Of the remaining three diatonic genera, the ditonic diatonic is the same as the *Timaeus* tetrachord, and the smooth diatonic consists of the only three superparticular proportions between which no other superparticular proportion exists and such that the three proportions added together equal a fourth, i.e., $(10:9 + 11:10 + 12:11)::4:3$.

Ptolemy provides very little information regarding the division of the soft diatonic genus. This probably was his arithmetic process:

$(4:3 - 8:7)::7:6$

$(7 \cdot 3):(6 \cdot 3)::21:18$

If, at this point, Ptolemy had followed the method that he had already used four times, then 19:18 would define the lowest interval. But $(21:18 - 19:18)::21:19$, which is not a superparticular proportion, and Ptolemy has used superparticulars for every proportion except the limma of the ditonic diatonic (256:243).

Therefore, Ptolemy takes 21:20 as the lowest interval:

(21:18 – 21:20)::10:9

and (8:7 + 10:9 + 21:20)::4:3

Ptolemy's divisions of the tetrachord, arranged according to decreasing size of the highest interval, are shown in Table 2.

An incomplete account of Thrasyllus's formation of the Greater Perfect System is handed down by Theon of Smyrna (second century A.D.), and from this can be deduced a sketch of Thrasyllus's tetrachord divisions.³¹ The diatonic genus is that of the *Timaetus* (9:8 + 9:8 + 256:243), and the enharmonic genus is equivalent to that of Nicomachus (mese:lichanos::81:64; lichanos:parhypate::16:9 $\sqrt{3}$; parhypate:hy-pate::16:9 $\sqrt{3}$). All we learn of the chromatic genus, however, is that mese:lichanos::32:27, leaving 9:8 as the pyknon.

In contrast with Theon, Gaudentius provides us, in effect, with two sets of tetrachord divisions. As we have seen, one set is equivalent to the genera of Aristoxenus. The other set, given in numerical proportions, can be derived from Gaudentius's formation of the Greater Perfect System. At odds with the Aristoxenian division of the semitone in half, Gaudentius explicitly states that the semitone is represented by the proportion 243:256 and thus cannot be halved. He derives this proportion by noting that the ditone is 81:64 and a fourth is 85 $\frac{1}{3}$:64. Therefore, the limma is 85 $\frac{1}{3}$:81—i.e., 256:243—and the diatonic genus is defined as 9:8 + 9:8 + 256:243 (*JanS* 342–44).³²

The other genus for which Gaudentius provides proportions is the syntonic chromatic (*JanS* 344). He notes that:

limma + apotome = tone, thus

256:243 + apotome = 9:8

and apotome = 2187:2048

	syntonic chromatic	diatonic
Mese	32:27	9:8
Lichanos	2187:2048	9:8
Parhypate	256:243	256:243
Hypate		

Obviously Gaudentius wanted the pyknon of the syntonic chromatic to be 9:8 and the lowest interval to be the limma. Thus, mese:lichanos::(4:3 – 9:8) = 32:27.

Table 2. Ptolemy's divisions of the tetrachord

	enharmonic	soft chromatic	syntonic chromatic	soft diatonic	tonic diatonic	ditonic diatonic	syntonic diatonic	smooth diatonic
Mese	5:4	6:5	7:6	8:7	9:8	9:8	10:9	10:9
Lichanos	24:23	15:14	12:11	10:9	8:7	9:8	9:8	11:10
Parhypate	46:45	28:27	22:21	21:20	28:27	256:243	16:15	12:11
Hypate								

Finally we come to the sixth-century A.D. theorist and translator Boethius. His *De Institutione Musica* is, as it has been since the Middle Ages, a major source for the study of Greek music theory. In the sixth chapter of Book IV Boethius divides the monochord, defining all the notes of the Greater and Lesser Perfect Systems for each of the three genera. For any given genus, each tetrachord of the system is divided similarly—e.g., the intervals of the diatonic hyperboleon are the same as those of the diatonic diezeugmenon. Boethius proceeds by assigning the number 2304 to the nete hyperboleon³³ and divides the hyperboleon tetrachord thus:³⁴

	enharmonic	chromatic	diatonic
Nete hyperboleon	2304	2304	2304
Paranete hyperboleon	2916	2736	2592
Trite hyperboleon	2994	2916	2916
Nete diezeugmenon	3072	3072	3072

In the diatonic genus the paranete hyperboleon is $2304 + \frac{1}{8}(2304) = 2592$. The trite hyperboleon is $2592 + \frac{1}{8}(2592) = 2916$. Boethius lets 2916 also represent the trite of the chromatic genus and the paranete of the enharmonic genus. The nete diezeugmenon is $2304 + \frac{1}{3}(2304) = 3072$.

