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I

PYTHAGORAS

THE BIRTH OF MATHEMATICAL SCIENCE

TO ANCIENT Greece the world owes many things, but perhaps our greatest debt to her is for founding that peculiar cultural achievement of man known as Science. Science in the general sense means knowledge, but in the restricted sense it means rigorous and rationally organized knowledge which, because it is rigorous and rationally organized, can progress in an uninterrupted forward direction. It is the type of knowledge we have been able to attain in the field of natural phenomena, and it contrasts very distinctly with other forms of knowledge, such as philosophy and art, where progress from age to age is never certain, and where, indeed, the subjective expression of the individual or the people plays a dominant role. Although science, too, is the creation of individuals and peoples, yet its results are objective and available to all. Orientals, who have not shared the cultural development of the West, may appropriate science for their purposes and even participate in its further advancement without assimilating Western religion, art, philosophy or social customs.

Everybody is familiar with the tremendous increase in man's control over the forces of nature which has followed in the wake of science. But this has been the fruit of science rather than its conscious goal. There are still to be found primitive peoples who try to control the forces of nature by prayer and symbolic dancing. And contrary to the general impression, they are very practical-minded, more practical-minded than Europeans and Americans:

all that they lack is scientific knowledge. They cannot acquire this scientific knowledge because they have never learned to woo nature disinterestedly, without wheedling and bargaining. The Greeks set the example of disinterested scientific research, and they "carried on" without any of the material rewards which have fallen to the lot of modern science. Science was for them its own reward.

THE CULTURAL BACKGROUND OF GREEK SCIENCE

How did this remarkable attitude develop? What was the recipe, the method which crowned it with success? The attitude was to a large extent characteristic of Greek culture as a whole. It was an attitude of serene, secular naturalism, accompanied by a habit—which they often carried to extremes—of dialectical discussion. The historian Reinach sums up the Greek attitude to religion with the words which Voltaire puts into the mouth of Spinoza addressing the Almighty: "I think, between ourselves, you don't exist!" The Greeks as a whole did not take their gods very seriously. Already in the sixth century B. C. the Ionian philosopher Xenophanes observed with a keen irony (and one that still holds good) that "the Ethiopians make their gods black and snub-nosed; the Thracians say theirs have blue eyes and red hair. And if oxen and horses or lions had hands and could paint with their hands, horses would paint the forms of gods like horses, and oxen like oxen, and make their bodies in the image of their several kinds."

It is because they did not accept the religious or superstitious explanation of phenomena that the Greeks sought to find the *physis*—the nature or mechanical organization of things. (From this word *physis* are derived our terms physics, physician, and physiology—terms which reflect the Greek attitude of mind that constituted science.) And being a slave-holding, prosperous people, they could discuss the causes of things quite leisurely, and not in fear and haste, as is the case with a superstitious people, who become interested in things only when there is a sudden calamity, or else for the purpose of preventing an imaginary disaster. Before the Greeks the Babylonians cultivated astronomy, or rather astrology, as a science born of fear; and they even

learned, by means of a rough rule of thumb, to predict eclipses, so that the king might heed in time the warning from on high. When the Greeks turned to astronomy they expressed their naturalistic habit of mind by dubbing the heavenly bodies, which the Babylonians regarded as full of foreboding for human affairs, with the irreverent name *planetoi*—"tramp stars."

Nonetheless it was not the mere attitude of naturalism which brought success to Greek science. Naturalism by itself was largely expressed in the tendency to physical philosophizing, in which the scientific aspirations of the Greeks outran their positive achievements. The Greeks invented the theory of atoms as well as the concept of matter, but they did not found either physics or chemistry. For one thing they were not very much interested in experimentation, although in the biological sciences (which we shall discuss in a succeeding chapter) they showed themselves to be remarkable observers as well as rational analysts, who banished the idea of magic in healing. In the second place the development of physical science had to wait the prior development of the science of mathematics, which was to prove such a fruitful model and instrument for all the exact sciences.

It is this science of mathematics which it was the good fortune of the Greeks to found. Yet curiously enough, the development of mathematics, which has proved so fateful for all science and which from one point of view fits in so well with Greek dialectical genius, is in its origins almost an un-Greek phenomenon. It is the work of a leader and a sect, who, contrary to the prevailing tone of Greek secularism, sought a religious revival. Mathematics was cultivated by them as a form of religious expression!

The man who may be said to have founded mathematics is Pythagoras, one of the strangest and greatest figures in all history. His life, with the legends that cluster around him, and with that curious mixture of the miracle worker and the veritable prophet, reminds one strangely of Moses. And like Moses he was a leader who impressed his powerful personality upon a group and made that group dramatize a doctrine. He was worshipped as a god by his followers, and his words were taken literally as

the utterances of an oracle. The master himself said it, αὐτὸς ἔφα, and yet the master was no charlatan imposing himself upon the credulity of his disciples. We find the philosopher Empedocles, who lived a half century after Pythagoras, saying of him that "he was a man of prodigious knowledge who acquired the profoundest wealth of understanding and was the greatest master of skilled arts of every kind; for, whenever he willed with his whole heart he could with ease discern each and every truth in his ten—nay, twenty—men's lives." Extravagant praise, but it is confirmed by the more sober-minded Plato, who says that Pythagoras won the affection of his followers in an *unusual degree* by teaching them a "way of life."

