

NUMBER I. IRRATIONAL NUMBERS

JOHN N. CROSSLEY

Part 2*

4579
CROSSLEY, J.N.
1778

9. Approximations

So far in all the mathematics we have discussed we have avoided mention of irrationals or even fractions.

In the time of Pythagoras and even until later

"... no example of simple division nor any rules for division are found in Greek arithmetical literature. The operation must have been performed by subtracting the divisor or some easily ascertained multiple of the divisor from the dividend and repeating this process with the successive remainders. The several quotients were then added together. But the Greeks had no name for a *quotient* and did not conceive the result of a division as we do. To a Greek 5 was not the *quotient* of $\frac{35}{7}$. The operation did not discover the fact that 5 times 7 is 35 but that a seventh part of 35 contains 5, and so generally in Greek a division sum is not stated in the form "Divide a by b ", but in the form "Find the b th part of a ."⁷⁹

On the other hand, the whole thrust of Pythagorean number theory as we know it (principally through Euclid) is in terms of submultiples. Now in Euclid Book VII Definitions we read:

3. A number is a *part* of a number, the less of the greater, when it measures the greater;
4. but *parts* when it does not measure it.
5. The greater number is a *multiple* of the less when it is measured by the less⁸⁰.

And Heath is generally followed in believing these definitions to be Pythagorean⁸¹. Moreover, Nicomachus⁸² also, in discussing the concept of number as a multitude⁸³, puts his whole discussion in the context of a "Pythagorean doctrine"⁸⁴. Moreover, even Iamblichus who wrote a commentary on Nicomachus⁸⁵ referred to the writings of many "ancient Pythagoreans ... and all the books which they published" as being such that

*. Part 1 appeared in Vol. 4 No. 3 of the Gazette. Footnote 70 in Part 1 should read "See above, Section 5."

79. Gow [6], p.51. (References are to the bibliography in part 1.)

80. Book VII Euclid [2], vol. II, p.279.

81. Thomas [23], vol. I, p.66n. and Heath, see n. 80.

82. Nicomachus lived in the second century A.D. See Nicomachus [15], p.71.

83. *ibid.*, p.190.

84. *ibid.*

85. Iamblichus [8].

"most of [them] have been preserved even to our time"⁸⁶. So it is possible that Nicomachus had such resources too. Thus we have grounds for regarding Heath as expounding Pythagorean doctrine when he says:

"By a *part* Euclid means a submultiple, as he does in v. Def. 1, with which definition this one is identical except for the substitution of *number* (ἀριθμός) for *magnitude* (μέγεθος); cf. note on v. Def. 1. Nicomachus uses the word "submultiple" (ὑποπολλαλόσιος) also. He defines it in a way corresponding to his definition of multiple (see note on Def. 5 below) as follows (I. 18, 2): "The submultiple, which is by nature first in the division of inequality (called) less, is the number which, when compared with a greater, can measure it more times than once so as to fill it exactly (πληροῦντως)." Similarly *sub-double* (ὑποδιπλασίος) is found in Nicomachus meaning *half*, and so on."⁸⁷

and

"The definition of a *multiple* is identical with that in v. Def. 2, except that the masculine of the adjectives is used agreeing with ἀριθμός understood instead of the neuter agreeing with μέγεθος understood. Nicomachus (I. 18, 1) defines a multiple as being "a species of the greater which is naturally first in order and origin, being the number which, when considered in comparison with another, contains it in itself completely more than once."⁸⁷

Thus the whole basis of the Pythagorean treatment of what we could call 'fractions' appears to consist in a reduction to integral multiples of a basic unit. In Egypt, however, a tremendous procedure is needed to deal with (what to us is) the simplest addition of fractions; about 20% of the RMP is occupied with tables for $2/n$ (n an odd number from 3 to 101, inclusive), thus indicating that adding say $1/7$ to $1/7$ was non-trivial to the Egyptians. In this example $1/7 + 1/7 = 1/4 + 1/28$, as the Egyptians eschew numerators greater than one (except in the case of $2/3$) and always insist on sums of distinct submultiples⁸⁸. Whether it was so difficult for the Babylonians is difficult to say, for non-sexagesimal fractions do not seem to be very common in cuneiform texts.

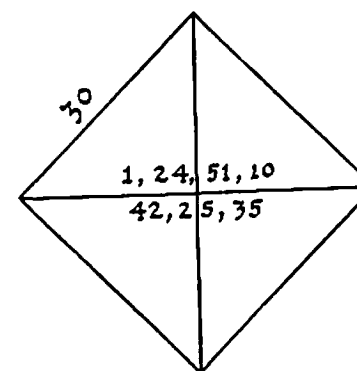
In Egypt, as we noted above, mathematics was of a very practical bent and although we have much less evidence than we have for Babylonian mathematics the difficulties the Egyptians experienced with the representation of fractions argues against their concern with irrational number, though there is good evidence for $4(8/9)^2$ as a value of π in the Rhind Papyrus⁸⁹. Further, as we have argued above, the idea of anything related to Pythagoras' theorem in Egypt seems to be a late fabrication. However, we do not know of any actual approximate calculations in Egypt where exact answers would require the use of irrationals.

