

NUMBER
I. IRRATIONAL NUMBERS¹

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1. Introduction

In the first scholium to Book X of Euclid's elements³ we read

"The Pythagoreans were the first to make inquiry into commensurability, having first discovered it as a result of their observation of numbers; for though the unit is a common measure of all numbers they could not find a common measure of all magnitudes. ... there is a story of the Pythagoreans that the man who was the first to bring examination of these matters out into the open suffered shipwreck, and perhaps they were hinting at the fact that everything that is irrational in every case is accustomed to be hidden (sc. so that it remains) irrational and formless, and if anyone ($\psi\upsilon\chi\eta$) were to make an assault on such a form of existence as this, he would simply make obvious and clear the fact that he is being carried under into the sea of creation and is being overwhelmed by the unstable surges of it. Such awe did these men have for examination of the irrational."

These remarks, written later than the time of Theon of Alexandria⁴ (end of the fourth century A.D.⁵) form perhaps the most explicit description of the discovery of irrational (or incommensurable) numbers that we possess.

In this paper we shall consider the mathematical background against which this discovery took place after looking at the Pythagoreans and their mathematics. Then we shall report on some earlier suggestions of the mode of production of irrational numbers and present some of our own. In doing this we wish to stress the difficulty on the one hand of trying to read the minds of individuals of whom we have virtually no record and on the other hand to attempt to show a possible coherent picture. We close by pointing out aspects of the discovery and why it was so disturbing. These are as much concerned with the philosophical side as the mathematical.

Our basic conclusion is that after the connection between (natural⁶) numbers and measurement of, for example, the length of the side of a field or geometrical figure had been made⁷, the direction was reversed and a connection sought in the

1. This is the first in a proposed series of papers on the development of the concept of number, taking into account historical, philosophical, anthropological and psychological aspects. Part 2 will appear in Vol. 5 No. 1 of the Gazette.
2. I am deeply indebted to Gordon Smith for his trenchant constructive criticism and advice.
3. Translated from the Greek of Heiberg's edition of Euclid [3] vol. V, p.417, . 22.12-20 by A.S. Henry to whom I am grateful.
4. See Heath's edition of Euclid [2], vol. I, p.66.
5. Kline [11], p.25.
6. Here we mean 1, 2, 3, ...; zero being specifically excluded.
7. That is, numbers had been correlated with lengths.

direction from lengths to numbers. To us of this era it seems natural to make both these connections. However, the (logical) necessity of the second connection⁸ depends on the philosophical view (of mathematics in particular) which is held.

2. Sources

If we wished to know exactly what Pythagoras did and taught we should have to travel back in time. This is because the sources of information which we possess at present are at most about a thousand years old while the age of Pythagoras was the sixth century B.C.. And indeed those sources are late transcriptions of earlier writings. According to the Oxford Classical Dictionary⁹, Pythagoras "wrote probably nothing (although works were later fathered on him) and already in Aristotle's day his life was obscured by legend." For us the *Vita Pythagoricae* of Iamblichus [9] is a major source but Iamblichus lived in the fourth century A.D.¹⁰, so the reliability of the text (let alone its content) cannot be guaranteed. Even the little which is told of Pythagoras by Aristotle (384-322 B.C.) was written originally more than a hundred years after Pythagoras' death (490 B.C.¹¹). Again Aristotle's work, like Iamblichus', was transcribed and the originals are not now available, though in this case we have a number of (late) manuscripts.

It is impossible to say how much distortion has been introduced by this transcription process. There is no doubt that much has been. However, as in the case of the Christian story, a picture of a certain, very pronounced and charismatic character with exceptional gifts does come through. Moreover, although the three biographies of Pythagoras which have been preserved in some form, in Diogenes Laertius' *Vitae Philosophorum* [1], Iamblichus, *op. cit.* and Porphyry [19] "are still considered as belonging rather to the genre of hagiography than to history"¹², nevertheless a picture of considerable coherence does emerge. In order to give sufficient indication at this time of the sort of individual Pythagoras was, we shall shortly turn to Plato's writings.