PYTHAGORAS AND HIS ORDER

PYTHAGORAS was born on the island of Samos, off the coast of Asia Minor (or as it was then known, Ionia), about the year 572 B. C. His father, Mnesarchos, was a signet engraver, one of those crafts which the Greeks cultivated as a fine art. Samos was a commercial center of some importance, ruled, as were so many Greek cities, by a tyrant, Polykrates. Greece was at that time in the first flush of its prosperity, in which the Ionian city-states took the lead. It was from these cities that had come a school of secular physical philosophers—Thales, Anaximander, Anaximenes. The school was shortly to die out because of the incursions of the Medes and the Persians, who at first harassed and then overthrew the Ionian civilization. In the fifth century they crossed the Aegean and threatened Greece proper, but fortunately the Greek city-states united under the leadership of Athens and saved Greece—and civilization.

Pythagoras' youth and manhood were passed with wars and rumors of wars on the frontier. Whether this turned his mind to religion, we cannot tell. But in any case we know that the Ionians on the islands, in contrast to those on the mainland, conserved many of the religious traditions that went back to pre-historic Mycenaean times. Pythagoras himself worshipped at one of the oldest altars of Apollo at Delos, the altar of Apollo the Father,

where no animal sacrifices were offered but only barley and wheat-cakes. Also, he was the disciple of a soothsayer, Pherekydes, whose cultural outlook had little in common with the new secular philosophy. There is a tradition that Pythagoras completed his education in Egypt and Babylonia, but although he may very well have travelled to these countries, since the island of Samos had extensive commercial relations, it is doubtful whether he derived his scientific knowledge from these countries. The Egyptian rudiments of geometry—the rules of thumb developed by the "rope-stretchers" or land-surveyors—had already been picked up by the Greeks, and we find several geometric propositions (without the rational demonstrations) attributed to Thales.

It seems that Pythagoras tried to establish a school in his native Samos. Whether he was successful as a teacher we do not know, but it is a fact that in the year 529 he got into difficulties with the tyrant Polykrates and had to leave the island. The Greek city-state was a very tight affair, in which politics, social relations, family, friends, and philosophic ideas were all interlocked. If one did not fit in, one simply had to get out. The philosopher, Heraclitus, when he could no longer stomach the narrow ideas that prevailed in Ephesus, resigned his hereditary magistracy, and retired to live as a hermit, penning bitter satirical darts against his compatriots. Pythagoras did better: he emigrated. But he did not emigrate to the metropolis, as one might do at the present time—he went to found a school and a society in the new regions of Southern Italy.

Pythagoras selected the seaport of Croton, located on the site of the present fishing-village of Cotrone. At that time Croton was famous for its athletes (the Crotoniates carried off a long string of Olympic victories) and for its medical school. This emphasis on physical culture was quite in keeping with the ideas of Pythagoras, who, if we may believe Diogenes Laertius, was a remarkable athlete himself. In fact, Pythagoras saw no more of a contrast between physical and mental education than he did between science and religion; or between a school and a political order. His program cut across all of these conventional modern antitheses, and it is only when we visualize it as a whole that we

can get at the context of ideas responsible for the rise of mathematics as a rational science.

We are to imagine the Pythagorean society first of all as a secret society, more or less on the model of the secret and benevolent orders like the Masons, which function today. In those days, however, when things were less centralized nationally, a society of this sort could exercise real political power. It not only ruled Croton, but soon established branch lodges in many other cities, where the governments also fell under this "rule of the saints," as it might be called.

This is of course only the external phase. Internally the order was held together by the bond of friendship and of life and work in common. We may judge of the extent of the ascendancy of the order over the minds of the members from the fact that the wives were allowed to listen to the master's lectures and that women in general played a prominent part in the affairs of the society. This breaking down of the family shell is exceptional for ancient Greece, and it is doubtless the source whence Plato derived his feminist ideas.

MYSTICISM, MUSIC AND MATHEMATICS

WHEN we try to reconstruct the actual doctrines of Pythagoras and his group—a task of some difficulty in view of the fragmentary nature of the sources—we have to utilize the analogy with the Greek Orphic sects. About the same time as Pythagoras founded his order, there was a great religious revival, ushered in by the Orphic groups with their cult of the Thracian god Dionysos or Bacchus. The purpose of the Orphic worship was to purify the soul, which was regarded as a fallen god, and to escape from the "wheel of birth" by way of orgiastic rites. In this way the Orphics introduced a serious emotional element into Greek religion, which was lacking in the ordinary Olympian or Apollonian worships. Now the Pythagoreans did not worship Dionysos—they went back instead, as we have already said, to an old worship of Apollo—but their religious doctrines were otherwise quite similar. Like the Orphics they believed in the divinity

and transmigration of the soul and the kinship of men and animals. Hence the famous story of Pythagoras interrupting a man who was beating a dog and exclaiming, "Stop! Don't beat it! It's the soul of a friend that I recognized when I heard its voice."