In Babylon the situation is quite different. The cuneiform tablets from Babylon that have been studied "can be classified into two major groups: 'table texts' and 'problem texts'."⁹⁰ Some of these latter are concerned with calculating the lengths of sides of geometric figures (e.g. BM34568 above⁹¹), but nowhere is there anything seen to be close to what we regard as the geometry of Thales and his successors. On the other hand there are two very important features of the tablets.

One is the presence of tables of reciprocals and the other the calculation of approximations to irrational numbers.

The Babylonians of about 1600 B.C. used a sexagesimal system (i.e. used 60 as a base instead of the 10 of the decimal system). However, they did not have a well-defined place system, thus a number (now represented) $1,30$ could be interpreted as 90 or $1\frac{1}{2}$ or $1/40$, etc. ... Presumably the context (not necessarily written) determined the desired value. However, because $60 = 2^2 \cdot 3 \cdot 5$ has a large number of factors (12), the reciprocals of many numbers with only 2, 3 and 5 as factors can be calculated exactly. Neugebauer and Sachs⁹² present many tables of reciprocals from cuneiform texts. These extend the numbers considered beyond simple ones like $1/6 = 10$, $1/12 = 5$ to ones even more complicated than $1/8,38,24 = 6,56,40$ ($1/0.144 = 6.94$ in modern notation)⁹³. Although these examples give neat fractions, YBC 10529⁹⁴ contains a list in which the reciprocals are approximations to three or four sexagesimal places. Thus approximations of reciprocals were used in Babylon.

The second important feature is the approximation of irrational numbers. Perhaps the most striking is one example for $\sqrt{2}$, though YBC 7302⁹⁵ gives the approximation 3 for π . The tablet YBC 7289⁹⁶ shows a square of side 30 ($= \frac{1}{2}$) and diagonal $42,25,35 = 30 \times (1,24,51,10)$. This last factor $(1,24,51,10)$ is a very good estimate of $\sqrt{2} = 1.4142 \dots$



(Neugebauer and Sachs mention in a footnote that there is a value of $1,25 = 1.41\bar{6}$ in a later, Seleucid, tablet⁹⁷.) It is impossible, without further evidence, to show how such a good value was obtained, but Neugebauer and Sachs give the following proposal which has (at the very least) the merit of giving exactly $1,24,51,10$.

"The procedure in question consists in the alternating approximation of a by arithmetic and harmonic means of previously found approximations. Let α_1 be any

86. Iamblichus [9], ch. XXIII, pp.55-56.

87. Euclid [2], vol. II, p.280.

88. Peet [17], p.15.

89. Rhind Mathematical Papyrus, problems S.41-43, 48, 50; Peet [17], p.81, 89.

90. Neugebauer [12], p.30.

91. Neugebauer [13], vol. III, p.17-19.

92. Neugebauer and Sachs [14], p.11 ff.

93. From the reverse of CBS 29.13.21 column III quoted in Neugebauer and Sachs [14], p.14.

94. Neugebauer and Sachs [14], p.16.

95. *ibid.*, p.44.

96. *ibid.*, p.42.

97. *ibid.*, p.43.

approximation of $\sqrt{a}$ such that $\alpha_1 > \sqrt{a}$. Then $\beta_1 = \frac{a}{\alpha_1}$ is also an approximation of $\sqrt{a}$ but deviates from the true value in the opposite direction because it follows from $\alpha_1 > \sqrt{a}$ that $\beta_1 = \frac{a}{\alpha_1} < \sqrt{a}$. We now derive a new pair of approximations, α_2 and β_2 , by

$$\alpha_2 = \frac{\alpha_1 + \beta_1}{2} \quad \beta_2 = \frac{a}{\alpha_2}$$

and continue this process by computing

$$\alpha_3 = \frac{\alpha_2 + \beta_2}{2} \quad \beta_3 = \frac{a}{\alpha_3}$$

etc.⁹⁸ It is evident that all the α 's are greater than $\sqrt{a}$, all β 's less than $\sqrt{a}$, but that each step diminishes the difference between corresponding approximations. We apply this method to $\sqrt{2}$ by starting with $\alpha_1 = \frac{3}{2}$ as the first rough approximation (α_1^2 would be 2;15). Then we obtain for the first pair

$$\alpha_1 = 1;30 \quad \beta_1 = \frac{2}{1;30} = 1;20.$$

The next step leads to

$$\alpha_2 = \frac{1}{2}(1;30 + 1;20) = 1;25$$

$$\beta_2 = \frac{2}{1;25} = 1;24, 42, 21, \dots$$

Here we have already reached the above mentioned value 1;25 as an approximation to $\sqrt{2}$. The very next step leads to

$$\alpha_3 = \frac{1}{2}(1;25 + 1;24, 42, 21, \dots) = 1;24, 51, 10, \dots$$

the value given in (1)⁹⁹. The fact that both values for $\sqrt{2}$ found in our texts are links of the same chain seems to be a rather strong argument in support of our explanation¹⁰⁰.