To derive the chromatic paranete, Boethius takes the number for the diatonic paranete (2592) and adds to it one half of the difference between the diatonic paranete and the nete hyperboleon, i.e., $2592 + \frac{1}{2}(2592 - 2304) = 2736$. To derive the enharmonic trite, Boethius adds to the number of the enharmonic paranete (2916) one half of the difference between the nete diezeugmenon and the enharmonic paranete, i.e., $2916 + \frac{1}{2}(3072 - 2916) = 2994$.³⁵

Thus Boethius has derived the chromatic paranete and the enharmonic trite by taking arithmetic means. Not surprisingly this process produces some unique proportions, which I have represented in standard format:

	enharmonic	chromatic	diatonic
Mese	81:64	19:16	9:8
Lichanos	499:486	81:76	9:8
Parhypate	512:499	256:243	256:243
Hypate			

It is worth noting that Boethius uses the limma (256:243) in all three of his tetrachord divisions, i.e., as the interval from parhypate to hypate in the diatonic and chromatic genera and as the enharmonic pyknon. Nicomachus does the same, but he refers to the limma as the "semitone" and never does provide a numerical characterization. In fact, Thrasyllus may also employ 256:243 in all three genera; but we cannot be certain about this since we do not know the positioning of his chromatic parhypate. This method also is used by Didymus, but instead of 256:243 he gives 16:15 as the interval from parhypate to hypate in the diatonic and chromatic genera and as the enharmonic pyknon.

With the exception of Didymus, it may be the case that Boethius, Nicomachus, and Thrasyllus are dividing the tetrachord similarly.³⁶ If this is true, then Nicomachus is referring to an arithmetic mean when he divides the interval of a semitone in half (see above, p. 303), and such may also be the case with Thrasyllus (see above, p. 306). Such a division is particularly interesting because it apparently has little or nothing to do with sound. In other words, when Boethius divides the enharmonic pyknon in half by taking an arithmetic mean, he produces two different intervals, 499:486 and 512:499.

Equally interesting is the fact that these arithmetic means produce curious looking proportions, and yet these proportions necessarily must have been considered before determining that 2304 is the smallest integer such that all of the intervals of the Greater and Lesser Perfect Systems could be characterized by integers. From these numerical manipulations, all in the name of music theory, we can gain an insight into the religiously motivated intermingling of Pythagorean numerology and music.

Perhaps the most informative, and surely the most graphic, representation of the distinction between the arithmetic and the geometric approaches to music theory lies in the divisions of the tetrachord. On the one hand there is the Pythagorean method with its dependence upon superparticular proportions as the definers of intervals. On the other hand there is the spatially oriented, geometric approach first expounded by Aristoxenus and found again in the works of Cleonides, Gaudentius, and Aristides Quintilianus.

In order to highlight this distinction I have graphed a two-place function with genera as its domain. The function divides the difference between the lichanoi of its first and second argument places by the difference between the parhypatai of its first and second argument places. This shows the rate of change from genus to genus of the lichanoi in relation to the rate of change of the parhypatai.³⁷ It was necessary, first of all, to convert the various presentations of tetrachord divisions into a unified format. Table 3 shows these divisions in terms of cents, with 0 cents assigned to the hypate and the mese therefore equal to 498 cents.

The tetrachord divisions of Aristoxenus, Eratosthenes, and Ptolemy are displayed graphically in Figures 1–3. I have not graphed the tetrachord divisions of Archytas; since he keeps the parhypate fixed in all three genera, the rate of change of the lichanoi divided by the rate of change of the parhypatai is undefined because the latter rate is zero ($x/0$ is undefined). Since only two tetrachord divisions can be determined completely for Thrasyllus and for Gaudentius, no true rate of change from genus to genus can be determined for their divisions, and so they have also been omitted. In general I have chosen the genera for the function in an order such that the value of the function is positive. I have therefore omitted Nicomachus's tetrachord divisions from the graphs, even though he changes the lichanoi and parhypatai in all three genera, because his chromatic parhypate, being higher in pitch than his diatonic parhypate, results in a negative value for the function at this point. Finally, when an author stabilizes a movable note, as Aristoxenus does with the parhypate in the tonic chromatic, soft diatonic, and syntonic diatonic genera, I have eliminated from consideration the genera occurring after the stabilization—e.g., Aristoxenus's diatonic genera and Ptolemy's smooth diatonic. This has been done because, after the point of stabilization, the function is disrupted and this disruption could obscure a functional constancy before the stabilization. This consideration, combined with the reason for eliminating the divisions of Thrasyllus and Gaudentius, effectively eliminates those of Didymus and Boethius, and leaves the divisions of Aristoxenus, Eratosthenes, and Ptolemy to be graphed. The divisions of Cleonides and Aristides are the same as those of Aristoxenus.