As a corollary to the doctrine of transmigration, the Pythagoreans observed a large number of abstinences, or taboos, as they are called by modern anthropologists. They abstained from certain, if not all, flesh diets and from beans, and they had a large number of miscellaneous prohibitions (*e. g.*, not to pick up what has fallen, not to break bread, not to stir the fire with iron, not to pluck a garland, not to sit on a quart measure, etc.).

Taboos such as these form a strange background for the birth of mathematical science. Yet there is a definite connection between the mystical abstinences and the Pythagorean scientific interests. The connection is the doctrine of the soul and its purification. Aristoxenus, one of the later Pythagoreans, said that Pythagoras used medicine to purge the body and music to purge the soul. By music we have to understand, not the simple music of entertainment, but that sort of ritual ecstatic music which Plato alludes to in describing the rites of the Korybantes, another mystical sect. Plato compares these rites with the practice of mothers who rock and sing their children to sleep. Music and singing, he explains, have the effect of curing the passions of the soul, for "when one applies external agitation to affections of this sort, the motion coming from without gets the better of the terrible and violent internal one, and produces a peace and calm in the soul . . . sending some to sleep and making others who are awake dance to the pipe with the help of the gods." It is the principle of religious catharsis—a principle still exemplified to a certain extent in modern church music. But among the Greeks it was a much more violent and bacchanalean affair.

We must conceive the Pythagoreans as starting out by using music as an emotional therapeutic, as did the Orphic and other religious groups. But they must soon have passed from music in a quasi-physical sense to music as the symbol for all spiritual and intellectual activity. Socrates in the Platonic dialogue *Phaedo* calls philosophy—that is to say, what we should nowadays call

the scientific and intellectual life—"the highest music." In this statement he was repeating a Pythagorean doctrine. Also Pythagorean in its origins was the doctrine—made famous by Plato and Aristotle—of the three kinds of lives, with the contemplative philosopher or scientist as leading the best and most worthy form of existence. As the historian Burnet describes this doctrine, "there are three kinds of men, just as there are three kinds of people who come to the Olympic games. The lowest class is made up of those who come to buy and sell, and next above them are those who come to compete. Best of all, however, are those who come to look on. The greatest purification of all is, therefore, science, and it is the man who devotes himself to that, the true philosopher, who has most effectually released himself from the 'wheel of birth.'"

In infusing into his disciples the sense of an intimate connection between music and philosophy and between religion and science, Pythagoras was aided by a very startling discovery which he made one day on the nature of musical intervals. Tradition represents him as walking on a road and stopping to listen to the musical sounds made by a blacksmith's hammers. He noticed that the sounds of two hammers were separated by an interval of an octave, and he had the presence of mind to weigh the hammers. He found that the weights were in the ratio of one to two! He went home and repeated the experiment with string instruments. And by measuring the strings which produced tones at a distance of an octave, he found that the lengths were again in the ratio of one to two; also, an interval of a fifth had a string ratio of three to two; a fourth, four to three.

This discovery of the law of simple numbers is perhaps the greatest scientific discovery that has ever been made—if not for its concrete significance, at least for its symbolic meaning. That such a subtle art as music should be governed by numerical relations was sufficient to evoke the faith that "numbers rule the universe." It has proved the guiding faith of all exact science from the days of Pythagoras down to the present.

After such a discovery we are not surprised to learn that Pythagoras centered the scientific activities of the society around

the theory of number. We are told that he was the first to raise the study of number from the plane of commercial utility and make it into a liberal science. And this science was cultivated not merely on its own account, but for its immediate ramifications into music, into astronomy, and, not least, into geometry.

Pythagoras organized his society on a hierarchical basis. There was the order of the *mathematikoi*, the mathematicians or scholars whose function it was to advance science. And then there were the *akousmatikoi*, the listeners to whom the results of science were made available. The plan of organization also served to maintain a balance between the scientific and religious interests of the society. We are not surprised to learn that this balance proved unstable and could not be maintained after the death of the master. The society, in fact, split up into a scientific and popular wing. The eating habits of the acousmatic sect of the Pythagoreans later became the favorite sport of the Greek comic poets, who of course had never heard of the Pythagorean scientific achievements.