98. From $\alpha_2 = \frac{1}{2}(\alpha_1 + \beta_1)$ and $\beta_2 = \frac{a}{\alpha_2}$ it follows that

$$\beta_2 = \frac{2a}{\alpha_1 + \beta_1} = \frac{2\alpha_1\beta_1}{\alpha_1 + \beta_1}.$$

This expression is known as the "harmonic mean" of α_1 and β_1 . (Footnote of Neugebauer and Sachs.) (1) is the value on the Tablet.

99. More accurately: $\alpha_3 = 1;24, 51, 10, 35, \dots$. (Footnote of Neugebauer and Sachs.)

100. It should be remarked that the expansion of $\sqrt{2}$ into a continued fraction also leads to (1), but not until the seventh step. (Footnote of Neugebauer and Sachs.)

Thus the Babylonians were either capable of calculating lengths without having to reduce figures to multiples of a basic unit or else did not strive to get complete accuracy in their calculations. In either case the arithmetic involved in Babylonian mathematics is markedly different from the Greek arithmetic of natural numbers before the discovery of the irrationality of $\sqrt{2}$. This appears to support the view that Pythagoras did not learn mathematics from Babylon or at least was not strongly influenced by Babylonian mathematics, especially as these tablets date from both the Old Babylonian period (circa 1900 B.C. to 1600 B.C.)¹⁰¹ and the later period (700 B.C. - A.D. O.)¹⁰². We have no evidence that Pythagoras was in fact involved in the Babylonian style of mathematics at all.

Thus we see that both in Babylon and in Egypt there was concern with approximate values, but that this was not so in Greece. For our present concern it is immaterial whether the Babylonians and Egyptians were aware of the possibility of (real) numbers which could not be finitely expressed as fractions (or finite decimals or sexagesimals) and which therefore could never be written down exactly.

On the other hand the "Greeks followed the ancient plan of avoiding, by the use of submultiples, the difficulty of computing with fractions; but in due time [about the 3rd century B.C.] the need for a fraction symbolism became so apparent that they developed a system that served their purposes fairly well."¹⁰³

"The natural numbers seem to have served the purposes of the world until about the beginning of the historic period. Men broke articles and spoke of the broken parts, but even after weights came into use it was not the custom to speak of such a fraction as $\frac{3}{4}$ of a pound. The world avoided difficulties of this kind by creating such smaller units as the ounce and then speaking of the particular number of ounces."¹⁰⁴

Even in the fifth century the Greeks were using a system of literals for numerals which did not employ a place notation and indeed it is not clear how much they were using symbols (letters) for numbers at the time of Pythagoras. Thus Heath¹⁰⁵ gives various not completely conclusive arguments for dates from 750 B.C. on for the introduction of such symbols and notes that "it was a long time before the alphabetic numerals found general acceptance. They were not officially used until the time of Ptolemies [3rd century B.C.]."¹⁰⁶ According to Heath it is not until Aristarchus (c. 310-230 B.C.) that we find fractions used in Greek mathematics¹⁰⁷.

10. *Figurate numbers*

What we believe to be Pythagorean arithmetic as handed down to us by, for example, Nicomachus¹⁰⁸ certainly depends heavily on the representation of numbers by arrays of points (or pebbles or etc.).

"For example, the writing of one unit by means of one alpha will be the sign for 1; two units side by side, that is, a series of two alphas, will be the sign for 2; when three are put in a line it will be the character for 3, four in a line for 4, five for 5, and so on. For by means of such a notation and indication alone could

101. Neugebauer and Sachs [14], p.39.

102. *ibid.*, p.37.

103. Smith [21], vol. II, p.214.

104. *ibid.*, p.208.

105. Heath 7, p.33 ff.

106. *ibid.*, p.34.

107. *ibid.*, p.43.

108. Nicomachus [15], ch. XII, p.246 ff.

the schematic arrangement of the plane and solid numbers mentioned be made clear and evident, thus:

The number 1, α
 The number 2, $\alpha \alpha$
 The number 3, $\alpha \alpha \alpha$
 The number 4, $\alpha \alpha \alpha \alpha$
 The number 5, $\alpha \alpha \alpha \alpha \alpha$

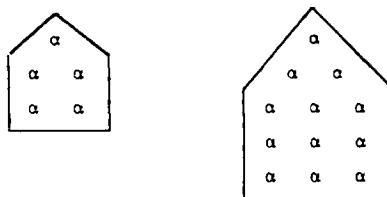
and further in similar fashion."¹⁰⁹

"To illustrate and classify, linear numbers are all those which begin with 2 and advance by the addition of 1 in one and the same dimension; and plane numbers are those that begin with 3 as their most elementary root and proceed through the next succeeding numbers. They receive their name also in the same order; for there are first the triangles, then the squares, the pentagons after these, then the hexagons, the heptagons, and so on indefinitely, ..."¹¹⁰

For a further example we have:

"The pentagonal number is one which likewise upon its resolution into units and depiction as a plane figure assumes the form of an equilateral pentagon. 1, 5, 12, 22, 35, 51, 70, and analogous numbers are examples. Each side of the first actual pentagon, 5, is 2, for 1 is the side of the pentagon potentially first, 1; 3 is the side of 12, the second of those listed; 4, that of the next, 22; 5, that of the next in order, 35, and 6 of the succeeding one, 51, and so on. In general the side contains as many units as are the numbers that have been added together to produce the pentagon, chosen out of the natural arithmetical series set forth in a row."¹¹¹

For example, 5 and 12 are represented as



The theory of proportions is regarded even by as late an exponent as Nicomachus as being something quite distinct.