3. Pythagoras

Plato was a Pythagorean in many of his views¹³ and although Aristotle says Plato's philosophy "had peculiarities that distinguished it from the Italians [i.e. Pythagoreans]"¹⁴, in particular "in making the One and the Numbers separate from things, and his introduction of the Forms"¹⁵, it is reasonable to assume that the basic tenor of Pythagoras' philosophy in the oldfashioned sense of attitude to, and view of, the world was strongly reflected in Plato's philosophy. Thus as in Plato we read that the reality of the good is what 'every soul of man pursues and makes the end of all his actions'¹⁶ and also that the pursuit of philosophy involves 'that perfect training which alone can lead to a satisfactory vision of the truth'¹⁷, so in Iamblichus [9] Chapter XII, p.28 we read:

8. We intend to deal with the first connection in a later paper.
9. Oxford Classical Dictionary (O.C.D.) [16], Pythagoras (1), pp.903-4.
10. c. A.D. 250-325 (ibid., p.538).
11. de Vogel [24], pp.21, 24.
12. ibid., p.5.
13. Aristotle, *Metaphysics* 987a-988a.
14. ibid., 987a.
15. ibid., 987b.
16. Plato, *Republic* 505e.
17. Plato, *Parmenides* 136c.

"But the most pure and unadulterated character, is that of the man who gives himself to the contemplation of the most beautiful things, and whom it is proper to call a philosopher."¹⁸ He adds, that the survey of all heaven, and of the stars that revolve in it, is indeed beautiful, when the order of them is considered. For they derive this beauty and order by the participation of the first and the intelligible essence. But that first essence is the nature of number and reasons [i.e. productive principles], which pervades through all things, and according to which all these [celestial bodies] are arranged, and fitly adorned. And wisdom indeed, truly so called, is a certain science which is conversant with the first beautiful objects¹⁹, and these divine, undecaying and possessing an invariable sameness of subsistence; by the participation of which other things also may be called beautiful. But philosophy is the appetition of a thing of this kind. The attention therefore to erudition is likewise beautiful, which Pythagoras extended, in order to effect the correction of mankind."

However, Pythagoras' and the Pythagoreans' main interest was number in all its forms.

"... the so-called Pythagoreans, who were the first to take up mathematics, not only advanced this study, but also having been brought up in it they thought its principles were the principles of all things."²⁰

4. Ideas of mathematics

In ancient Greek mathematics the drawing of distinctions around the concept of number was somewhat different from what it is today. It is easy to forget this though happily not everyone does. Nowadays we do use the word 'number' in many different ways, in particular, for counting and for measuring we use numbers though we also use the concept of number in a more vague way - a large number of people, several pounds of beans. For the ancient Greeks of the fifth or fourth century B.C. 'number' was used in ways that are, to us, so different that it is difficult for us to comprehend that there is even what Wittgenstein calls a "family resemblance"²¹.

In the Philosophical Investigations, Wittgenstein says:

"And for instance the kinds of number form a family Why do we call something a "number"? Well, perhaps because it has a - direct - relationship with several things that have hitherto been called number; and this can be said to give it an indirect relationship to other things we call the same name. And we extend our concept of number as in spinning a thread we twist fibre on fibre. And the strength of the thread does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres.

But if someone wished to say: "There is something common to all these constructions - namely the disjunction of all their common properties" - I should reply: Now you are only playing with words -."²²

18. Iamblichus derived this very beautiful passage from Heraclides Ponticus, as is evident from Cicero, *Tusc. Quaest. lib. v. 3*, who relates the same thing of Pythagoras, from the aforesaid author. (Footnote of T. Taylor.)
19. i.e. With intelligibles properly so called. (Footnote of T. Taylor.)
20. Aristotle, *Metaphysics* 985b.
21. Wittgenstein [26], no. 67.
22. ibid.

According to Aristotle "the so-called Pythagoreans, who were the first to take up mathematics ... thought its principles were the principles of all things."²³ Even in Plato the pre-eminence of number is still evident. This is well illustrated by the following extract from the *Republic* (524e - 525b). Here Socrates is describing the education of the proposed philosopher-rulers:

"... the soul in perplexity, is obliged to rouse her power of thought and to ask: 'What *is* absolute unity?' This is the way in which the study of the one has a power of drawing and converting the mind to the contemplation of true being.