DEATH OF PYTHAGORAS AND COLLAPSE OF HIS ORDER

BEFORE taking up the concrete scientific achievements of the Pythagoreans—in which it is practically impossible to distinguish between the work of the master and the work of the disciples—we may cast a glance at the external and political fortunes of the order. For a while the order waxed in strength and controlled the political destinies of a number of cities. Then gradually jealousy developed among the outsiders who were excluded from the society. There was a revolt in 509, which compelled the master to retire to the city of Metapontion, where the Pythagoreans had strong lodges. It is there that the master spent his remaining years until his death in the year 492. The house in which he lived was still standing in Cicero's time (four and a half centuries later), and the Roman orator tells us how, although wearied from his travels, he would not repair to his hotel until he had first visited the house where Pythagoras had died. He found the house consecrated as a temple by the Metapontionites—showing that a

prophet is not always without honor in his own country, especially when he is dead.

Returning to the fifth century, we find that a generation or two after Pythagoras' death—the dates on this point are rather obscure—there were new and more serious outbreaks against the Pythagorean society. A state of civil war ensued, in which the colonies sought aid from their mother cities in Greece. The Pythagorean lodges were burnt down, and in one case, in Croton, all the Pythagoreans assembled in the lodge perished in the flames, with the exception of two young and strong men, who managed to escape. One of these, Lysis, after whom Plato named his dialogue on friendship, went to Thebes and became the teacher of the famous Greek general Epaminondas.

The effect of the expulsion of the Pythagoreans from Italy was to disseminate their doctrines and make them the public property of Greek thought. Most of the Pythagorean scientific doctrines were taken over by the school that gathered around Plato, who, though not a mathematician himself, shared the Pythagorean admiration for mathematics as the central science and encouraged his pupils to solve mathematical and astronomical problems.

THE STUDY OF NUMBERS AND GEOMETRY

THE study of music, which gave the Pythagoreans their justification for scientific activity, probably also suggested to them the order of research. As we have said, the discovery of the law of musical intervals gave the strongest possible incentive for the investigation of the nature of numbers. Now the study of number was carried on by the Pythagoreans in close relationship with the study of geometrical forms. Indeed, our word figure as denoting numerals is reminiscent of this relationship between number and geometry. It goes back to the Pythagorean practice of representing numbers by an arrangement of dots or pebbles in certain geometrical figures. The reason for this practice probably lay in the clumsiness of the ordinary methods of representing numbers. The Greeks at that time had two methods of writing

numbers, both equally unsatisfactory for scientific purposes. One was by a system of signs made from the initial letters of the numeral adjectives, and the other was by the use of the letters of the alphabet—*alpha* for one, *beta* for two, etc.—a method which the Greeks shared in common with the Syrians and Hebrews. The so-called Arabic numerals had not yet been invented, although for practical purposes of computation the counting board, with its place system of tens and hundreds, was generally employed. The Pythagoreans were not interested, however, in practical computation but in theoretical analysis of the properties of number.

For purposes of science what was more convenient than to represent 10, for instance, in the figure of a *tetrakys*, which

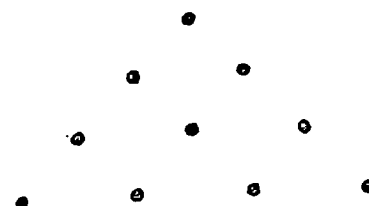
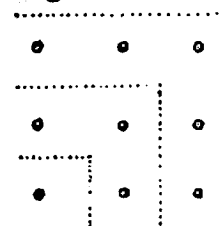
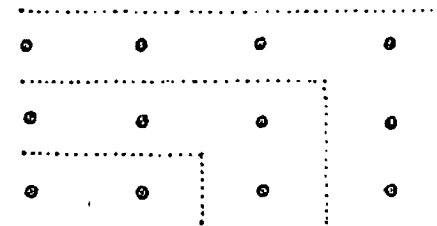


Figure of Tetrakys

showed at once that 10 was the sum of $1 + 2 + 3 + 4$? By the use of this method the Pythagoreans came to represent all numbers which were sums of successive integers as "triangular" numbers. They discovered, too, that numbers which are formed by adding the series of odd numbers, as 1 , $1 + 3$, $1 + 3 + 5$, $1 + 3 + 5 + 7$, arrange themselves in squares, while the sums of even numbers 2 , $2 + 4$, $2 + 4 + 6$ arrange themselves in oblongs.



Square Number



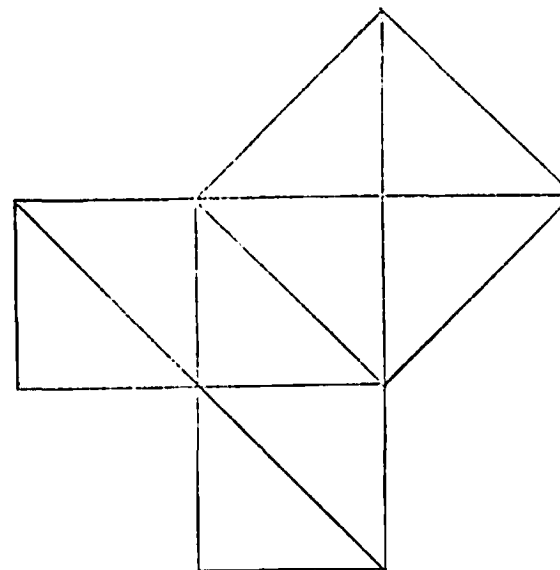
Oblong Number

We cannot discuss here the number of interesting theorems which the Pythagoreans discovered about the abstract properties of numbers. A great deal of their work served as the basis for the modern science of number, which was revived in the seventeenth century with the aid of algebra. What is important to note here is the way in which the number theory suggested problems in geometry.