"After this", he says at the beginning of chapter XXI, "it would be the proper time to incorporate the nature of proportions, a thing most essential for speculation about the nature of the universe and for the propositions of music, astronomy, and geometry, and not least for the study of the words of the ancients,

and thus to bring the *Introduction to Arithmetic* to the end that is at once suitable and fitting."¹¹³ Further, all the proportions he discusses are presented in integers. Indeed the whole book contains no reference to irrationals. Of course it could be argued that since the book is entitled *Introduction to Arithmetic* there is no need to mention numbers other than integers. However, Robbins and Karpinski in their introduction to the translation of this work discount the influence of Euclid on its layout.

"It cannot even be positively stated that Nicomachus used Euclid's *Elements*. There can be no question of course that he knew it. But the two works are of entirely different character, Euclid defining and demonstrating, Nicomachus defining and laying down general principles with abundant illustration and explanation. If in any respect Euclid could have served as a basis for Nicomachus, we should at once think of his definitions, but even here it will be observed that there are many divergences between the two. In Euclid's *Elements*, however, there were at least a pattern of arrangement and an example showing what subjects needed exposition. It is perhaps in this general way, if at all, that we are to look for a relation between the two. It may be confidently stated, nevertheless, that Euclid did not serve as the only model, nor even as the principal model, for Nicomachus."¹¹⁴

Thus there is a clear distinction drawn here between the mathematics of Nicomachus (meaning, in our terminology, arithmetic and the geometry of rectilinear plane figures) and the theory of mensuration and proportion (as treated in, say, Euclid's Books VII, VIII and IX). If we regard Nicomachus as proceeding in the Pythagorean spirit as he appears to do¹¹⁵ then it is reasonable to conclude that the orientation of the Pythagoreans was very much in terms of representation of numbers by geometrical forms. Such a geometrical representation is, in our terms, closely akin to working in a discrete space rather than a continuous one. Now we know that it is quite possible to do topology in such a space¹¹⁶ and it follows that there is no need, from the philosophical point of view, to insist on a continuous space. It therefore also follows that the move from discrete geometry, if we may call Nicomachus' graphical presentation of number in that way, to the comparison of lengths is just as inessential.

However, let us now turn to the discovery (or invention) of irrational numbers.

11. Irrational numbers

The earliest explicit account of irrationals that is extant is in Plato's *Theaetetus*:

Theaetetus. Theodorus was writing out for us something about roots, such as the sides of squares three or five feet in area, showing that they are incommensurable by the unit: he took the other examples up to seventeen, but there for some reason he stopped. Now as there are innumerable such roots, the notion occurred to us of attempting to find some common description which can be applied to them all.

¹⁰⁹. *ibid.*, ch. VI, p.237.

¹¹⁰. *ibid.*, ch. VII, p.239.

¹¹¹. *ibid.*, ch. X, p.243.

¹¹². *ibid.*, p.264 n.1.

¹¹³. *ibid.*, ch. XXI, p.264.

¹¹⁴. *ibid.*, p.34.

¹¹⁵. See e.g. *ibid.*, pp. 254, 266, 283.

¹¹⁶. See, for example, B. Abramenko, On Dimensionality and Continuity of Physical Space and Time, *Brit. J. Phil. Sci.* 34 (1958), p.95 ff.

Socrates. And did you find any such thing?

Theaetetus. I think that we did; but I should like to have your opinion.

Socrates. Let me hear.

Theaetetus. We divided all numbers into two classes: those which are made up of equal factors multiplying into one another, which we compared to square figures and called square or equilateral numbers; - that was one class.

Socrates. Very good.

Theaetetus. The intermediate numbers, such as three and five, and every other number which is made up of unequal factors, either of a greater multiplied by a less, or of a less multiplied by a greater, and, when regarded as a figure, is contained in unequal sides; - all these we compared to oblong figures, and called them oblong numbers.

Socrates. Capital; and what followed?