Table 3. Tetrachord divisions in terms of cents

Author	Notes	Genera		
Archytas		enharmonic	chromatic	diatonic
	M	498	498	498
	L	112	204	294
	P	63	63	63
	H	0	0	0
Aristoxenus (Aristides, Cleonides, Gaudentius)			soft	hemioolic
		enharmonic	chromatic	chromatic
	M	498	498	498
	L	100	132	149
	P	50	66	75
	H	0	0	0
		tonic	soft	syntonic
		chromatic	diatonic	diatonic
	M	498	498	498
	L	199	249	299
	P	100	100	100
	H	0	0	0
Boethius		enharmonic	chromatic	diatonic
	M	498	498	498
	L	90	200	294
	P	44	90	90
	H	0	0	0
Didymus		enharmonic	chromatic	diatonic
	M	498	498	498
	L	112	182	294
	P	55	112	112
	H	0	0	0
Eratosthenes		enharmonic	chromatic	diatonic
	M	498	498	498
	L	89	182	294
	P	44	89	90
	H	0	0	0

Table 3 (continued)

Author	Notes	Genera		
			syntonic chromatic	diatonic
Gaudentius	M		498	498
	L		204	294
	P		90	90
	H		0	0
Nicomachus ³⁸		enharmonic	chromatic	diatonic
	M	498	498	498
	L	100	199	299
	P	50	100	100
	H	0	0	0
Plato				diatonic
	M			498
	L			294
	P			90
	H			0
Ptolemy			soft chromatic	syntonic chromatic
	M	498	498	498
	L	112	182	231
	P	38	63	81
	H	0	0	0
		soft diatonic	tonic diatonic	ditonic diatonic
	M	498	498	498
	L	267	294	294
	P	85	63	90
	H	0	0	0
		syntonic diatonic		smooth diatonic
	M	498		498
	L	316		316
	P	112		150
	H	0		0

Table 3 (continued)

Author	Notes	Genera		
		enharmonic	chromatic	diatonic
Thrasyllus	M	498	498	498
	L	90	204	294
	P	45	?	90
	H	0	0	0

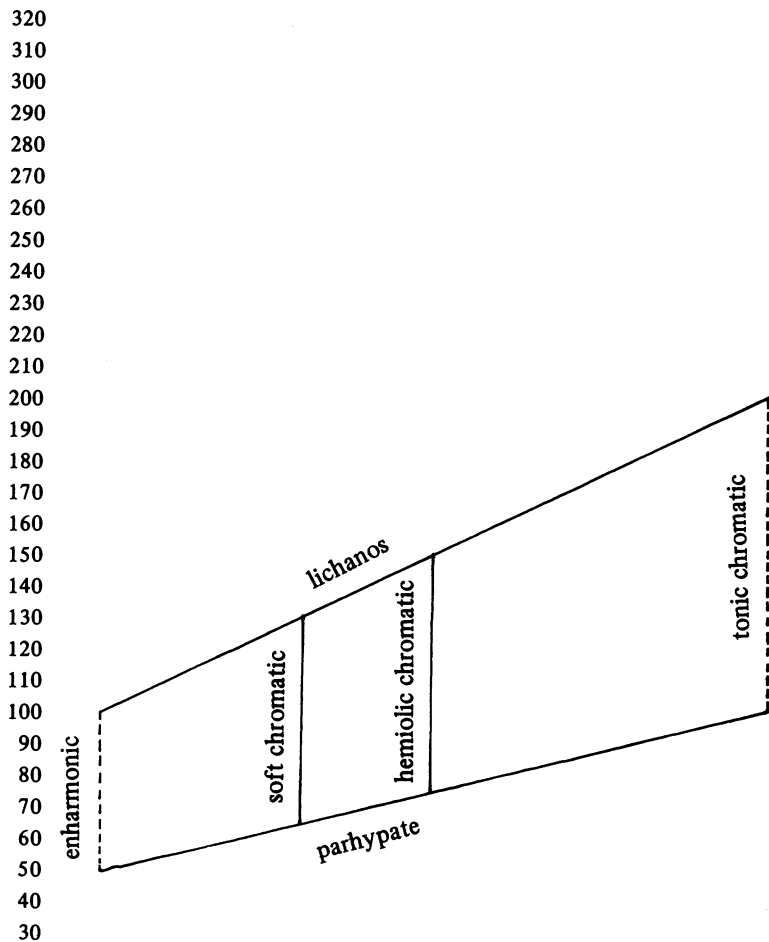
I have used solid lines to plot the curves of the movable notes and to indicate the relationships between these notes when they are both changing. I have used dotted lines to indicate the arbitrary relationships of starting and ending points. The mese and the hypate have been omitted from the graphs because they are stationary and therefore do not partake in the function.