The Egyptian rope-stretchers knew that a triangle with sides of three, four, and five units has a right angle opposite the largest side. Other primitive peoples also were familiar with this curious property, but nobody sought to go any further with the discovery. There is a dialogue written by the Chinese Emperor Tchau-Kong, in which the Emperor himself is one of the characters. When his interlocutor tells him about the properties of this famous triangle, the Emperor replies, "Truly that is marvelous!" But he does not think to ask the reason why. When Pythagoras and his young men took hold of this piece of knowledge it was immediately incorporated in a whole series of abstract relations which they had studied and analyzed. Thus not only did they know that $3^2 + 4^2 = 5^2$, but by their number analysis they also learned that there are other numbers whose squares make a square, *e. g.*, $5^2 + 12^2 = 13^2$, $15^2 + 8^2 = 17^2$. Trying out triangles with sides based on these dimensions, they discovered that the triangles were right triangles. Also by studying the diagonal of a square they discovered that here, too, despite the fact that the relationship between the diagonal and the sides cannot be stated numerically as the sum of square numbers, inspection shows that the square on the diagonal is equal to the sums of the squares on the arms.

(As anyone may see from the figure on page 13, a square on the diagonal is made up of four of the isosceles right triangles, while the squares on any of the sides comprise only two such triangles.)

In this way the general and universal character of the famous Pythagorean proposition was made obvious to the intuition. One felt that in the case of any right triangles whatsoever, the square on the hypotenuse was equal to the sum of the squares on the arms. It remained to demonstrate it logically. This achievement, too, was Pythagorean. Pythagoras, we are told, was the first to



Arrangement of squares which suggested Pythagorean theorem.

"probe the theorems in an immaterial and intellectual manner," which is to say, that instead of using his eyesight to establish geometric propositions, as every schoolboy still wants to do, he used his reason. The importance of rational demonstration is basic in geometry—it is not the individual truths that count, but the system of truths. And to establish a system of truths requires an effort of no mean ability. In the case of the Pythagorean theorem, the systematic proof had to be made by connecting the problem of the right triangle with the theory of similar triangles, while the theory of similar triangles was in turn tied up with the propositions about parallel lines and the theory of proportions.

From what we can gather, the greater part of the plane geometry afterwards codified in the textbook of Euclid was developed in the school of Pythagoras. Besides the subject of parallel lines, and the propositions about similar triangles and proportions, the Pythagoreans originated the very important subject of equivalent areas—to construct a figure equivalent in area to

one figure and similar in form to another. This must not be confused with the computation or calculation of areas. Even as late as Euclid there was no theorem for calculating the area of a triangle or the area of a circle (although all the basic propositions preliminary to these theorems were worked out). All the Greek geometers before Archimedes eschewed mensuration as too practical and illiberal. Thus the chief ground by which many a modern educator seeks to justify the utility of geometry to high school students is really an accidental interpolation foreign to the Greek conception of the science.

In solid geometry the Pythagoreans discovered the five regular solids, even learning to construct the dodecahedron, or 12-faced solid, given the radius of the circumscribed sphere. According to tradition, one of the members of the Pythagorean school, Hippasus, was drowned at sea for revealing this construction. The dodecahedron was a subject of peculiar reverence to the Pythagoreans because its faces were regular pentagons, and the pentagon was their symbol of health. Plato took over from them the reverence for the dodecahedron, and in the *Timaeus*, where he gives us a fascinating story of the logical generation of the universe from pure mathematics, the sphere of the universe is generated from the dodecahedron by the faces falling out and making the solid circular.

THE FIRST SCIENTIFIC CRISIS: THE DISCOVERY OF IRRATIONALS

THE movement of mathematical thought among the Pythagoreans was not altogether smooth sailing. They had their share of theoretical difficulties, and indeed it rather testifies to the high quality of their mathematical work that they had to meet the first great crisis in intellectual history. Ignorance never finds itself with any problems, it never leads to any revolutionary overturns. In the case of Pythagorean mathematics, it was their greatest achievement, the discovery of the proposition about the square of the hypotenuse, which put them face to face with a logical

scandal of the first magnitude, the problem of irrational numbers and incommensurable lines.