Theaetetus. The lines, or sides, which have for their squares the equilateral plane numbers, were called by us lengths; and the lines whose squares are equal to the oblong numbers, were called powers or roots; the reason of this latter name being, that they are commensurable with the former not in linear measurement, but in the area of their squares. And a similar distinction was made among solids.¹¹⁷

The *Theaetetus* is thought to have been written about 368 B.C.¹¹⁸, in other words about 200 years after Pythagoras' birth. Given that Theodorus starts off with $\sqrt{3}$ and $\sqrt{5}$ and does not mention $\sqrt{2}$ it is reasonable to assume that the incommensurability of $\sqrt{2}$ was well-known at that time. Conflicting views on the name of the exact discoverer persist and even the approximate date of the original discovery continues to be in doubt. However, ingenious arguments for the discoverer being Hippasus and the date being about 450 B.C. have been made. The best of these arguments may well be those of von Fritz¹¹⁹. We shall give some of his arguments a little later.

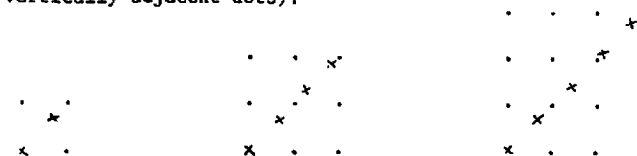
Let us consider two very different ways of displaying irrational numbers. The first may be called algebraic. This is the proof perhaps referred to by Aristotle:

"... the fallacy is obvious; as for example that if the diagonal of a square is taken to be commensurable, odd numbers are equal to even ones."¹²⁰

The more modern version proceeds: Suppose $\sqrt{2} = m/n$, where m, n are positive integers with no common factor. Then $m = n\sqrt{2}$ and $m^2 = 2n^2$. Hence 2 divides m^2 and therefore 2 divides m . But then $m = 2p$, say, and therefore $(2p)^2 = 4p^2 = 2n^2$ and $2p^2 = n^2$. The same argument now yields 2 divides n . Therefore 2 divides both

m and n contrary to our hypothesis. Thus $\sqrt{2}$ is not expressible as m/n , that is to say, $\sqrt{2}$ is irrational.¹²¹

Whether, in fact, the incommensurability of $\sqrt{2}$ was established inductively rather than deductively is by no means clear. Now von Fritz¹²² also points out how certain dot figures (or figurate numbers) corresponding to integers can be used to demonstrate a significant difference between 'oblong' and 'square' numbers, that is, numbers representable only by configurations of unequal sides, as for example :: and those which can be represented by squares, for example :: . As we have seen this is very much in the Pythagorean tradition (see §9 above). It is by no means impossible that by taking larger and larger squares people came to realize that the super-position of a row of similarly spaced dots along the diagonal never exactly fits the diagonal. Thus one may, for a start, consider the following figures (where we have marked the diagonal dots with crosses and the distance between crosses is the same as between horizontally or vertically adjacent dots):



The other approach is a geometric one. One geometric proof is presented very nicely by von Fritz¹²³. This approach depends on the discovery of incommensurables via five-sided figures rather than squares and triangles. The pentagram (or five-pointed star) which can be drawn without removing pencil from paper is very much associated with the Pythagoreans¹²⁴. Moreover, von Fritz points out that dodecahedral crystals of pyrite with its regular pentagonal faces occur naturally in Italy¹²⁵. The basis of von Fritz's theory comes from Euclid.

Almost immediately after the definitions of number cited above from Euclid Book VII we find the Euclidean algorithm.

"Proposition I. Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another."¹²⁶

"Proposition II. Given two numbers not prime to one another, to find their greatest common measure."¹²⁷

The essence of the proofs is perhaps aptly described as by means of measuring off lengths representing numbers against each other. The use of this technique is what von Fritz suggests.

121. Heath says in Euclid [2], vol. III, p.2: "The actual method by which the Pythagoreans proved the incommensurability of $\sqrt{2}$ with unity was no doubt that referred to by Aristotle [see note 120] ... The proof formerly appeared in the texts of Euclid as X.117, but it is undoubtedly an interpolation."

122. von Fritz [4], p.254.

123. *ibid.*, pp.257 ff.

124. de Vogel [24], pp.29 ff, p.156.

125. von Fritz [4], p.256.

126. Euclid 2, vol. II, p.296.

127. *ibid.*, p.298.

117. Plato, *Theaetetus* 147d-148b.

118. Jowett [10], vol. III, p.192.

119. von Fritz [4].

120. Aristotle, *Prior Analytics* i. 23, 41a 26-27 quoted in Thomas [23], vol. I, p.111.

"... if one looks at the pentagram or at a regular pentagon with all its diameters filled in - and we have seen that the Pythagoreans were interested in diameters - the fact that the process of mutual subtraction goes on infinitely, that therefore there is no greatest common measure, and that hence the ratio between diameter and side cannot be expressed in integers however great, is apparent almost at first sight. For one sees at once that the diameters of the pentagon form a new regular pentagon in the centre, that the diameters of this smaller pentagon will again form a regular pentagon, and so on in an infinite process."¹²⁸

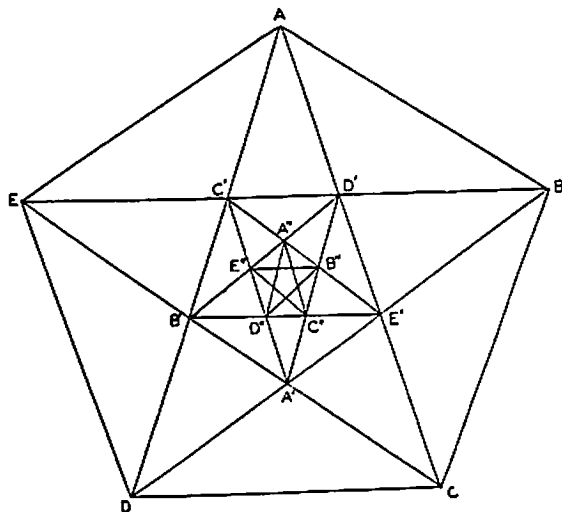


Figure 2.