"And surely, he said, this occurs notably in the visual perception of unity; for we see the same thing at once as one and as infinite in multitude?

"Yes, I said; and this being true of one must be equally true of all number?

"Certainly.

"And all arithmetic and calculation have to do with number?

"Yes.

"And they appear to lead the mind towards truth?

"Yes, in a very remarkable manner.

"Then this is a discipline of the kind for which we are seeking ..."

But although number was the basis of the Pythagorean philosophical system, it is not clear that mathematics was regarded as a coherent whole. From the works of Anatolius²⁴ we learn:

"The Pythagoreans are said to have given the special name *mathematics* only to geometry and arithmetic; previously each had been called by its separate name, and there was no name common to both."²⁵

Elsewhere mathematics is said to have comprised four or five divisions. Plato speaks of the five mathematical studies: arithmetic, geometry, solid geometry, astronomy and harmony (or harmonics)²⁶. On the other hand Porphyry²⁷ cites Archytas who lived in the first half of the fourth century B.C.²⁸ listing four studies.

"Let us now cite the words of Archytas the Pythagorean, whose writings are said to be mainly authentic. In his book *On Mathematics* right at the beginning of the argument he writes thus:

'The mathematicians seem to me to have arrived at true knowledge, and it is not surprising that they rightly conceive the nature of each individual thing; for, having reached true knowledge about the nature of the universe as a whole,

they were bound to see in its true light the nature of the parts as well. Thus they have handed down to us clear knowledge about the speed of the stars, and their risings and settings, and about geometry, arithmetic and sphaeric, and, not least, about music; for these studies appear to be sisters."

In this paper we shall only treat arithmetic and geometry. What is important here is the view that there was a rather more tenuous nature in the connection between arithmetic and geometry than there is for us today. And also let us not forget that the very root of the word mathematics has nothing to do with number but comes from *μάθημα*: that which is learnt²⁹. However, there are grounds for believing that the connection was strengthened in Pythagorean times. Indeed Aristotle shortly after the above quotation writes of the Pythagoreans:

"... since, then, all other things seemed in their whole nature to be modelled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number. And all the properties of numbers and scales which they could show to agree with the attributes and parts and the whole arrangement of the heavens, they collected and fitted into their scheme; and if there was a gap anywhere, they readily made additions so as to make their whole theory coherent."³⁰

It must be added that he then gives an example to show how he disapproves of their additions at times. (At this point he cites the addition of a 'counter-earth' to the planets in order that there should be the perfect number, ten.)

Here we are only considering two areas of mathematics: arithmetic and geometry. In order to put these in perspective, we shall consider not only what Pythagoras and his followers learnt and taught but also whence they learnt these things.

5. Pythagoras' education

According to Iamblichus, Pythagoras' teacher was Thales of Miletus³¹.

"Thales ... was born about B.C. 640 at Miletus, ... , and died at the same place about B.C. 542. The main facts of his life are given by Diogenes Laertius ([1], vol. I, pp.23-47), who cites Apollodorus as authority for the birth of Thales in the 35th Olympiad, and Socrates, for his death in the 58th."³²

Thales was a Phoenician by remote descent³³ who had some interest in politics for Herodotus quotes an excellent political proposal made by Thales "that the Ionians should set up a common centre of government at Teos, as that place occupied a central position; the other cities would continue as going concerns, but subject to the central government"³⁴. On the mathematical side Thales is chiefly memorable for the following assertions which are attributed to him - all in the realm of geometry.

23. Aristotle, *Metaphysics* 986a.

24. Anatolius was bishop of Laodicea about A.D. 280 (Thomas [23], vol. I, p.2).

25. Anatolius, cited by Heron, *Definitions*, ed. Heiberg 160.8, translated in Thomas [23], vol. I, p.3.

26. Plato, *Republic* 525 ff.

27. A.D. 232/3 - c.305 (O.C.D. [16] p.864).

28. Thomas [23], vol. I, p.4.n.

29. *ibid.*, vol. I, p.2.n.