As one of the immediate consequences of the Pythagorean proposition, it became evident that in the case of an isosceles right triangle, the square of the hypotenuse is equal to twice the square of either arm. We may express the same result in saying that if the side of a square is 1 unit, the diagonal of the square or hypotenuse of the triangle, is equal to $\sqrt{2}$. Now what kind of a number is the square root of two? Mathematicians still call such a number an irrational, just as they call $\sqrt{-1}$ an imaginary, but these appellations have now only a technical significance. At the time of their discovery, however, they were the subject of much intellectual heartburning, as their very names indicate.

The difficulty with the diagonal of the square and the $\sqrt{2}$ arose from the fact that the Pythagoreans had assumed a direct correspondence between the numbers of arithmetic and physical and geometrical magnitudes. Numbers governed the universe because they were imbedded in things. A line was made up of number points, and a surface was made up of a combination of lines. This may seem naive to us, and yet it provides a working basis for explaining the ordinary facts of experience. We compare lengths by numbers, and although we may use any numbers we please, depending on the size of the units, the ratio of the numbers is an absolute relationship. We may compare two pieces of cloth and say that the lengths are as one yard to two, or three feet to six, but in both cases the ratio is the same. For the Pythagoreans this ratio was somehow imbedded in the nature of the cloths. Vice versa, in comparing two lengths, they acted on the assumption that in all cases their ratio should be expressible in whole numbers. It was on this assumption that they had built up their theory of proportions and similar triangles. Now here in the case of the diagonal and the side of a square was a relationship which could not be expressed in an ordinary numerical ratio; there was no common measure which could go into both lengths a whole number of times. In fact the Pythagoreans even proved logically that the thing was impossible. The number of the

diagonal would have to be both odd and even—it had no civil status in the number system, and so they gave it the name of irrational.

But it wasn't simply the question of where to catalogue this new phenomenon that worried the Pythagoreans. The discovery of the irrationals upset the logical foundations of most of their geometry and scientific philosophy of the universe. Not only was the diagonal an exception to their logical principles, but the side of a pentagon was also incommensurable with the radius. In fact, the more they thought it over, the more they came to realize that pairs of lines for which numerical ratios could be found were the exception rather than the rule. But in that case the whole theory of proportion which they had built up was invalid—and invalid, too, was the proof of the Pythagorean theorem. It was a crisis not unlike the crisis of modern physics when the Michelson-Morley experiment threw a shadow of doubt over the whole system of Newton.

EUDOXUS SOLVES THE PROBLEM OF IRRATIONALS

THE mathematical Einstein who met the crisis and allowed geometry to resume its progress was Eudoxus of Cnidos (408–355). Eudoxus was a member of the school of Plato, where after the break-up of the Pythagoreans the best mathematicians of Greece gathered. It is said that when he first came to Athens he was so poor that he could not afford to live in the capital, but had to find lodgings in Piraeus, the seaport and suburb of Athens. From there he trudged on foot every day to listen to Plato's lectures. Later he went with a letter of introduction to the King of Egypt, and after many travels and adventures he finally gathered a school around him in Athens. At this date philosophers and scientists were already permitted to accept payment for their lectures, and so we may surmise that the poor student reached a stage of comfortable prosperity. Before he died he returned to his native city, and was elected to legislative office by popular vote, which shows that mathematicians were considered quite respectable in ancient Greece.

The achievement of Eudoxus was to reconstruct the theory of proportions on a new basis so that it was applicable to commensurable and incommensurable cases alike. He did this by a new and masterful definition which avoided conceiving ratios and equalities of ratios in terms of definite whole numbers, but defined them rather in terms of certain rules of operation. According to this definition (Euclid V, Def. 4) "Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when if any equi-multiples whatever be taken of the first and third and any equi-multiples whatever of the second and fourth, the former equi-multiples alike exceed, are alike equal to, or alike fall short of the latter equi-multiples respectively taken in corresponding order."

This definition of equal ratios has practically been taken over in the definition given by modern mathematicians to equal numbers. What is significant about it is that it detaches the conception of number and ratio from the limited horizon fixed by the ordinary arithmetic of whole numbers. In terms of ordinary arithmetic incommensurable magnitudes and irrational numbers seemed to be logical contradictions; but in terms of the wider realm of number which the new rules of reasoning enabled man to enter, it is rather ordinary commensurable magnitudes which are seen to be an exceptional and limited case. This solution is typical of the method by which science achieves progress: without losing touch with sense experience, science manages continually to de-anthropomorphize its conceptions and its point of view, and in so doing it attains its increasing power over the world of nature.

The same order of ideas as involved in the solution of irrationals also suggested to Eudoxus a method for the problem of "squaring the circle," and for dealing in general with curved surfaces and solids. It is the method of exhaustion, or what we nowadays call the method of limits, one of the most potent intellectual instruments ever developed in mathematics.