However, he then provides a somewhat more elaborate proof of the relations between the various lengths.

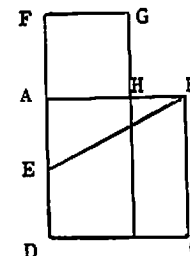
"It is then also very easy to see that in the pentagons produced in this way $AE = AB'$ and $B'D = B'E'$ and therefore $AD - AE = B'E'$, and likewise $AE = ED' = EA'$ and $B'E' = B'D = B'E$ and therefore $AE - B'E' = B'A'$, and so forth ad infinitum, or, in other words, that the difference between the diameter and the side of the greater pentagon is equal to the diameter of the smaller pentagon, and the difference between the side of the greater pentagon and the diameter of the smaller pentagon is equal to the side of the smaller pentagon, and again the difference between the diameter of the smaller pentagon and its side is equal to the diameter of the next smaller pentagon and so forth in infinitum. Since ever new regular pentagons are produced by the diameters it is then evident that the process of mutual subtraction will go on forever, and that therefore no greatest common measure of the diameter and the side of the regular pentagon can be found.

One may, of course, still ask how the Pythagoreans could prove that $AE = AB'$ and $B'D = B'E'$, etc. Now Proclus, probably getting his information from Eudemus of Rhodes, states¹²⁹ that Thales was the author of the theorem that

in an isosceles triangle the base angles are equal. In connection with this it is important to note that Aristotle¹³⁰ refers to an archaic proof of this proposition. He does not quote all the steps of this proof, but what he quotes shows that 'mixed angles', i.e. angles formed by a straight line and the circumference of a circle, were used in the demonstration, and that in all likelihood the proof was based on a rather primitive method of superimposition¹³¹. It is clear that with this latter method the converse of the proposition could be proved without difficulty. It follows that the equality of AE with AB' and of $B'D$ with $B'E'$ could be derived from the equality of $\angle AEB'$ with $\angle AB'E$ and of $\angle B'DE'$ with $\angle B'E'D$, if these angles could be proved to be equal respectively.

As to this latter proof, the evidence is somewhat less definite. But Eudemus of Rhodes¹³² attributes to the early Pythagoreans the proof that the sum of the internal angles in any triangle is equal to two right angles. From this theorem the general theorem that in any polygon the sum of the internal angles is equal to $2n - 4$ right angles can very easily be derived, if one divides the polygon into triangles¹³³, and we know¹³⁴ that the Pythagoreans constantly experimented with dividing polygons into triangles. The proposition further more that in any polygon the sum of the external angles is equal to four right angles is a mere corollary of the preceding proposition¹³⁵. On the basis of these propositions, finally, the equality of the angles figuring in the demonstration suggested above can be very easily shown¹³⁶.

There are simpler ways of showing the existence of irrationals using the golden ratio. The golden ratio is the ratio $AB' : AD$ (see Fig. 2). Here B' divides AD in 'extreme and mean ratio'¹³⁷. The problem of locating B' is "a particular case of the problem of 'application of areas', and therefore was obviously within the power of the Pythagoreans"¹³⁸. The golden ratio features extensively in Euclid¹³⁹ and it is very easy to construct by simply drawing a square



130. Aristotle, *Prior Analytics* 41b, 13 ff. (von Fritz's footnote.)

131. For details see [Euclid [2]] I, 253 (von Fritz's footnote.)

132. Quoted by Proclus [[20], p.298] (von Fritz's footnote.)

133. The proof is quoted by Proclus, op.cit.. After the polygon has been divided into triangles, the proposition about the sum of the angles of a triangle being known, the remainder of the proof is a simple addition. (Footnote of von Fritz.)

134. von Fritz refers to his earlier note: "The Pythagoreans were, of course, aware that triangles are the only rectilinear figures whose shape is determined by the proportionate length of their sides. That they realized the importance of this fact for their theory seems proved by Theon's statement [Theon Smyrnaeus *Expos. Rerum Mathem.*], p.40 ff [Hiller] that they divided all other rectilinear figures into triangles.

135. Quoted by Proclus [[20], p.298] (von Fritz's footnote.)

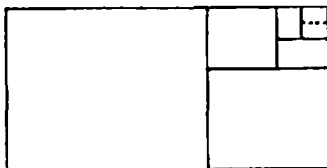
136. von Fritz [4], p.258.