30. Aristotle, *Metaphysics*, 985b-986a.

31. Iamblichus [9], ch. II p.6. However, Diogenes Laertius [1], p.322, gives several other teachers.

32. Gow [6], pp.138-9.

33. Herodotus, I. 169.

34. *ibid.*

- "(1) The circle is bisected by its diameter.
- (2) The angles at the base of an isosceles triangle are equal.
- (3) If two straight lines cut one another, the opposite angles are equal.
- (4) The angle in a semicircle is a right angle.
- (5) A triangle is determined if its base and base-angles be given (practically Euc. I, 26)."³⁵

(1) is "merely stated as a fact [by Euclid] in I, Def. 17. Thales therefore probably observed rather than proved the property."³⁶ But it is attributed to Thales by Proclus³⁷. Gow writes:

"The language of Proclus also³⁷ seems to hint that Thales proved the proposition (2), our old friend, the *Pons Asinorum*, by taking two equal isosceles triangles and applying them to one another as in Euc. I. 4, another case of experiment. But the theorems [(4) and (5)] are obviously incapable of such treatment [i.e. by superposition], and must have been supported either by deduction or at least by very wide induction. The last of them (Euc. I, 26) is attributed to Thales by Eudemus (Proclus, p. 65), apparently on the ground that Thales invented a mode of discovering the distance of a ship at sea, in which the proposition was used."³⁸

However, on (4) Diogenes Laertius writes:

"Pamphila says that, having learnt geometry from the Egyptians, he was the first to inscribe in a circle a right-angled triangle, whereupon he sacrificed an ox. Others say it was Pythagoras, among them being Apollodorus the calculator."³⁹

According to Proclus, Thales

"first went to Egypt and thence introduced this study (geometry) into Greece. He discovered many propositions himself, and instructed his successors in the principles underlying many others, his method of attack being in some cases more general (i.e. more theoretical or scientific), in others more empirical (*αἰσθητικώτερον*, more in the nature of simple inspection of observation)."⁴⁰

Pythagoras, on the other hand, went to Thales first. Iamblichus writes:

"But after he had attained the eighteenth year of his age, about the period when the tyranny of Polycrates first made its appearance, foreseeing that under such a government he might receive some impediment in his studies, which engrossed the whole of his attention, he departed privately by night with one Hermodamas ... to Pherecydes, to Anaximander the natural philosopher, and to Thales at Miletus. He likewise alternately associated with each of these philosophers, in such a manner, that they all loved him, admired his natural endowments, and made him a partaker of their doctrines."⁴¹

6. Pythagoras' later life

Iamblichus goes on to say:

"Indeed, after Thales had gladly admitted him to his intimate confidence, he admired the great difference between him and other young men, whom Pythagoras left far behind in every accomplishment. And besides this, Thales increased the reputation Pythagoras had already acquired, by communicating to him such disciplines as he was able to impart: and, apologizing for his old age, and the imbecility of his body, he exhorted him to sail into Egypt, and associate with the Memphian and Diospolitan⁴² priests. For he confessed that his own reputation for wisdom, was derived from the instructions of these priests; but that he was neither naturally, nor by exercise, endued with those excellent prerogatives, which were so visibly displayed in the person of Pythagoras. Thales, therefore, gladly announced to him, from all these circumstances, that he would become the wisest and most divine of all men, if he associated with these Egyptian priests."⁴³

Pythagoras

"spent therefore two and twenty years in Egypt, in the adyta of temples, astronomicalizing and geometrizing, and was initiated, not in a superficial or casual manner, in all the mysteries of the Gods, till at length being taken captive by the soldiers of Cambyzes, he was brought to Babylon. Here he gladly associated with the Magi, was instructed by them in their venerable knowledge, and learnt from them the most perfect worship of the Gods. Through their assistance likewise, he arrived at the summit of arithmetic, music, and other disciplines; ..."⁴⁴

Whether this account is accurate or not, it is known that precise astronomical observations were made in Babylon⁴⁵ and Thales is believed to have predicted the year of a solar eclipse⁴⁶. Moreover, Thales is designated as being 'of Miletus' (*v. supra*) and Miletus is in Asia Minor. He therefore came from a place as close to Babylon as Italy, Pythagoras' main abode⁴⁷, was to his birth place, Samos⁴⁸ (a small island not far from Miletus). *Prima facie*, therefore, it is possible that Thales was familiar with some Babylonian mathematics, but we shall see below that there are difficulties in arguing that either Thales or Pythagoras was very familiar with Babylonian mathematics.