As in the theory of proportion, Eudoxus' part consisted in formulating rigorous notions by means of which the long chain of reasoning which constitutes geometry could continue its progress. The Sophist Antiphon, a contemporary of Socrates, had

suggested that if we inscribe a polygon in a circle, and then successively double the number of sides, we shall finally get a polygon whose sides will coincide with the circle. Antiphon's statement is formally incorrect, although it contains the germ of an important idea. It was specially attacked by the dialectic emanating from the school of Zeno, the author of the famous paradoxes of motion. In fact, by the same method with which he punched holes in the Pythagorean philosophy that lines are made up of points, Zeno proceeded to show that it was absurd to regard a circle as made up of an infinite number of infinitesimal straight lines. But whereas the paradoxes of motion (which incidentally the Pythagoreans could never answer) concerned the philosophic interpretation of scientific laws, the attack on the mathematical notion of infinity concerned the internal progress of mathematical science itself. As such it had to be met and could not be dodged if geometry were to move forward.

The reader who recalls high school geometry will remember that our present method for dealing with such a problem is to make two variable magnitudes approach two constant magnitudes as limits, and then to observe that if the variable magnitudes are always in the relation of equality, therefore the limits are also equal. We inscribe, for example, two regular polygons in a pair of circles, and then continually double the number of sides. Then we find that the quotients of the areas of the polygons over the squares of their respective radii are always equal ($P/R^2 = P'/R'^2$). On the other hand these two quotients each approach as their limit the quotient of the area of its respective circle over the square of the radius (P/R^2 approaches C/R^2 as limit; P'/R'^2 approaches C'/R'^2 as limit). Whence we conclude that if the variables are equal, the limits are equal ($C/R^2 = C'/R'^2$) and that the areas of two circles are to each other as the squares of their radii.

Our method is a simplification of the method of Eudoxus, who had to be over-rigorous on account of the fear of Zeno and his group. What Eudoxus does is first to formulate the lemma—we give it in Euclid's version—that if from any magnitude we subtract not less than half and then from the remainder not less than half, and so on continually, there will sometimes be left a

magnitude less than any assigned magnitude x , of the same kind, however small. He then proceeds to say that of the two quantities which are to be proved equal (say as in the above, C/R^2 and C'/R'^2) one of three alternatives must be true: either they are equal, or the first is greater than the second, or the first is less than the second. But by reasoning from the lemma he shows that certain dire contradictions will result from the acceptance of the second or third alternatives. Ergo the first alternative is true, and the quantities in question are equal.

Eudoxus' method of exhaustion contains in it the germ of the differential and integral calculus, and, as we shall see, in the hands of a great mathematician like Archimedes, it was actually used to solve problems in integral calculus.

PYTHAGOREAN ASTRONOMY—ANTICIPATIONS OF COPERNICUS

Astronomy shares with mathematics the honor of being the cradle of science. And as in the case of mathematics it was cultivated not for any practical utility—the application of astronomy to navigation is rather mythical—but for its relative scientific simplicity. It offered to human reason a field in which it could train its analytical powers. Poincaré has well remarked that were it not for his initial successes in organizing celestial phenomena, man might have given up natural science as a hopeless task.

One might expect that astronomy would be intensively cultivated by the Pythagoreans, and our expectations are not disappointed. We are told that for the Pythagoreans astronomy was regarded as a sister science to music! When Pythagoras and his circle looked at the heavens they saw there a confirmation of their ideas about number and harmony. Each constellation had not only a certain geometrical form, but it had a definite number of star points—it was in fact the best illustration of their theory of "figured numbers." Numbers were omnipresent and ruled the heavens as well as the earth. Not only was number at the basis of each constellation, but by a daring feat of the imagination, the Pythagoreans came to conceive of the distances between the constellations and the distances between the planets as being

related by a mathematical law. And the best mathematical law they could conceive was the law of musical intervals. The heavens, in short, were a musical scale, and the spheres in their motions were always producing music. If we did not hear the music the fault was our own—our soul was out of tune with the universe and needed to be attuned by science.

An extravagant conception, but not one without results. Putting aside the influence of this doctrine of the music of the spheres on world literature—traces of which are to be found in Shakespeare's *Merchant of Venice* and Milton's *Hymn on the Nativity*—the mathematical conception was to serve as the basis for Kepler's discoveries and for Bode's law of planetary distances in modern astronomy.

More directly constructive were the Pythagorean attempts to untangle the motions of the heavens. Here again the basic tools were mathematical conceptions, the notion of spheres and of circular motions. Pythagoras was the first man to conceive of the earth and planets as spheres. This at once put an end to the anthropomorphic cosmogonies which made the earth a flat mass resting on the back of a god or an elephant. The earth was suspended in space. It was now a question of explaining the heavenly motions so as to account for the paradoxical shiftings to and fro as they appear to an earthly observer. This task of finding a composition of the heavenly movements adequate to "save the phenomena" we are now beginning to realize is an endless one, but in Pythagoras' time scarcely the beginning had been made.