Table 4 shows the defined values of the function for the tetrachord divisions of Figures 1–3. The relationships between lichanoi and parhypatai are represented by solid vertical lines in the graph of the Aristoxenian tetrachord divisions (Figure 1). This indicates that in these cases the value of the function is a constant or nearly a constant.

The function for Eratosthenes's tetrachord divisions and for some of those of Ptolemy (Figures 2 and 3), however, does not produce solid vertical lines as it did for the Aristoxenian divisions. This is so because the graph and the function are designed to highlight tetrachord divisions that are conceived in a spatial, geometric fashion. Neither Eratosthenes nor any of the authors omitted from the graphs divide the tetrachord from this point of view. In fact, with the possible exception of Gaudentius, they all are Pythagoreans or neo-Pythagoreans firmly entrenched in the ideal and obsessed with the power of numerical representation of the physical world—or, more severely, they see the physical world as a material representation of number.

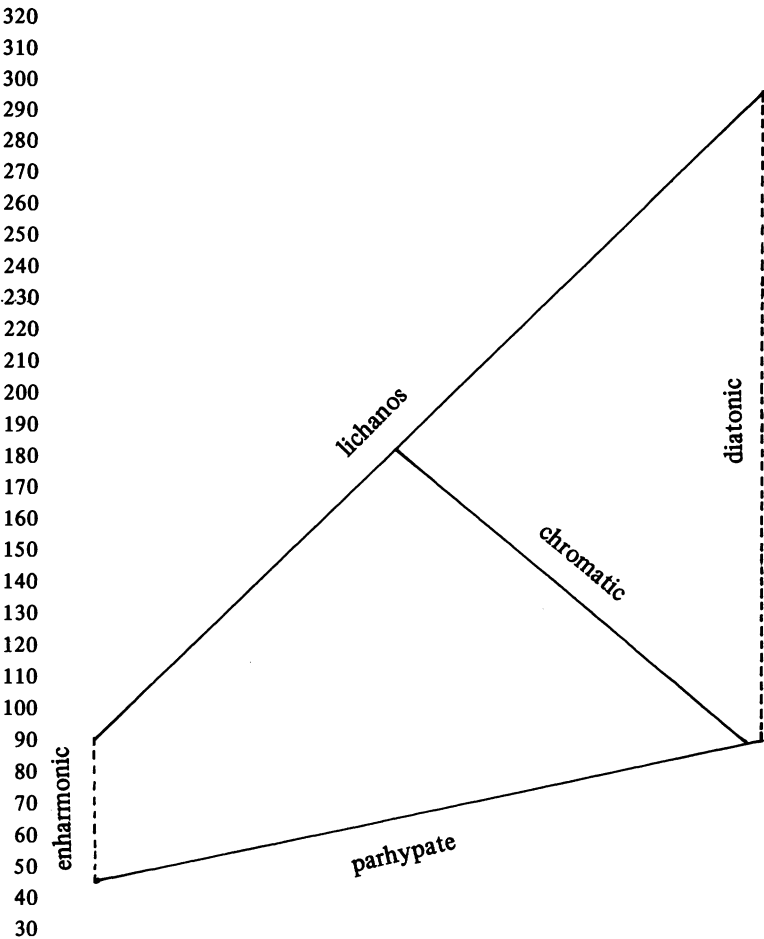
With Ptolemy the case is different. As mentioned before, in his *Harmonics* (I 6, 13–15 and I 9–10, 19–24) Ptolemy points out inconsistencies in the theories of both the Pythagoreans and the Aristoxenians. In Book I he presents his tetrachord divisions, which are characterized completely by superparticular proportions with only one exception, 256:243. In Book II, 2, however, Ptolemy constructs the helicon, a geometric figure that can represent through proportions of line segments all of the consonant intervals of Greek music theory as well as the tone. This figure can be put together without reference to numbers and yet, after its construction, Ptolemy assigns 12, 9, 8, 6, 4, and 3 to the various line segments. He then represents through arithmetic as well as geometric proportions the musical intervals obtainable from the helicon.