137. Euclid [2], vol. II, p.11.

128. von Fritz [4], p.257.

129. [Proclus [20], p.195] (Equivalent of von Fritz's footnote.)

$ABCD$, bisecting AD at E giving BE such that $AE : BE = 1 : \sqrt{5}$. Thence $\frac{1}{2}(\sqrt{5} - 1)AE$ can be obtained by a simple construction and bisection, or as in Euclid Book II, Proposition 11, by setting $EF = BE$, completing the square $AFGH$ whence $AH : AD = \frac{1}{2}(\sqrt{5} - 1)$. Moreover, all that is required here is the construction of right angles and midpoints of lines. Thus there was no difficulty in constructing the golden ratio and from that pentagrams, or *vice versa*. However, once the golden ratio has been constructed, a simpler argument in the style of von Fritz's will yield the incommensurability of the golden ratio by using the figure¹⁴⁰



What we are at pains to point out here is not exactly what methods the Pythagoreans used but merely the possibility of at least two quite distinct approaches to the problem, which were, so far as we know, feasible for them.

12. Conclusion

As we have seen, Egyptian mathematics (and Babylonian, too, for that matter) was very much concerned with geometry in an arithmetic framework, that is to say, with problems of mensuration. The time of Thales is generally regarded as heralding the start of Euclidean style geometry, though it is by no means clear how close to the presentation in the books of Euclid the studies were in Thales' time. Further, in Babylon and Egypt (as is obvious from the problems considered on many cuneiform tablets and in the Rhind Papyrus¹⁴¹) and presumably in Greece too, natural numbers were being used in commercial transactions and in particular the dimensions of land were being measured¹⁴². Thus there was an application of (natural) numbers to concrete geometric figures (fields, etc.). So long as there is only this one way application the correlation works well. But once this idea that corresponding to each (even merely straight) edge of a field there should correspond a number, a whole new area is opened up. So long as it is possible, by shortening the basic unit of length, to measure each line as an integral multiple there is no need to extend the number system beyond the natural numbers. It is, in fact, not necessary to employ even fractions explicitly. But when it is insisted that just as numbers may be applied to lengths, so too must a length have a number corresponding to it, then is the point at which the possibility of incommensurables arises¹⁴³.

Now there is, as we have just pointed out, no *necessity* to make the connection (between numbers and lengths) in *both* directions.

Now, according to Aristotle, the Pythagoreans regarded numbers (meaning natural numbers rather than fractions or real numbers or etc.) as being "the first things in the whole of nature"¹⁴⁴. But by Aristotle's time the interconnection between lines and numbers had been hypostatized. Aristotle saw the difficulty in the Pythagorean view that all things are number but had already opted for the view which regards real space as continuous.

"For it is not true to speak of indivisible spatial magnitudes; and however much there might be magnitudes of this sort, units at least have not magnitude; and how can a magnitude be composed of indivisibles? But arithmetical number, at least, consists of units, while these thinkers identify number with real things; at any rate they apply their propositions to bodies as if they consisted of those numbers."¹⁴⁵

Considerably later Proclus¹⁴⁶, writing on Euclid, also points out how the existence of irrationals is what distinguishes geometry and arithmetic.

"If there were no infinity, all magnitudes would be commensurable and there would be nothing inexpressible or irrational¹⁴⁷, features that are thought to distinguish geometry from arithmetic; ..."¹⁴⁸.

Thus this later view concerned the division of mathematics into arithmetic and geometry, but there is a close connection. Gow goes so far as to state

"*Number absolute* was the field of arithmetic: *number applied* of music: *stationary magnitude* of geometry, *magnitude in motion* of spheric or astronomy. But they did not so strictly dissociate discrete from continuous quantity. An arithmetical fact had its analogue in geometry and *vice versa*; a musical fact had its analogue in astronomy and *vice versa*. Pythagorean arithmetic and geometry should therefore be treated together, ..."¹⁴⁹.

But a most important question is how soon the "*vice versa*" analogue of geometry in arithmetic was appreciated. I suggest it was closely associated with the discovery of irrational numbers for it is then that the question becomes significant.

At this discovery the foundation of the Pythagorean philosophy, namely that the whole essence of the world is number¹⁵⁰, meaning natural number, is destroyed. I therefore suggest that when we read in the Scholium to Book X of Euclid¹⁵¹ that "if any psyche were to make an assault on such a form of existence as this, he would simply make obvious and clear the fact that he is being carried under into the sea of

138. Heath[7], vol. I, p.161.

139. See especially Books II and IV.

140. This observation is due to C.J. Ash.

141. Neugebauer [12], *passim*.

142. Peet [17], p.94 Problem no. 52 "Examples of reckoning a truncated triangle of land."

143. This point has also been made in an informal way in a paper of J. McGeachie (unpublished).

144. Aristotle, *Metaphysics* 986a.

145. *ibid.*, 1083b.