At this date we find it difficult to imagine what an effort of human thought was required to come to the conclusion that it was one and the same sun which set the night before and rose the next morning. This apparently simple inference was not reached by primitive man except by way of personifying the sun as a god in heaven. Easier to believe that the sun was one if it had a soul and went to sleep like a man, than to suppose it was simply a natural mass which went around in an orbit! A century after Pythagoras, Anaxagoras, the protégé of the great Pericles, was thrown into prison because he said that the sun was "a red-hot stone larger than the Peloponnesus" and that the moon was made

of earth. So fixed was the divinity of the sun, even in Greece, that Epicurus (342–270 B. C.), who wanted to save men from superstition, was ready to go to the other extreme and entertained the belief that the sun was just the size it appeared on the horizon and that for all he knew it was a new fire that was daily kindled in the east and extinguished in the west.

In the school of Pythagoras, the analysis of the heavenly movements was carried on quietly under the auspices of number and geometry. It was Pythagoras himself who observed that the morning and the evening star are one and the same, and it was either Pythagoras or his immediate disciple, Alcmaeon, who first attempted to provide a system for the heavenly movements. According to this system the earth was still in the center of things, and the sun, moon, and planets moved in independent orbits around it from west to east. This accounted for the seasonal movements, but as all the heavenly bodies were seen to partake also of a common motion in the opposite direction, the whole sphere of the universe including the fixed stars was conceived as having a daily rotation from east to west. In this way, however crudely, the attempt was made to account for all the phases of the visible astronomical phenomena.

This system was only the beginning. It was soon felt that a simpler and more adequate scheme of things could be organized if all the heavenly motions were to be plotted in the same direction. In order to accomplish this, it was necessary to sacrifice the earth's position of primacy, and for the sake of mathematics this was actually done. In the system taught by Philolaus, a Pythagorean who was a contemporary of Socrates, the earth is displaced for the first time from its position as the center of the universe. All the planets travelled from west to east around a body called the Central Fire. Nobody had ever seen this Central Fire, but this was explained on the assumption that the known parts of the earth were situated on the outward hemisphere which is always turned away from the center of the orbit. The earth made its revolution in 24 hours, and further outward were the moon, the sun, and the five planets, which made more extended circuits. The moon went around in 29½ days, the sun in a year, and the

planets in still longer periods. In addition to postulating a Central Fire, Philolaus postulated a body called the counter-earth, which moved inside the earth's orbit and was thus invisible to us. Two reasons are given for postulating this body. One is that the Pythagoreans wanted to bring the number of heavenly bodies up to the sacred number of ten (the fixed stars being counted as one), the other is that it was designed to explain the greater frequency of lunar over solar eclipses. The eclipses of the moon could under this theory be caused sometimes by the interposition of the earth between the sun and moon, sometimes by the interposition of the counter-earth.

The really remarkable feature about this later Pythagorean system was that it contained the germ of the modern Copernican hypothesis. It did not matter so much that the earth and planets went around a Central Fire, instead of around the sun. The precedent had been set for regarding the earth in the same light as any other planet, and this victory over anthropomorphism and human pride is to be credited as an achievement of the mathematical method. And we shall see in a succeeding chapter, it was a non-mathematical philosopher, Aristotle, who not only threw in the weight of his authority in favor of a rival geocentric theory but encased it in a metaphysical system which stifled any germs of mathematical progress that it contained.

Historically, the system of Philolaus was developed even further in the Pythagorean school. Two Pythagoreans of Syracuse, Hiketas and Ekphantos, took the step of substituting for the Central Fire the fire in the interior of the earth, thus evolving the idea of the rotation of the earth on its own axis. Heraclides of Pontus, outside of the Pythagorean school, revived certain traits of this doctrine and passed it on to Aristarchus of Samos, the Copernicus of antiquity, whose system combined the rotation of the earth around its axis and the revolution of all the planets, including the earth, around the sun. Aristarchus, however, had no successors in antiquity. But two thousand years later, the Polish monk, Nicholas Copernicus, coming to realize that something was wrong with the system of Ptolemy, went to Italy to find out whether any other system had ever been taught. He

learned about the Pythagorean doctrines and of the still-born attempt of Aristarchus. This gave him enough courage to work out his own theory.

We have completed our survey of Pythagorean achievements in mathematics and astronomy. In the field of science it has been well remarked that the first steps are the most difficult, but even without taking into account the difficulty of the first steps one stands amazed at the amount of progress achieved in so short a time. From Pythagoras to Eudoxus it is only a matter of two centuries and the edifice of geometry is completed, while in the three centuries that elapse from Pythagoras to Aristarchus, astronomy has advanced to the point where the Copernican theory, in its essential aspects, is envisaged. It is a record without comparison, and it does honor to the genius of Pythagoras, who laid the foundations for this progress and laid them well.

BIBLIOGRAPHICAL NOTES

CHAPTER I

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