Figure 1. Aristoxenus



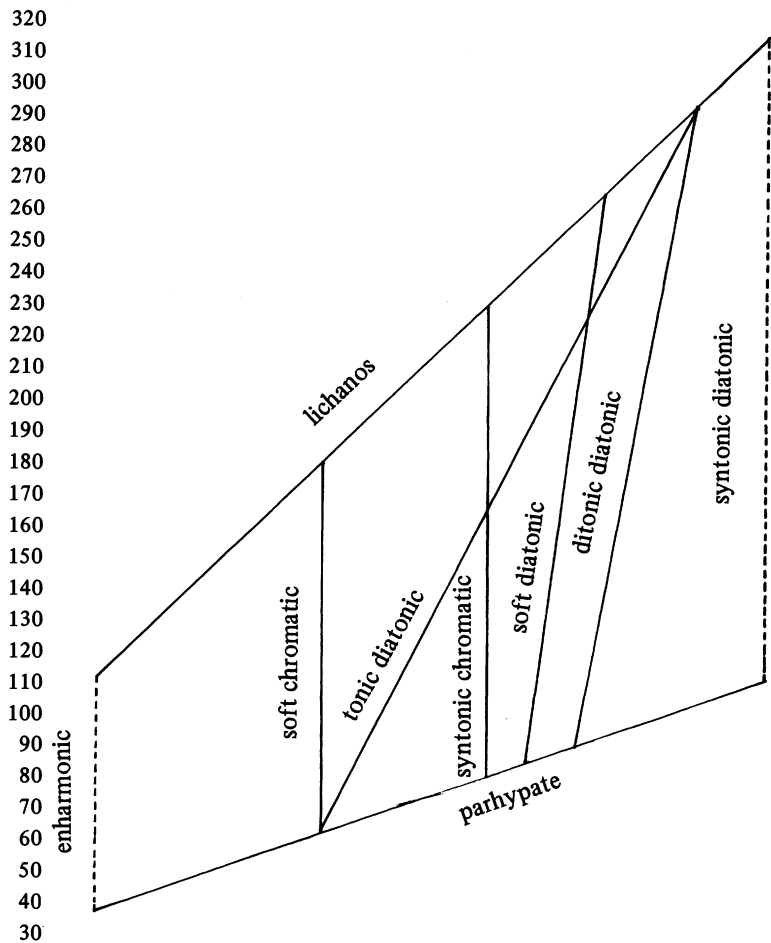
	enharmonic	soft chromatic	hemiolic chromatic	tonic chromatic
lichanos	100	132	149	199
parhypate	50	66	75	100

Figure 2. Eratosthenes



	enharmonic	chromatic	diatonic
lichanos	89	182	294
parhypate	44	89	90

Figure 3. Ptolemy



	enharmonic	soft chromatic	syntonic chromatic	soft diatonic	tonic diatonic	ditonic diatonic	syntonic diatonic
lichanos	112	182	231	267	294	294	316
parhypate	38	63	81	85	63	90	112

Table 4

f = function

g[1], g[2] = variables with genera as domain

L = lichanos

P = parhypate

$$(Lg[2] - Lg[1]) \div (Pg[2] - Pg[1]) = f(g[2], g[1])$$

Aristoxenus³⁹

1. g[1]=enharmonic
g[2]=soft chromatic $(132 - 100) \div (66 - 50) = 2.00$
2. g[1]=soft chromatic
g[2]=hemioctic chromatic $(149 - 132) \div (75 - 66) = 1.89$
3. g[1]=hemioctic chromatic
g[2]=tonic chromatic $(199 - 149) \div (100 - 75) = 2.00$