According to Iamblichus⁴⁹, Pythagoras had brought his knowledge of some parts of arithmetic and geometry from Egypt, but whether Pythagoras actually visited Egypt or Babylon is questionable. Thus J.A. Philip writes:

⁴². i.e. the priests of Jupiter. (Footnote of T. Taylor.)

⁴³. Iamblichus, ch. II, p.6.

⁴⁴. *ibid.*, ch. IV; p.9.

⁴⁵. Neugebauer [12], p.114: "The ephemerides alone are never a reliable source for the investigation of the basic empirical facts. At present it is completely impossible to write a "history" of Babylonian astronomy in its latest phase. All we do have is the ephemerides in a form excellently adapted to practical computation and to predicting new moons, eclipses, etc."

⁴⁶. Herodotus, I.73.

⁴⁷. de Vogel [24], p.24.

⁴⁸. Philip [18], p.185.

⁴⁹. See above n.43.

³⁵. Gow [6], pp.140-1.

³⁶. Heath [7], vol. I, p.131.

³⁷. Proclus [20], p.124.

³⁸. Gow [6], p.141. Gow's reference to p.65 (of Friedlein's edition) appears to be incorrect. It should be 352 (see Proclus [20] p.275).

³⁹. Diogenes Laertius [1], pp.25-27.

⁴⁰. See Proclus [20], p.52.

⁴¹. Iamblichus [9], ch. II, pp.5-6.

"The voyages of Pythagoras were a favourite theme of the biographical tradition. In early times philosophers and sophists, as they shared the name of "sophist", shared also the characteristic of travelling from city to city and from country to country. In that he remained all his life in Athens, Socrates was an exception to the rule. His pupil Plato, however, in his first voyage to Sicily undertook what was a voyage of instruction in the strict sense, and thereafter it was common for philosophers to embark on a journey abroad to learn what there was to be learned in foreign lands. They naturally imagined their predecessors to have made similar journeys, and when it came to writing the life of Pythagoras they credited him not only with migration from Ionia to Magna Graecia but also with preceding voyages of instruction. They did so the more readily because, in the Academy of Plato's later years, a lively interest in the East developed, and the tendency arose to seek the origins of Greek religious and philosophical doctrines there.⁵⁰ That this tendency was not restricted to the Academy is shown by the fact that Isocrates⁵¹ alludes to a journey made by Pythagoras to Egypt.

In the biographies of Pythagoras this theme of the voyages of instruction and initiation is treated in the manner and in the stages we have observed in the case of other themes. In the fourth century and in the Peripatetic school we find the voyages used to explain how Pythagoras came by esoteric wisdom, foreign to Greece. Here, as in so many instances, Aristoxenus is the first to make the suggestion. He has him travel not only to Egypt but also to the East, to Zartas to whom a doctrine of immortality was (falsely) ascribed.⁵² The purpose of his journey (that is, what they are meant to explain) was the study of mathematics and priestly lore in Egypt and in Asia Minor and instruction in a way of life from the Magi.^{53,54}

But there are many more questions to be asked as far as both Egypt and Babylon are concerned. First let us turn to Egypt.

7. *Egyptian mathematics*

It is ironic that although Western mathematics has remained incredibly dependent on Greek mathematics, indeed on Euclid's Elements, no original documents of early Greek mathematics have survived, while in the case of Egypt in the same period we have less mathematics and more documents.⁵⁵ But even here these only number a handful.