146. Fifth century A.D. (Paulys Realencyclopädie der Classischen Altertumswissenschaft).

147. ἄπρητον, ἄλογον. This is a reflection of Euclid's distinction between two orders of irrationals (Bk. X. Def. III and IV). ἄπρητον denotes a line incommensurable in length with a given (rational) line, ἄλογον a line which is commensurable neither in length nor in square with the given line (Footnote of Morrow).

148. Proclus [20], p.5.

149. Gow [6], pp.71-72.

150. See above p.5 ff.

151. Euclid [3], vol. V, p.417.

creation and is being overwhelmed by the unstable surges of it", it merely means what it says. I suggest that the discovery had a profoundly disturbing effect on the consciousness of the discoverer (be it Hippasus, as suggested by von Fritz [4] or someone else). Moreover, to overthrow the accepted doctrine by means which were acceptable to the givers of the doctrine must have made the effect all the more profound and disturbing. For, surely, in the discovery of incommensurables some proof¹⁵² must have been necessary, though, of course, it might have borne the merest shadow of resemblance to our styles of proof. Indeed, even if the 'proof' consisted of demonstrating, say using pebbles or dots, that the ratio of the diagonal of a square (or the ratio of a pentagon or pentagram's diagonal) to its side was incommensurable, the result *was* accepted or rejected with such ferocity as to generate a legend of quite dramatic nature.

Thus we conclude that the introduction of an analogue in arithmetic of an element of geometry was philosophically necessary, psychologically disturbing and from a mathematical point of view, both immensely stimulating and creative.

152. Thus Proclus [20], p.28 writes: "Every man who knows his science or his art should make his arguments appropriate to the things with which he is dealing. ... Even in mathematics we cannot demand the same degree of accuracy in all parts."

THE 21ST ANNUAL MEETING OF THE AUSTRALIAN MATHEMATICAL SOCIETY

16-20 May 1977, La Trobe University

Two distinguished mathematicians, Professor Karl Hofmann (Tulane University) and Dr. Shmuel Winograd (IBM, Yorktown Heights), visited Australia especially for the Annual Meeting. Dr. Winograd, whose visit was wholly supported by IBM (Australia) Ltd., spent about ten days in Australia during which he lectured in Sydney and Canberra as well as participating in the Meeting. Professor Hofmann spent about a month in Australia during which time he lectured at New South Wales, Sydney, Monash, Western Australia, Canberra, Adelaide and Flinders, and gave an extensive series of lectures at La Trobe. There is no doubt that the visits of these two mathematicians were a great success and the organisers of this meeting feel that future such visits should be encouraged.

The number of participants, 285, is apparently the greatest number ever to attend an AMS meeting and this large number of participants, together with a number of generous donations from companies, enabled the Meeting to return a profit of about \$3,400.

In addition to the usual invited general addresses and short contributed papers the meeting included about 20 invited specialist addresses. These covered the full spectrum of mathematics and proved very popular.

The conference was organised by S. Morris, K. Pearson, D. Ross and P. Stacey with substantial help from I. Robinson, A. Gray, D. Scott, R. Rimmer, B. Davey, J. Strantzen, B. Mond, G. Davis, C. Fox, S. Loch, A. Andrew and G. Elton. This long list indicates the large number of people involved in the organisation and the reason why the attention to detail, which helped to make the Meeting a success, was possible.

RECONSTRUCTION

D.A. HOLTON

Now that the Four Colour question has been settled it may appear that graph theorists have no major problem to work on. This article is an attempt to show that this is not the case.

1. *Ulam's Conjecture*

Perhaps the big problem of graph theory at the moment is Ulam's conjecture. Many of the "big guns" of the field are at present aimed at its downfall.

Take any old graph G with vertex set VG . Now remove a vertex v and all edges of G that are incident with v and call the graph obtained G_v . An example is given in Figure 1. Now form the collection $\bar{G} = \{G_v : v \in VG\}$. This is just all possible graphs G_v obtainable from G by removing the various vertices of G , one at a time.

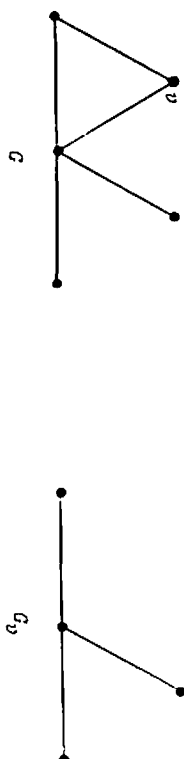


Figure 1.

But we can take some other graph H and repeat the process above to give the collection of subgraphs of H , $\bar{H} = \{H_w : w \in VH\}$. It may well be that $\bar{G} = \bar{H}$. Ulam conjectured that if this is so, then $G = H$. In other words if the two collections $\bar{G}$ and $\bar{H}$ are the same, then they must have come from the same graph.

The first counterexample to this conjecture is easily found. Consider the graphs G and H of Figure 2. They are different, but they give rise to the equal collections $\bar{G}$, $\bar{H}$. So we must amend the conjecture so that we only deal with graphs on more than two vertices.



Figure 2.