Eratosthenes

1. g[1]=enharmonic
g[2]=chromatic $(182 - 89) \div (89 - 44) = 2.07$
2. g[1]=chromatic
g[2]=diatonic $(294 - 182) \div (90 - 89) = 112.00$

Ptolemy

1. g[1]=enharmonic
g[2]=soft chromatic $(182 - 112) \div (63 - 38) = 2.80$
2. g[1]=soft chromatic
g[2]=syntonic chromatic $(231 - 182) \div (81 - 63) = 2.72$
3. g[1]=syntonic chromatic
g[2]=soft diatonic $(267 - 231) \div (85 - 81) = 9.00$
4. g[1]=soft diatonic
g[2]=tonic diatonic $(294 - 267) \div (63 - 85) = -1.23$
5. g[1]=tonic diatonic
g[2]=ditonic diatonic $(294 - 294) \div (90 - 63) = 0.00$
6. g[1]=ditonic diatonic
g[2]=syntonic diatonic $(316 - 294) \div (112 - 90) = 1.00$
7. g[1]=syntonic chromatic
g[2]=syntonic diatonic $(316 - 231) \div (112 - 81) = 2.74$

A similar combination of arithmetic and geometric conceptions can be seen in Ptolemy's tetrachord divisions by considering only the enharmonic, soft chromatic, syntonic chromatic, and syntonic diatonic genera. Graphically these four genera are represented by vertical lines (Figure 3) indicating that the values of the function of rates between these genera are constant, or nearly so (see above, p. 318). It turns out that these are exactly the four genera that can be determined by the arithmetic formula detailed above (see p. 304). Generalizing this method, let $(x + 1):x$ be the superparticular proportion that defines the interval from mese to lichanos.

$$(4:3 - (x + 1):x)::4x:(3x + 3)$$

Reduce $4x:(3x + 3)$ to its lowest terms, $(y + 1):y$

$$((y + 1) \cdot 3):(y \cdot 3)::(3y + 3):3y$$

Thus, $(3y + 1):3y$ is the lowest interval

$$((3y + 3):3y - (3y + 1):3y)::(3y + 3):(3y + 1)$$

$$\text{and } ((x + 1):x + (3y + 3):(3y + 1) + (3y + 1):3y)::4:3$$

In his tetrachord divisions Ptolemy has relied exclusively upon superparticular proportions in all but the ditonic diatonic genus. This veneration of superparticulars, combined with the above method for determining the lower intervals of four of his genera, is pure Pythagoreanism. Obscured by Ptolemy's arithmetic and formulaic procedure, however, is a geometric conception of music theory analogous to that of Aristoxenus. The values of the function for the enharmonic, soft chromatic, syntonic chromatic, and syntonic diatonic genera (2.80, 2.72, 2.74) show that Ptolemy combines the Pythagorean, arithmetic and the Aristoxenian, geometric approaches to music theory and transcends their seeming incompatibility.

NOTES

1. Armand Delatte, *Études sur la littérature pythagoricienne* (1915; repr. Geneva: Slatkine Reprints, 1974), pp. 253–55.
2. For example, both 12:9 and 8:6 represent a fourth. (I use “:” to represent a proportion, or in more modern terminology, a ratio. Thus, where x and y are numbers, $x:y$ may be represented as a fraction, x/y . I use “::” to represent the identity relation between two proportions. Thus, where x, y, z, w are numbers, $x:y::z:w$ may be represented as $x/y = z/w$. As is traditionally done, I use “+” to represent the addition of proportions and “-” to represent the subtraction of proportions. Thus, in numerical terms, $x:y + z:w$ may be represented as $x/y \cdot z/w$ —multiplication of fractions—and $x:y - z:w$ may be represented as $x/y \div z/w$ —division of fractions.) B. L. van der Waerden, in “Die Harmonielehre der Pythagoreer,” *Hermes*, LXXVIII (1943), p. 175, observes that Pythagorean numerical proportions are not based solely on string length since the higher pitch often is associated with the larger number. On the other hand, it seems unlikely that the proportions are based solely on frequencies because, on occasion, the lower number is associated with the higher pitch and because of erroneous understandings of the physics of sound. For an introduction to Pythagorean mathematics and music see Richard L. Crocker, “Pythagorean Mathematics and Music,” *Journal of Aesthetics and Art Criticism*, XXII (1963–64), 189–98 and 325–35. A more general and more extensive orientation to scientific thought in antiquity can be found in Walter Burkert’s *Lore and Science in Ancient Pythagoreanism*, trans. Edwin L. Minar, Jr. (Cambridge, Mass.: Harvard University Press, 1972).
3. A genesis of the tetrachord is given by Boethius in *De Institutione Arithmetica libri duo, De Institutione Musica libri quinque. Accedit Geometrica quae fertur Boetii*, ed. Godofred Friedlein (Leipzig: Teubner, 1867), pp. 205–6.
4. For instance see Aristotle, *Metaphysics* (986a, 2–7), Loeb Classical Library, vol. XVII, trans. Hugh Treddennick (Cambridge, Mass.: Harvard University Press, 1933), I, p. 33.
5. These proportions are detailed by Nicomachus in bk. I, chaps. 17–23 of his *Introduction to Arithmetic*, trans. Martin Luther D’Ooge (New York: Macmillan, 1926), pp. 212–29. In chap. 19 Nicomachus defines a superparticular proportion such that the greater term “is a number that contains within itself the whole of the number compared with it, and some one factor of it besides.” Such a definition does not exclude 2:1 from the class of superparticulars. However, Theon of Smyrna effectively excludes 2:1 in his definition. “The proportion is called superparticular when the greater term contains the lesser once and one part of the lesser, that is to say when the greater term surpasses the lesser by a certain quantity