The documents we do possess indicate an overwhelming concern with mensuration and, indeed, very 'concrete' mathematics. Van der Waerden judges Egyptian geometry as "not a science in the Greek sense of the word, but merely applied arithmetic".⁵⁶ Peet, in his edition of the Rhind Mathematical Papyrus, perhaps our most important surviving Egyptian mathematical work, puts it better than some later writers:

50. Jaeger, *Aristotle* [1948], 131-137. (Phillip's note.)
51. Bus. 28 = Vors. 14.4 (Phillip's note.)
52. Hehrl, *Aristoxenus* fr. 13 and comment; Bidez-Cumont, *Les Mages Hellénisés* [Paris 1938], 28, 38 ff. (Phillip's note.)
53. Popphry, *Vita Pythagoricae* 6. (Phillip's note.)
54. Phillip [18], p. 189.

"The outstanding feature of Egyptian mathematics is its intensely practical character. This is not peculiar to mathematics, for it is typical of all the sciences in Egypt. As Plato alone of the Greeks seems to have realized⁵⁷, the Egyptians were essentially a "nation of shopkeepers", and interest in or speculation concerning a subject for its own sake was totally foreign to their minds.

To realize this we have only to take a glance through the problems of the Rhind Papyrus. Here everything is expressed in concrete terms. The Egyptian does not speak or think of 8 as an abstract number, he thinks of 8 loaves or 8 sheep. He does not work out the slope of the sides of a pyramid because it interests him to know it, but because he needs a practical working rule to give to the mason who is to dress the stones ... If he resolves $\frac{2}{13}$ into $\frac{1}{8} + \frac{1}{52} + \frac{1}{104}$ it is not because this fact in itself appeals in any way to his curiosity, but simply because sooner or later he will come across the fraction $\frac{2}{13}$ in a sum, and since he has no machinery for dealing with fractions whose numerators are greater than unity he will then urgently need the resolution above stated.

Perhaps it is in keeping with this attitude that there is in our papyrus practically no instance of the use of a general formula (Nos 61B and 66 are perhaps the only exceptions), each case being worked out on its own merits, and cases which to us seem analogous being sometimes dealt with by totally different methods.⁵⁸

The actual nature of the problems given in the Rhind Mathematical Papyrus makes one question just how 'practical' the Egyptian mathematicians were. It is very noticeable that all the answers work out neatly. But then most mathematics (as opposed to engineering) textbooks of the present era have this same tendency. However, there is what appears to be a very modern (20th century) tradition that Egyptian mathematics was not only practical even in this restricted sense. Gillings dissects

55. Neugebauer [12] supplies the following list:

1. "Our knowledge of Egyptian mathematics is primarily based on the following texts, all of which were written in the Middle Kingdom or the Hyksos period.
 1. *Mathematical Papyrus Rhind*, published first by Eisenlohr in 1877. Modern publication by T. E. Peet, London, 1923; additional material and photographs in *Chace-Bull-Manning-Archibald*, The Rhind Mathematical Papyrus, Oberlin, Ohio 1 1927, II 1929.
 2. *Moscow Mathematical Papyrus* published by H. Strube, Quellen und Studien zur Geschichte der Mathematik, ser. A, vol. 1 (1930).
 3. *P. Berlin 6619*. Published by Schack-Schackenburg, Zeitschr. f. ägyptische Sprache 38 (1900), p. 135 ff. and 40 (1902), p. 65 f.
 4. *P. Kahun*. Published by F. Ll. Griffith, Hieratic Papyri from Kahun and Gurob, London 1898 P. VII and p. 15 ff.
 5. *Leather roll British Museum 10250* by S. R. K. Glanville in J. Egyptian Archaeology 13 (1927), p. 232 ff.
 6. *Wooden tablets Cairo 25367 and 25368*. Recueil de travaux relatifs à la philologie et à l'archéologie égyptiennes 28 (1906), p. 62 ff. and Catalogue générale ... du Musée du Caire, Ostraca, 1901 pl. 62-64 and p. 95 f.
- For the late period, Demotic papyri should be added."
56. van der Waerden [25], pp. 31, 36 (van der Waerden's italics).
 57. [Problem] nos. 61B and 66 are perhaps the only exceptions. (Peet's footnote.)
 58. Peet [17], p. 10.

this view nicely. Amongst other justified comments of his we find:

... a sober-minded person as H.W. Turnbull writes, "Their land surveyors were known as rope stretchers, because they used ropes with knots or marks at equal intervals to measure their plots of land. By this simple means, they were able to construct right angles, for they knew that three ropes of lengths three, four, and five units respectively, could be formed into a right-angled triangle."⁵⁹

It is, however, nowhere attested that the ancient Egyptians knew even the very simplest case of Pythagoras' theorem! But Turnbull goes further: "As Professor D'Arcy Thompson has suggested, the very shape of the Great Pyramid indicates a considerable familiarity with that [*sic*] of the regular pentagon. A certain obscure passage in Herodotus, can, by the slightest literal emendation, be made to yield excellent sense. It would imply that the area of each triangular face of the Pyramid, is equal to the square of the vertical height. If this is so, the ratios of height, slope, and base, can be expressed in terms of the *golden section*, or of the ratio of a circle to the side of the inscribed decagon."

I am unable to understand exactly what Turnbull means by this last sentence. But whatever it means, with further slight emendations, the dimensions of the Eiffel Tower or Boulder Dam could be made to produce equally vague and pretentious expressions of a mathematical connotation⁶⁰.

Let us not, however, be careless in distinguishing the dross of later imputations of mathematics to the Egyptians from the substance. Egyptian mathematics with its supposed connections with the pyramids has attracted wild speculation⁶¹, while the actual results contained in the Rhind Mathematical Papyrus are very interesting and substantial (see below).

Thus there is, to our knowledge, nothing even remotely approaching Pythagoras' theorem. Gillings has five quotations⁶²; all indicate the falseness of another modern tradition, namely that the Egyptians at least knew of the (3,4,5) right-angled triangle, but one author he quotes only indirectly is Peet, who says:

"... an interesting problem is raised by Democritus' reference to the *harpedonaptai* of Egypt. The philosopher boasted that no one of his time had surpassed him in constructing figures from lines and in proving their properties, not even the so-called *harpedonaptai* of Egypt. Who were these *harpedonaptai*? More than one historian of mathematics has supposed that they were land-measurers. The literal meaning of the word is "rope-stretchers", and it is suggested [by Heath⁶³] that they were acquainted with the fact that a triangle whose sides were 3, 4 and 5 contained a right-angle, and that they constructed right-angles accordingly, as did the Chinese and the Indians.

For this last statement I can find no foundation whatsoever: nothing in Egyptian mathematics suggests that the Egyptians were acquainted even with special cases of Pythagoras' theorem concerning the squares on the sides of a right-angled triangle. That the *harpedonaptai* were land-measurers on the other

hand is most probable, indeed we can even see such persons at work in the pictures on the walls of Egyptian tombs⁶⁴.

Now the Egyptian Rhind Papyrus⁶⁵ is dated about 1650 B.C.⁶⁶, so it is possible that geometry did develop significantly in Egypt in the millennium preceding the time of Pythagoras. Moreover, Heath gives details of other documentation for the development of geometry in Egypt, but says that "[the] statements [Heath quotes] may all be founded on the passage of Herodotus, and Herodotus may have stated as his own inference what he was told in Egypt."⁶⁷ In Herodotus we read:

"It was this king [Sesostris]⁶⁸, moreover, who divided the land into lots and gave everyone a piece of equal size, from the produce of which he exacted an annual tax. Any man whose holding was damaged by the encroachment of the river would go and declare his loss before the king, who would send inspectors to measure the extent of the loss, in order that he might pay in future a fair proportion of the tax at which his property had been assessed. Perhaps this was the way in which geometry was invented, and passed afterwards into Greece - for knowledge of the sundial and the gnomon and the twelve divisions of the day came into Greece from Babylon."⁶⁹

Thus Egyptian mathematics (in the millennium preceding Pythagoras) as we know it, is concerned with specific contexts, as is all mathematics in this period. Later we shall see that it was concerned with calculations in what to us is a geometrical context but there is no record of anything approaching even the geometric assertions attributed to Thales⁷⁰.