- which is part of the lesser." (*Theon de Smyrne philosophe platonicien exposition des connaissances mathématiques utiles pour la lecture de Platon*, trans. Jean Depuis [1892; repr. Brussels: Culture et Civilisation, 1966], p. 125.)
6. See chaps. 18, p. 88 and 34, p. 247 of Iamblichus, *De Vita Pythagorica*, ed. Augustus Nauck (1884; repr. Amsterdam: A. M. Hakkert, 1965), pp. 65–66 and 172. For a thorough discussion of the discovery of incommensurability see Kurt von Fritz, "The Discovery of Incommensurability by Hippasus of Metapontum," *Annals of Mathematics*, 2nd Ser., XLVI, 2 (1945), 242–64.
 7. Plato, *Theaetetus*, trans. John McDowell (Oxford: Clarendon Press, 1973), p. 9.
 8. Hermann Diels, *Die Fragmente der Vorsokratiker* (Berlin: Weidmannsche Buchhandlung, 1922), I, pp. 334–35.
 9. Ingemar Düring, *Ptolemaios und Porphyrios über die Musik* (Göteborg: Elanders, 1934), p. 47.
 10. I shall represent the tetrachord divisions in the same way as these divisions appear in the sources. Since a degree of standardization may be useful here, however, whenever possible I shall also provide these divisions in terms of numerical proportions based on frequencies if the author has not done so.
 11. A pyknon is a composite interval from the hypate to the chromatic or enharmonic lichanos, such that this interval is smaller than the incomposite interval from the lichanos to the mese. An apyknon is a composite interval from the hypate to the lichanos such that this interval is not smaller than the incomposite interval from the lichanos to the mese.
 12. Plato, *Timaeus*, trans. H. P. D. Lee (Baltimore: Penguin, 1965), pp. 47–48.
 13. This genus also can be derived from Philolaus's remarks on harmony (5th century B.C.) (see Diels, *Die Fragmente*, I, pp. 311–12).
 14. François Lasserre, *The Birth of Mathematics in the Age of Plato* (Larchmont, N.Y.: American Research Council, 1964), p. 185.
 15. Aristoxenus, *Aristoxeni elementa harmonica*, ed. and trans. Rosetta Da Rios (Rome: Typis Publicae Officinae Polygraphicae, 1954). For a translation into English of the *Elements* see Aristoxenus, *The Harmonics of Aristoxenus*, ed. and trans. Henry S. Macran (Oxford: Clarendon Press, 1902).
 16. Aristoxenus's proof that two and a half tones equal a fourth (*Elem.* II 55) is, of course, bogus. Ptolemy (*Harm.* I 10, 21–24) has a good time with this at Aristoxenus's expense.
 17. Euclid, *The Thirteen Books of Euclid's Elements*, trans. with introduction and commentary by Sir Thomas L. Heath (Cambridge: University Press, 1908; rev. 2nd ed., New York: Dover Publications, 1956).
 18. Plato, *Philebus and Epinomis*, trans. A. E. Taylor, ed. Raymond Klibansky (1956; repr. New York: Barnes & Noble, 1972), pp. 249–50.