3. Babylonian mathematics

Turning now to Babylon the situation is somewhat different. The "oldest preserved document in number theory" ... "tabulates the answers to a problem containing Pythagorean numbers (or Pythagorean triangles)"⁷¹. This document is the cuneiform tablet Plimpton 322 of the Plimpton collection in Columbia University, New York. "It falls in the period between 1900 and 1600 B.C."⁷² and thus is from the same era as the Rhind Mathematical Papyrus. Now, according to Neugebauer and Sachs, the "terminology [of cuneiform tablets in general] is geometrical, [but] the whole treatment is strongly algebraic"⁷³. (3,4,5) triangles specifically occur: thus Plimpton 322 2.13 contains the numbers 45, 75 and Neugebauer and Sachs interpolate the 60 and in the British Museum table BM 34568 we have

"[4 ist die Lnge und 5 die Diagonale. Was ist die Breite? Die Grsse ist nicht bekannt. 4 mal 4 (ist) 16. 5 mal 5 (ist) 25. Von 16 (bis) 25 steigst Du auf und es bleibt 9.
Was mal was soll ich nehmen, um 9 (zu erhalten)?
3 mal 3 (ist) 9.
3 (ist) die Breite."⁷⁴

59. H.W. Turnbull, *The Great Mathematicians*, 4th edition, Methuen, London, 1951, pp.2 f. (Gillings's footnote.)

60. Gillings [5], p.238.

61. See, for example, T. Bruns: *The Secrets of Ancient Geometry* (2 volumes), Rhodos, Copenhagen 1967.

62. Gillings [5], Appendix 5, p.242.

63. Heath [7], vol. I, p.122.

64. Peet [17], pp.31-32.

65. BM10057 and 10058.

66. Peet [17], p.3 adds "there is no reason to doubt the scribe's own statement that it was a copy of an older document [of the nineteenth century B.C.]."

67. Heath [7], vol. I, p.121.

68. That is, Ramesses II c. 1300 B.C. (Heath, [7], vol. I, p.121).

69. Herodotus, II, 109.

70. See above, p.11.

71. Neugebauer and Sachs [14], p.37.

72. *ibid.*, p.39.

73. *ibid.*, p.37.

74. Neugebauer [13], vol. III, p.17.

There are other similar examples given by Neugebauer. Thus numerical versions of Pythagoras' theorem were certainly known from the Old Babylonian period of 1600 B.C. up to the Seleucid period which started about 300 B.C.⁷⁵ However, there is no evidence at all of what we would call well-developed general methods, though specific problems may be construed as general. Thus on BM 34568 we find Neugebauer commenting:

"Die Beispiele 14 und 17 rechnen gemäß (4) wie üblich in konkreten Zahlen. Von großem grundsätzlichen Interesse ist aber die Tatsache, da Nr. 18 die Formel (4) ganz ohne spezielle Zahlen beschreibt, also wirklich als *all-gemeine Formel*."⁷⁶

But all of these are in a tradition which is concerned with mensuration. There appears to be no evidence at all of synthetic geometry in Babylon or Egypt before the time of Thales⁷⁷.

Thus synthetic geometry seems to have sprung up very rapidly indeed according to the evidence we have. It is claimed by van der Waerden:

"... what is characteristic and absolutely new in Greek mathematics, is the advance by means of demonstration from theorem to theorem. Evidently, Greek geometry has had this character from the beginning, and it is Thales to whom it is due."⁷⁸

However, while the claim that Greek mathematics from Thales proceeded from theorem to theorem by demonstration is not as clear to the present author as it appears to be to van der Waerden, there does appear to be a clear qualitative difference between early Greek geometry and the early geometry of Egypt and Babylon. In the latter two countries geometry is always intuitively connected with mensuration and calculation in all the records we possess. In Greece, from the time of Euclid at least, geometry seems to have existed in a form much closer to that part of pure mathematics as we know it today.

75. Neugebauer [12], p.14.

76. Neugebauer [13], vol. III, p.21.

77. cf. Neugebauer [12], p.44, Neugebauer and Sachs [14], p.37.

78. van der Waerden [25], p.89. However, Diogenes Laertius [1], p.331, attributes it to Moeris who is reputed to have lived about the same time in Egypt (Smith [22], vol. 2, p.109).

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