19. Heath does not include this proof in his translation of Euclid's *Elements*; for such a translation see K. von Fritz, "The Discovery of Incommensurability," pp. 254–55, n. 60.
20. The numerical representation of Aristoxenus's division of the enharmonic genus is obtained by letting the smallest interval ($\frac{1}{4}$ tone) be the unit of measurement. Since there are 10 units in Aristoxenus's fourth ($10 \cdot \frac{1}{4}$ tone = $2\frac{1}{2}$ tones), and since by hypothesis 4:3 represents a fourth, $\sqrt[10]{4:3}$, or $\sqrt[10]{4}:\sqrt[10]{3}$, represents $\frac{1}{4}$ tone.
21. Karl von Jan, *Musici scriptores graeci. Aristotelis. Euclides. Nicomachus. Bacchius. Gaudentius. Alypius et melodiarum veterum quidquid exstat* (1895; repr. Hildesheim: G. Olms, 1962), pp. 181–82. Further references to this work will be abbreviated as *JanS*. For a translation into English of Cleonides's *Harmonic Introduction* see *Source Readings in Music History*, ed. Oliver Strunk (New York: W. W. Norton, 1950), pp. 34–46. A few lines of text as edited by Jan are not included in the Strunk translation.
22. Düring, *Ptolemaios*, p. 87.
23. Aristides Quintilianus, *De musica libri tres*, ed. R. P. Winnington-Ingram (Leipzig: Teubner, 1963), pp. 17–18. For a translation into German of Aristides's *De musica*, see *Von der Musik*, trans. Rudolph Schäfke (Berlin: M. Hesse, 1937).
24. Also, in French, Gaudentius, *Alypius et Gaudence*, trans. Charles-Émile Ruelle (Paris: Firmin-Didot, 1895), pp. 60–61.
25. Thomas J. Mathiesen, "An Annotated Translation of Euclid's Division of the Monochord," *Journal of Music Theory*, XIX (1975), 236–58.
26. R. P. Winnington-Ingram, "Aristoxenus and the Intervals of Greek Music," *Classical Quarterly*, XXVI (1932), p. 198, n. 2.
27. Aristoxenus, *Elem.* II 52. In the enharmonic and chromatic genera the interval from hypate to parhypate is equal to the interval from parhypate to lichanos. In the diatonic genera the lowest interval is the smallest. Never is the interval from hypate to parhypate larger than the other intervals making up the tetrachord.
28. Nicomachus, *Nicomachus. Manual of Harmonics*, trans. Flora R. Levin, Ph.D. diss., Columbia Univ., 1967.
29. Aristoxenus requires only that the interval from parhypate to hypate be less than or equal to the interval from lichanos to parhypate.
30. Eventually Ptolemy uses the first six superparticular proportions less than 4:3 to define the highest interval of the tetrachord (5:4, 6:5, 7:6, 8:7, 9:8, 10:9).
31. Theon of Smyrna, pp. 149–52.
32. Gaudentius, pp. 74–78.
33. 2304 is the smallest integer such that all of the proportions characterizing the intervals of the Greater and Lesser Perfect Systems can be represented by integers. If the intervals of the GPS and LPS are represented by fractions, then 2304 is the least common multiple of the denominators.

34. Boethius, p. 322.
35. Boethius, pp. 319-22.
36. This proposition, as well as the general correlation of Boethius and Nicomachus, is due to Calvin M. Bower, "Boethius and Nicomachus: An Essay concerning the Sources of *De Institutione Musica*," to appear in *Vivarium*, XXV (1977).
37. The idea of picturing the change of lichanoi and parhypatai from genus to genus is due to Richard L. Crocker, "Aristoxenus and Greek Mathematics," *Aspects of Medieval and Renaissance Music: A Birthday Offering to Gustave Reese*, ed. Jan LaRue (New York: W. W. Norton, 1966), p. 102.
38. The cents for Nicomachus's divisions of the tetrachord are based on the assumption that, in his discussion of the genera, he is referring to actual *musical* intervals and not to purely numerical (arithmetic) divisions of the intervals of the tetrachord.
39. Theoretically the values of the function should be identical in all three instances listed under Aristoxenus. The variance in values indicates the approximations required by representing intervals by cents.

